**Stat3021: Final**

**Summer 2002**

**Name:**

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**Problem 1 (8 points).** A particular airline has 10 A.M. flight from Chicago to New York, Atlanta, and Los Angeles. Let $A$ denote the event that New York flight is full and defines events $B$ and $C$ analogously for the other two flights. Suppose $P(A) = .6, P(B) = .5$ and $P(C) = .4$, and the three events are independent. What is the probability that

a) All three flights are full? That at least one flight is not full?

b) Only the New York flight is full? That exactly one of the three flights is full?

**Solution.**

a) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = 0.6 \cdot 0.5 \cdot 0.4$;

$$1 - P(A \cap B \cap C) = 1 - 0.6 \cdot 0.5 \cdot 0.4.$$

b) $P(A \cap B^c \cap C^c) = P(A) \cdot P(B^c) \cdot P(C^c) = 0.6 \cdot (1 - 0.5) \cdot (1 - 0.4)$.

$$P(A \cap B^c \cap C^c) + P(A^c \cap B \cap C^c) + P(A^c \cap B^c \cap C)$$

$$= 0.6 \cdot (1 - 0.5) \cdot (1 - 0.4) + (1 - 0.6) \cdot 0.5 \cdot (1 - 0.4) + (1 - 0.6) \cdot (1 - 0.5) \cdot 0.4.$$
Problem 2 (10 points). A chain of video stores sells three different brands of VCRs. Of its VCR sales, 50% are brand 1, 30% are brand 2, 20% are brand 3. Each manufacture offers a 1-year warranty on parts and labor. It is known that 25% of brand 1’s VCRs require warranty work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively. Randomly select a purchaser.

(a) Calculate $P$(the individual bought a brand 1 VCR that will need repair while under warranty);

(b) Calculate $P$(the individual bought a VCR that will need repair while under warranty);

(c) If a customer returns to the store with a VCR that needs warranty repair work, what is the chance that it is a brand 1 VCR?

Solution. (a) $0.5 \times 0.25$.

(b) $0.5 \times 0.25 + 0.3 \times 0.2 + 0.2 \times 0.1$.

(c) 

\[
\frac{0.5 \times 0.25}{0.5 \times 0.25 + 0.3 \times 0.2 + 0.2 \times 0.1}.
\]
Problem 3 (18 points). Let $X_1, X_2, \cdots, X_{100}$ denote the actual weights of 100 randomly selected 50-lb bags of fertilizer. If the expected weight of each bag is 50 and the standard deviation is 1.

(a) What are the mean and the variance of $\bar{X} = (X_1 + X_2 + \cdots + X_n)/n$?

(b) Calculate $P(49.75 \leq \bar{X} \leq 50.25)$ (approximately). We know $\Phi(2) = 0.9772; \Phi(2.2) = 0.9783, \Phi(2.5) = 0.9938$;

(c) What value is $X_1^2 + X_2^2 + \cdots + X_{100}^2$ around? Why?

Solution. (a) $E(\bar{X}) = 50$ and $Var(\bar{X}) = 1^2/100 = 0.01$.

(b) 

$$P(49.75 \leq \bar{X} \leq 50.25) = P \left( \frac{49.75 - 50}{\sqrt{0.01}} \leq Z \leq \frac{50.25 - 50}{\sqrt{0.01}} \right)$$

$$= \Phi(2.5) - \Phi(-2.5) = \Phi(2.5) - (1 - \Phi(2.5)) = 2\Phi(2.5) - 1.$$

(c) $E(X_1^2 + X_2^2 + \cdots + X_{100}^2) = 100 \times E(X_1^2)$. But $E(X_1^2) = (EX_1)^2 + Var(X_1) = 1^2 + 50^2 = 2501$. So $E(X_1^2 + X_2^2 + \cdots + X_{100}^2) = 250100$. 
**Problem 4 (15 points).** The weekly demand for propane gas (in 1000’s of gallons) from a particular facility is an random variable with probability density function

\[
f(x) = \begin{cases} 
2 \left(1 - \frac{1}{x^2}\right), & \text{if } 1 \leq x \leq 2; \\
0, & \text{otherwise.}
\end{cases}
\]

(a) Compute the cumulative distribution function of \(X\);

(b) What is the 95% percentile of \(X\)?

(c) If 1.5 thousand gallon is in the stock at the beginning of the week and no new supply is due in during the week, how much of the 1.5 thousand gallons is expected to be left at the end of the week?

**Solution.**

(a) For any \(x\) such that \(1 \leq x \leq 2\)

\[
F(x) = \int_1^x f(t) \, dt = 2 \int_1^x \left(1 - \frac{1}{t^2}\right) \, dt = 2 \left[ t + \frac{1}{t} \right]_1^x = 2 \left(x + \frac{1}{x}\right) - 4.
\]

Also \(F(x) = 0\) if \(x < 1\) and \(F(x) = 1\) if \(x > 1\).

(b) Set \(F(x) = 0.95\). That is

\[
2 \left(x + \frac{1}{x}\right) - 4 = 0.95.
\]

Simplify it. We obtain \(x^2 - 2.475x + 1 = 0\). Then

\[
x = \frac{2.475 \pm \sqrt{(-2.475)^2 - 4}}{2} = 0.51 \text{ or } 1.96.
\]

So the 95% percentile of \(X\) is 1.96.

(c) \(E \max(1.5 - X, 0)\). Now

\[
E \max(1.5 - X, 0) = \int_1^{1.5} (1.5 - x)f(x) \, dx = 1.5F(1.5) - \int_1^{1.5} xf(x) \, dx = 0.061.
\]
Problem 5 (19 points). Suppose $X$ and $Y$ have joint probability density function

$$ f(x, y) = \begin{cases} 
C(2x + 3y^2), & \text{if } 0 \leq x \leq 1, 0 \leq y \leq 1; \\
0, & \text{otherwise}. 
\end{cases} $$

(i) Calculate $C$;

(ii) Calculate the marginal probability density function $f_X(x)$ and $f_Y(y)$. Are $X$ and $Y$ independent? Why?

(iii) Calculate $EX$, $EY$, $Var(X)$ and $Var(Y)$;

(iv) Calculate $Cov(X, Y)$ and the correlation coefficient $Corr(X, Y)$.

Solution. (i) Use the fact $1 = \int_0^1 \int_0^1 f(x, y) \, dx \, dy$, and then evaluate the integral. This gives $C = 1/2$.

(ii) $f_X(x) = \int_0^1 f(x, y) \, dy = x + 0.5$ if $0 \leq x \leq 1$; $f_X(x) = 0$ otherwise. Similarly, $f_Y(y) = (3y^2/2) + (1/2)$, $0 \leq y \leq 1$. The random variables $X$ and $Y$ are independent iff $f(x, y) = f_X(x) \cdot f_Y(y)$ for any values $x$ and $y$. This is not the case. So $X$ and $Y$ are independent.

(iii) $EX = \int x f_X(x) \, dx = 7/12$. Similarly, $EY = 5/8$. Also, $EX^2 = \int x^2 f_X(x) \, dx = 5/12$; $EY^2 = 7/15$. So variance of $X$ is $Var(X) = E(X^2) - (EX)^2 = 11/144$ and $Var(Y) = 73/960$.

(iv) $Cov(X, Y) = E(XY) - (EX)(EY)$. But $E(XY) = \int \int xy f(x, y) \, dx \, dy = 17/48$. So $Cov(X, Y) = -1/96$ and

$$ Corr(X, Y) = \frac{Cov(X, Y)}{\sigma_X \cdot \sigma_Y} = -0.1367. $$
Problem 6 (15 points). Let \(X_1, X_2, \cdots, X_n\) be a random sample from the population \(N(\mu, \sigma^2)\) with \(\mu\) and \(\sigma^2\) unknown.

(i) Find estimators for \(\mu\) and \(\sigma^2\) by using the Moment method;
(ii) Find estimators for \(\mu\) and \(\sigma^2\) by using the Maximum Likelihood Estimation method;
(iii) Are the estimators obtained from (ii) biased? Why?

Solution. (i) Set \(\hat{\mu} = \bar{X}\) and \(E(X^2) = \mu^2 + \sigma^2 = (1/n) \sum_{i=1}^{n} X_i^2\). Solve the two equations, one obtains \(\hat{\mu} = \bar{X}\) and \(\hat{\sigma}^2 = (1/n) \sum_{i=1}^{n} X_i^2 - \bar{X}^2 = (1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2\).

(ii) By independence

\[
f(x_1, x_2, \cdots, x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n (\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right).
\]

Take log

\[
\log f = C(x_1, \cdots, x_n) = -\frac{n}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2.
\]

Take derivative

\[
\frac{\partial(\log f)}{\partial \mu} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu) = \frac{1}{\sigma^2}(n\mu - n\bar{X})
\]
\[
\frac{\partial(\log f)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (x_i - \mu)^2.
\]

Let the above be equal to zero and then solve for the two equations. We obtain that \(\mu = \bar{X}\)
and \(\hat{\sigma}^2 = (1/n) \sum_{i=1}^{n} X_i^2 - \bar{X}^2 = (1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2\).

(iii) Since \(E(\bar{X}) = \mu\) and \(E((1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2) = (n - 1)/n\sigma^2\), we have that \(\bar{X}\) is unbiased and \((1/n) \sum_{i=1}^{n} (X_i - \bar{X})^2\) is biased.
Problem 7 (15 points). A sample of 26 pieces of laminate used in the manufacture of circuit boards was selected and the amount of warpage (in.) under particular conditions was determined for each piece, resulting in a sample mean warpage of 0.0635 and a sample standard deviation of 0.0065.

(i) Suppose the amount of warpage follows a normal distribution. Construct a 95% confidence interval for the true mean warpage;

(ii) Suppose we do not know the distribution of the amount of warpage. A sample of 49 pieces of laminate was selected with the same sample mean and sample standard deviation. Construct a 95% confidence interval for the true mean warpage.

You may use the following information: \( t_{0.025,26} = 2.056, \ t_{0.025,25} = 2.06, \ z_{0.025} = 1.96. \)

Solution. (i) When the standard deviation for the normal distribution is unknown, use \( t \)-distribution to construct confidence intervals and do hypothesis test. In our case, the C.I. is

\[
\bar{X} \pm t_{1-\frac{\alpha}{2}, n-1} \cdot \frac{S}{\sqrt{n}} = 0.0635 \pm 2.016 \cdot \frac{0.0065}{\sqrt{26}} = (0.0687, 0.06613).
\]

(ii) Use large sample \( z \)-test.

\[
\bar{X} \pm z_{1-\frac{\alpha}{2}} \cdot \frac{S}{\sqrt{n}} = 0.0635 \pm 1.96 \cdot \frac{0.0065}{\sqrt{49}} = (0.06168, 0.06532).
\]