Supplementary material for wild residual bootstrap inference for penalized quantile regression with heteroscedastic errors

BY LAN WANG

School of Statistics, University of Minnesota, 224 Church Street South East,
Minneapolis, Minnesota 55455, U.S.A.
wangx346@umn.edu

INGRID VAN KEILEGOM

Research Centre for Operations Research and Business Statistics, KU Leuven,
Naamsestraat 69, B-3000 Leuven, Belgium
ingrid.vankeilegom@kuleuven.be

AND ADAM MAIDMAN

School of Statistics, University of Minnesota, 224 Church Street South East,
Minneapolis, Minnesota 55455, U.S.A.
maidm004@umn.edu

APPENDIX 1

Proofs of Lemma 1, Lemma 2 and Lemma A1

The proofs of Lemmas 1 and 2 combine the ideas in Wu & Liu (2009) and Wang et al. (2012). Section 3.3 of Wu & Liu (2009) considered an extension of the asymptotic theory of penalized quantile regression to the general heteroscedastic error setting but only a sketch of the derivation was provided in their online supplement. We provide a detailed derivation below for completeness.

Proof of Lemma 1. Write \( \delta = (\delta^T_1, \delta^T_2)^T \), where \( \delta_1 = (\delta_0, \delta_1, \ldots, \delta_q)^T \) and \( \delta_2 = (\delta_{q+1}, \ldots, \delta_p)^T \). Write \( \tilde{\delta} = (\tilde{\delta}_1, \tilde{\delta}_2)^T = n^{1/2}(\tilde{\beta} - \beta_0) \). Then \( \delta \) minimizes \( Q_n(\delta) \), where

\[
    Q_n(\delta) = \sum_{i=1}^{n} \left\{ \rho_\tau(\epsilon_i - n^{-1/2}x^T_i \delta) - \rho_\tau(\epsilon_i) \right\} + \lambda_n \sum_{j=1}^{p} w_j (|\beta_{0j} + n^{-1/2}\delta_j| - |\beta_{0j}|).
\]

It follows from Knight (1998) and Koenker (2005) that \( \sum_{i=1}^{n} \left\{ \rho_\tau(\epsilon_i - n^{-1/2}x^T_i \delta) - \rho_\tau(\epsilon_i) \right\} = -\delta^T H + \delta^T B_1 \delta/2 + o_p(1) \), where \( H \sim N(0, \tau(1-\tau)B_0) \). For the penalty term, we consider two cases. (i) For \( j = 1, \ldots, q, \beta_j \to \beta_{0j} \neq 0 \) in probability, and \( n^{1/2}(|\beta_{0j} + \delta_j/n^{1/2}| - \ldots

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Note that the above asymptotic normality result suggests that pr(\|β_0\| → δ_0 sign(β_0)). It follows that λ_n w_j (|β_0j| + δ_j/n^{1/2} - |β_0j|) = (n^{-1/2}λ_n)|β_0j| |γ_n^{1/2}((|β_0j| + δ_j/n^{1/2}) - |β_0j|)| → 0 as n^{-1/2}λ_n → 0. (ii) For j = q + 1, \ldots, p, λ_n w_j (|β_0j| + δ_j/n^{1/2} - |β_0j|) = (n^{-1/2}λ_n)|β_0j| |γ_n^{-1/2}δ_j|. Since n^{-1/2}λ_n → 0 and n^{-1/2}|β_0j| = O_p(1), the limit of λ_n w_j (|β_0j| + δ_j/n^{1/2} - |β_0j|) is zero if δ_j = 0 and is ∞ if δ_j ≠ 0. Hence

\[
Q_n(δ) → Q(δ) = \begin{cases} -δ^TH + δ^TB_1δ/2, & δ_{q+1} = \ldots = δ_p = 0, \\ +∞, & \text{otherwise}, \end{cases}
\]

in probability. Note that Q_n(δ) is convex in δ and its limit Q(δ) has a unique minimum (D_1^{-1}W, 0_q^T - q)^T, where W ∼ N(0, τ(1 - τ)D_0), D_0 = lim_{n→∞} n^{-1}∑_{j=1}^n x_jx_j^T and D_1 = lim_{n→∞} n^{-1}∑_{j=1}^n f_j(0)x_jx_j^T. It follows from the epi-convergence theory, see Geyer (On the asymptotics of convex stochastic optimization, technical report, 1996) and Knight (Epi-convergence in distribution and stochastic equi-continuity, technical report, 1999), that \( \tilde{δ} → \arg\min_δ Q(δ) \) in distribution. Hence \( \tilde{δ}_1 → D_1^{-1}W ∼ N(0, τ(1 - τ)D_1^{-1}D_1B_1D_1^{-1}) \) in distribution and \( \tilde{δ}_2 → 0 \) in distribution. This proves (ii).

Note that the above asymptotic normality result suggests that pr(j ∈ A) → 1 for j = 1, \ldots, q. To prove (i), it remains to show pr(j ∈ A) → 0 for j = q + 1, \ldots, p. For a given j ∈ \{q + 1, \ldots, p\}, let

\[
ξ_j(δ) = -τ n^{-1/2} \sum_{i=1}^n x_{ij}^T (\epsilon_i - n^{-1/2}x_i^T δ > 0) \\
+(1 - τ)n^{-1/2} \sum_{i=1}^n x_{ij}^T (\epsilon_i - n^{-1/2}x_i^T δ < 0) - n^{-1/2} \sum_{i=1}^n x_{ij}v_i + λ_n n^{-1/2} w_j sign(δ_j),
\]

where v_i = 0 if \( \epsilon_i - n^{-1/2}x_i^T δ \neq 0 \) and \( v_i ∈ [τ - 1, τ] \) otherwise. By the KKT optimality conditions (Boyd & Vandenberghe, 2004), if \( j ∈ A \), then there must exist some v_i^* such that v_i^* = 0 if \( \epsilon_i - x_i^T δ/n^{1/2} \neq 0 \) and \( v_i^* ∈ [τ - 1, τ] \) otherwise, such that for \( ξ_j(δ) \) with \( v_i = v_i^* \), \( ξ_j(δ) = 0 \). Hence

pr(j ∈ A) ≤ pr(ξ_j(δ) = 0). Note that λ_n n^{-1/2} w_j = (n^{−1/2}λ_n)(n^{1/2}|β_j|)^−γ → ∞ as n → ∞. Furthermore, we have

\[
-τ n^{-1/2} \sum_{i=1}^n x_{ij}^T (\epsilon_i - n^{-1/2}x_i^T δ > 0) + (1 - τ)n^{-1/2} \sum_{i=1}^n x_{ij}^T (\epsilon_i - n^{-1/2}x_i^T δ < 0) - n^{-1/2} \sum_{i=1}^n x_{ij}v_i^* \\
= n^{-1/2} \sum_{i=1}^n x_{ij}^T (I(\epsilon_i - n^{-1/2}x_i^T δ ≤ 0) - τ) - n^{-1/2} \sum_{i=1}^n x_{ij}^T (v_i^* + (1 - τ))
\]

where \( D = \{i : \epsilon_i - n^{-1/2}x_i^T δ = 0\} \). With probability one the number of elements in D is finite, following the same argument as in Section 2.2 of Koenker (2005). Therefore, \( n^{-1/2} \sum_{i∈D} x_{ij}^T (v_i^* + (1 - τ)) = O_p(n^{-1/2}) \). Similarly as in the proof of Lemma 4.3 of Wang et al. (2012), we can show that for any \( Δ > 0 \), as n → ∞,

\[
\sup_{\|δ^′ - δ\| ≤ Δ} n^{-1/2} \sum_{i=1}^n x_{ij}^T (I(\epsilon_i - n^{-1/2}x_i^T δ ≤ 0) - I(\epsilon_i - n^{-1/2}x_i^T δ^′ ≤ 0) - pr(\epsilon_i - n^{-1/2}x_i^T δ ≤ 0) + pr(\epsilon_i - n^{-1/2}x_i^T δ^′ ≤ 0)) = o_p(1),
\]
Therefore, \( \Pr(\cdot) \) denotes the \( L_2 \)-norm. As a result,

\[
\begin{align*}
    n^{-1/2} \left| \sum_{i=1}^{n} x_{ij} \{ I(\epsilon_i - n^{-1/2}x_i^T \hat{\delta} \leq 0) - \tau \} \right| \\
    \leq n^{-1/2} \sup_{|\delta' - \delta| \leq \Delta} \left| \sum_{i=1}^{n} x_{ij} \{ I(\epsilon_i - n^{-1/2}x_i^T \delta' \leq 0) - I(\epsilon_i - n^{-1/2}x_i^T \delta \leq 0) - \Pr(\epsilon_i - n^{-1/2}x_i^T \delta' \leq 0) \} \right| \\
    \quad + \Pr(\epsilon_i - n^{-1/2}x_i^T \delta \leq 0)) \right| + n^{-1/2} \sup_{|\delta' - \delta| \leq \Delta} \left| \sum_{i=1}^{n} x_{ij} \{ \Pr(\epsilon_i - n^{-1/2}x_i^T \delta' \leq 0) - \Pr(\epsilon_i - n^{-1/2}x_i^T \delta \leq 0) \} \right| \\
    \quad + n^{-1/2} \left| \sum_{i=1}^{n} x_{ij} \{ I(\epsilon_i - n^{-1/2}x_i^T \delta \leq 0) - \tau \} \right| \\
    = o_p(1).
\end{align*}
\]

Therefore, \( \Pr(j \in \hat{A}) \leq \Pr(\xi_j(\hat{\delta}) = 0) \to 0 \), for \( j = q + 1, \ldots, p \).

**Proof of Lemma 2.** Similarly as in the proof of Lemma 1, we can show that

\[
\begin{align*}
    \sum_{i=1}^{n} \{ \rho_{\tau}(\epsilon_i - x_i^T \delta/n^{1/2}) - \rho_{\tau}(\epsilon_i) \} + \lambda_n \sum_{j=1}^{p} (|\beta_{0j} + \delta_j/n^{1/2}| - |\beta_{0j}|) \\
    \to -\delta^T H + \delta^T B_1 \delta/2 + \lambda_0 \sum_{j=1}^{p} (|\delta_j|I(\beta_{0j} = 0) + \delta_j \text{sign}(\beta_{0j})I(\beta_{0j} \neq 0))
\end{align*}
\]

in distribution. The result then follows from epi-convergence theory. \( \square \)

**Proof of Lemma A1.** We have \( V_{1n}^*(\delta) = n^{-1/2} \sum_{i=1}^{n} x_i^T \delta \{ I(r_i|\epsilon_i| < 0) - \tau \} = -n^{-1/2} \sum_{i=1}^{n} x_i^T \delta \{ \tau - I(r_i < 0) \} \). Note that \( E^*\{ V_{1n}^*(\delta) \} = 0 \) and \( \text{var}^*\{ V_{1n}^*(\delta) \} = \tau(1 - \tau)n^{-1} \sum_{i=1}^{n} \delta^T x_i x_i^T \delta \to \tau(1 - \tau) \delta^T B_0 \delta \) in probability. To check the Lindeberg condition, it suffices to show that \( \forall \epsilon > 0, \)

\[
\begin{align*}
    n^{-1} \sum_{i=1}^{n} \left[ x_i^T \delta \{ I(r_i|\epsilon_i| < 0) - \tau \} \right]^2 I[|x_i^T \delta| > \epsilon \sqrt{n}] \to 0,
\end{align*}
\]

in probability. This holds by noting that the left side of the above expression is upper bounded by \( n^{-1} \sum_{i=1}^{n} (x_i^T \delta)^2 I[|x_i^T \delta| > \epsilon \sqrt{n}] \), which converges to zero in probability by the dominated convergence theorem. The result of the lemma follows from the Lindeberg central limit theorem. \( \square \)

**Appendix 2**

A Useful Lemma from Cheng & Huang (2010)

We use \( r = \{r_1, \ldots, r_n\} \) to denote the random bootstrap weights and \( z = \{z_1, \ldots, z_n\} \) to denote the random sample. Note that \( r \) and \( z \) induce two different sources of randomness. By the wild bootstrap mechanism, the distribution of \( r \) is independent of that of \( z \). We adopt the following notation from Cheng & Huang (2010). A random quantity \( R_n \) is said to be \( o_p^*(1) \) if for any \( \epsilon, \delta > 0 \), \( \Pr_z(\Pr_{r|z}(|R_n| > \epsilon) > \delta) \to 0 \), as \( n \to \infty \). Similarly, \( R_n \) is said to be \( \tilde{O}_p^*(1) \) if for all \( \delta > 0 \) there exists a \( 0 < M < \infty \) such that \( \Pr_z(\Pr_{r|z}(|R_n| > M) > \delta) \to 0 \), as \( n \to \infty \). And \( o_{p,z}(1) \), \( \tilde{O}_{p,z}(1) \) are the regular notion with respect to the joint probability distribution of \( r \) and \( z \).
The following lemma from Cheng & Huang (2010) will be used repeatedly in our proof. It allows
the transition of various stochastic orders in different probability spaces and leads to simplified
proofs in many places.

**Lemma B1.** (Lemma 3 of Cheng & Huang (2010)) Suppose that

\[ Q_n = o_{p_r}(1), \quad R_n = O_{p_r}(1) \]

We have

\[ A_n = o_{p_{r,z}}(1) \iff A_n = o_{p_r}(1), \]
\[ B_n = O_{p_r}(1) \iff B_n = O_{p_r}(1), \]
\[ C_n = Q_n \times O_{p_r}(1) \iff C_n = o_{p_r}(1), \]
\[ D_n = R_n \times O_{p_r}(1) \iff D_n = O_{p_r}(1), \]
\[ F_n = Q_n \times R_n \iff F_n = o_{p_r}(1). \]

**Appendix 3**

**Additional Examples of Random Weight Distribution**

The random weights used in the wild residual bootstrap procedure are generated from a
distribution \( G \) that satisfies Conditions 3–5 of the main paper. Two examples of such random
weight distributions were given in Feng et al. (2011). We propose below three new weight
distributions satisfying these conditions. Note that compared with the continuous distribution
in Feng et al. (2011), the new distributions given in Examples 1–2 below have no restrictions on
the value of \( \tau \).

**Example 1.**

\[ g_1(r) = G'_1(r) = \begin{cases} 
-\frac{r}{8v_1} \text{I}\{-2(\tau + v_1) \leq r \leq -2(\tau - v_1)\} \\
+\frac{r}{8v_2} \text{I}\{2(1-\tau - v_2) \leq r \leq 2(1-\tau + v_2)\},
\end{cases} \]

where \( 0 < v_1 < \tau \) and \( 0 < v_2 < 1 - \tau \).

**Example 2.**

\[ g_2(r) = G'_2(r) = \begin{cases} 
-\frac{r}{32v_1} \text{I}\{-4(a + v_1) < r < -4(a - v_1)\} \\
-\frac{r}{32v_2} \text{I}\{-4(\tau - a + v_2) < r < -4(\tau - a - v_2)\} \\
+\frac{r}{32v_3} \text{I}\{4(b - v_3) < r < 4(b + v_2)\} \\
+\frac{r}{32v_4} \text{I}\{4(1-\tau - b - v_3) < r < 4(1-\tau - b + v_2)\},
\end{cases} \]

where \( 0 < v_1 < a \), \( 0 < v_2 < \tau - a \), \( 0 < v_3 < b \), \( 0 < v_4 < 1 - \tau - b \), \( 0 < a < \tau \), and \( 0 < b < 1 - \tau \).
Example 3. The point mass distribution

\[ P(W = r) = a I \{ r = -4a \} + (\tau - a) I \{ r = -4(\tau - a) \} \]
\[ + b I \{ r = 4b \} + (1 - \tau - b) I \{ r = 4(1 - \tau - b) \}, \]

where \( 0 < a < \tau \) and \( 0 < b < 1 - \tau \).

Appendix 4

Additional Numerical Results

In Table 1, we summarize the simulation results on the comparison of empirical coverage probabilities (\( \times 100 \)) and average interval lengths (in parentheses) for 95% confidence intervals for \( \tau = 0.5 \), \( n = 250 \) and \( \tau = 0.7 \), \( n = 400 \) for the various methods described in Section 4.1 of the main paper. We note that the standard errors of the coverage probabilities are below 0.01 and the standard errors of the confidence interval lengths are below 0.005 for all cases. These results supplement those in Table 1 of the main paper and demonstrate further improvement with increased sample size.

Figure 1 displays the QQ plots of the quantiles of the wild residual bootstrapped estimator versus the empirical quantiles of the corresponding penalized estimator for estimating the smallest coefficient \( \beta_3 = 0.25 \) for both the \( L_1 \) penalty and the adaptive \( L_1 \) penalty when sample size \( n = 250 \) and 400, for \( \tau = 0.5 \) and 0.7, respectively. Overall, the wild residual bootstrapped distribution has satisfactory performance.

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References

Table 1: Empirical coverage probabilities (×100) and average interval lengths (in parentheses) for nominal 95% confidence intervals

<table>
<thead>
<tr>
<th>Method</th>
<th>$\tau = 0.5$</th>
<th>$\tau = 0.7$</th>
<th>Zeros</th>
<th>TP</th>
<th>FP</th>
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<td>$\beta_1 = \Phi^{-1}(\tau)$</td>
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<td>$\beta_2 = 0.25$</td>
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<td>$\tau$</td>
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<td>$n = 250$</td>
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<td>94.8 (0.09)</td>
<td>93.3 (0.09)</td>
<td>94.9 (0.07)</td>
<td>94.0 (0.08)</td>
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<tr>
<td>New AL2</td>
<td>91.4 (0.28)</td>
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<td>93.8 (0.08)</td>
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<td>94.3 (0.09)</td>
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<td>Full RS</td>
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<td>Oracle RS</td>
<td>-</td>
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<td>96.2 (0.12)</td>
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<td>95.0 (0.10)</td>
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<td>95.7 (0.09)</td>
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</table>

New AL1: adaptive $L_1$ method with wild residual bootstrap ($\gamma = 1$); New AL2: adaptive $L_1$ method with wild residual bootstrap ($\gamma = 2$); New L1: $L_1$ method with modified wild residual bootstrap (data-driven choice of $\alpha_n$); New L2: $L_1$ method with modified wild residual bootstrap ($\alpha_n = n^{-1/3}$); Full RS: full model with rank-score method; Full WB: full model with wild residual bootstrap; TS AL RS: two-step procedure, adaptive $L_1$ ($\gamma = 1$) followed by rank-score method for the refitted model; TS AL WB: two-step procedure, adaptive $L_1$ ($\gamma = 1$) followed by wild residual bootstrap for the refitted model; TS L RS: two-step procedure, lasso followed by rank-score method for the refitted model; TS L WB: two-step procedure, lasso followed by wild residual bootstrap for the refitted model; Oracle RS: oracle model with rank-score method; Oracle WB: oracle model with wild residual bootstrap; Zeros: the reported average coverage probability (length) is the average for all zero coefficients; TP: average number of true positives; FP: average number of false positives.
Fig. 1: QQ plots for the New AL1 (+), New AL2 (•), New L1 (△), and New L2 (×) methods for estimating $\beta_3 = 0.25$ when $n = 250$. (a) and (b) adaptive $L_1$ method when $\tau = 0.5$ and $\tau = 0.7$, respectively; (c) and (d) $L_1$ method when $\tau = 0.5$ and $\tau = 0.7$, respectively.