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Subgame-Perfect Equilibria for Stochastic Games

Ashok P. Maitra, William D. Sudderth
School of Statistics, University of Minnesota, Minneapolis, Minnesota 55455
{maitr001@umn.edu, bill@stat.umn.edu}

For an n-person stochastic game with Borel state space S and compact metric action sets $A^1, A^2, \ldots, A^n$, sufficient conditions are given for the existence of subgame-perfect equilibria. One result is that such equilibria exist if the law of motion $q(\cdot \mid s, a)$ is, for fixed s, continuous in $a = (a^1, a^2, \ldots, a^n)$ for the total variation norm and the payoff functions $f^1, f^2, \ldots, f^n$ are bounded, Borel measurable functions of the sequence of states $(s_1, s_2, \ldots) \in S^{\mathbb{N}}$ and, in addition, are continuous when $S^{\mathbb{N}}$ is given the product of discrete topologies on S.

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1. Introduction. The stochastic games we study have n players $1, 2, \ldots, n$. The state space S is a Borel subset of a Polish space. Each player i has a compact metric action set A^i. The set $\Delta(A^i)$ of probability measures defined on the Borel subsets of A^i is equipped with its usual weak topology and, hence, $\Delta(A^i)$ is also compact metrizable. Let $A = A^1 \times A^2 \times \cdots \times A^n$ have its product topology and it too is compact metrizable. The law of motion q is a conditional probability distribution on S given $S \times A$ with the interpretation that, if the players choose actions $a = (a^1, a^2, \ldots, a^n) \in A$ at state $s \in S$, then $q(\cdot \mid s, a)$ is the conditional distribution of the next state. We always assume that $q(B \mid s, a)$ is Borel measurable jointly in (s, a) for B a Borel subset of S and we will make further assumptions below.

The game begins at some initial state s_1. The players choose actions $a_1 = (a_1^1, a_1^2, \ldots, a_1^n)$ and the next state s_2 has distribution $q(\cdot \mid s_1, a_1)$. This process is iterated to generate a random *history* or *play*

$$h = (s_1, (a_1, s_2), (a_2, s_3), \ldots) \in H = S \times (A \times S)^{\mathbb{N}}.$$

Here, $a_k = (a_k^1, a_k^2, \ldots, a_k^n)$ is, for each k, the n-tuple of actions chosen by the players at stage k. Each player i has a bounded, Borel measurable payoff function $f^i\colon H \mapsto \mathbb{R}$ and receives $f^i(h)$ as payoff at history h. Let $f = (f^1, f^2, \ldots, f^n)$ be the n-tuple of payoff functions.

Denote by H^* the disjoint union of the sets $S, S \times (A \times S), S \times (A \times S)^2, \ldots$; that is,

$$H^* = S \cup \left(\bigcup_{k \geq 1} [S \times (A \times S)^k] \right).$$

The elements of H^* are called *partial histories*.

A *strategy* σ^i for player i assigns to each partial history $p = (s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k))$ in H^* the conditional distribution $\sigma^i(p) \in \Delta(A^i)$ for a_k^i given p. Formally, a strategy for player i is a Borel function from H^* into $\Delta(A^i)$. It is assumed that the players choose their actions independently. So the conditional distribution of a_k given p is the product measure

$$\sigma(p) = \sigma^1(p) \times \sigma^2(p) \times \cdots \times \sigma^n(p) \tag{1}$$

on A.

An n-tuple $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ consisting of a strategy for each player is called a *profile*. A profile σ together with an initial state s_1 determines the distribution $P_\sigma = P_{s_1, \sigma}$ of the history h. Note that, by an abuse of notation, we write $\sigma(p)$ for the n-tuple $(\sigma^1(p), \sigma^2(p), \ldots, \sigma^n(p))$ as well as for the product measure in (1). The meaning will always be clear from the context.

The stochastic game $\Gamma(f, s_1)$ begins at state s_1. The players select strategies to form a profile σ and each player i receives the expected payoff

$$E_\sigma f^i = \int f^i(h) P_\sigma(dh).$$

We write $E_\sigma f$ for the vector of expected payoffs $(E_\sigma f^1, E_\sigma f^2, \ldots, E_\sigma f^n)$.

To define a subgame of $\Gamma(f, s_1)$, let $p = (s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k))$ be a partial history. The length of p, written $lh(p)$, is defined to be $k-1$. (Thus $lh(s) = 0$ for $s \in S$.) Denote by $l(p)$ the last state s_k of p. The *subgame* $\Gamma(f, p)$ is the stochastic game with initial state $s_k = l(p)$, and payoff functions $f^i p$ defined for histories

$$h' = (s_k, (b_1, t_2), (b_2, t_3), \ldots)$$

to be

$$(f^i p)(h') = f^i(s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k), (b_1, t_2), (b_2, t_3), \ldots). \tag{2}$$

Thus $f^i p$ is the section of f^i at p. We use $\Gamma(f, \cdot)$ to denote the collection of all the games $\Gamma(f, p)$, $p \in H^*$. (Notice that the game $\Gamma(f, s)$ is itself the subgame $\Gamma(f, p)$ for which $p = s$.)

For a strategy σ^i and a partial history p, the *conditional strategy given* p is written $\sigma^i[p]$. If $\sigma = (\sigma^1, \sigma^2, \ldots, \sigma^n)$ is a profile of strategies, the *conditional profile* $\sigma[p]$ is just the profile of conditional strategies $(\sigma^1[p], \sigma^2[p], \ldots, \sigma^n[p])$.

Now, we can define a *subgame-perfect equilibrium* (SPE) for $\Gamma(f, \cdot)$ as being a profile σ such that, for all $p \in H^*$, the conditional profile $\sigma[p]$ is a Nash equilibrium for the subgame $\Gamma(f, p)$.

To prove the existence of an SPE, further conditions on the game are necessary. (Without additional assumptions, there need not exist an SPE even for one-person games.) To state one of the conditions, let $\Delta(S)$ be the set of probability measures defined on the Borel subsets of the state space S.

Condition 1 (Variation Norm Continuity). *For every fixed $s \in S$, the law of motion $q(\cdot \mid s, a)$ is a continuous function of a when $\Delta(S)$ is given the topology induced by the total variation norm.*

This is a very strong condition and it would be preferable to assume some sort of weak continuity for the law of motion such as Condition 3 in §3 below. However, as is explained in the introduction of Mertens and Parthasarathy [10], it seems difficult to do without Condition 1.

A key assumption on the payoffs needed for our results is inspired by Dubins and Savage [1], who study (finitely additive) probability measures on an infinite product of spaces, each of which is given the discrete topology.

Definition 1.1. Suppose that each of the nonempty sets $X_1, X_2, \ldots$ has the discrete topology and the product $Y = X_1 \times X_2 \times \cdots$ has the product topology. Then, a continuous function g defined on Y is called *DS continuous*.

It follows from Definition 1.1 that a function $g: Y \mapsto \mathbb{R}$ is DS continuous if and only if, for each $x = (x_1, x_2, \ldots) \in Y$ and $\epsilon > 0$, there exists n such that $|g(x) - g(x')| < \epsilon$ for every $x' = (x'_1, x'_2, \ldots)$ such that $x_i = x'_i$ for $1 \le i \le n$. If $Y = X_1 \times X_2 \times \cdots$ is a product of Borel sets, some of which are uncountable, then continuity in the product of the Borel topologies is a more stringent requirement than DS continuity. The function g is called *finitary* if it depends only on the coordinates $x_1, x_2, \ldots, x_t$, for some stop rule t such that $t(x) < \infty$ for all $x = (x_1, x_2, \ldots)$. Finitary functions are clearly DS continuous and, in fact, the DS-continuous functions are precisely those functions that can be uniformly approximated by finitary functions. (A Borel measurable version of this fact is Lemma 3.1 below. See Dubins and Savage [1] for a discussion of the properties of finitary functions.) Functions that depend on the tail of the sequence $x = (x_1, x_2, \ldots)$, such as $g(x) = \limsup_m u_m(x_m)$ for some functions $u_m: X_m \mapsto \mathbb{R}$, are typically not DS continuous.

Now, we can state our first theorem.

Theorem 1.1. *Assume that the payoff functions $f^1, f^2, \ldots, f^n$ depend only on the sequence of states and are DS continuous from H into $\mathbb{R}$. Assume also that Condition 1 holds. Then, $\Gamma(f, \cdot)$ has an SPE.*

Recall that we always assume that the payoff f is bounded and Borel measurable for the product of the original Polish topologies.

An example in Harris et al. [3] shows this theorem does not hold in general for payoff functions that depend on actions as well as states. However, the corresponding result is true when the action sets are finite.

Theorem 1.2. *Assume that the action sets $A^1, A^2, \ldots, A^n$ are finite and that the payoff functions f^1, $f^2, \ldots, f^n$ are DS continuous from H into $\mathbb{R}$. Then, $\Gamma(f, \cdot)$ has an SPE.*

Fudenberg and Levine [2] established the special case of Theorem 1.2 in which the state space is finite. See also §6.3 in Harris et al. [3].

Our last result is for games with additive payoffs.

Condition 2 (Additive Rewards). *Assume that, for $i = 1, 2, \dots, n$,*

$$f^i(h) = \sum_{k=1}^{\infty} r_k^i(s_k, a_k)$$

for all $h = (s_1, (a_1, s_2), (a_2, s_3), \dots) \in H$, where the functions $r_k^i\colon S \times A \mapsto \mathbb{R}$ are uniformly bounded, Borel in s for fixed a, and continuous in a for fixed s. Assume also that the convergence of the partial sums $\sum_{k=1}^{m} r_k^i(s_k, a_k)$ is uniform on H as $m \to \infty$.

Theorem 1.3. *Under Conditions 1 and 2, $\Gamma(f, \cdot)$ has an SPE.*

Theorem 1.3 implies the existence of an SPE for discounted games in which the daily reward function of each player is bounded, Borel in s, and continuous in a.

Antecedents. The papers of Nowak [12], and Rieder [17] contain early results on stochastic games with continuous payoffs when the state space is countable. Sengupta [18] treats zero-sum games with lower semicontinuous payoffs, finite action sets, and a compact metric state space. The arguments in this paper owe a great deal to the earlier work of Mertens and Parthasarathy [9, 10] and Solan [19]. Indeed, the method we use in the next section is abstracted from Solan's [19] proof that SPEs exist for discounted stochastic games. A crucial tool for us, as it was for Solan, is the "measurable 'measurable' choice theorem" of Mertens [8]. We also rely on concepts introduced in the gambling theory of Dubins and Savage [1].

Theorem 1.3 is very close in spirit to the results of Mertens and Parthasarathy [10] and Solan [19]. Our proof of Theorem 1.3 is similar to that of Solan [19], but our result is different from his in that the additive payoff functions of Theorem 1.3 need not be discounted.

The existence of stationary equilibria for discounted stochastic games has been proved under various assumptions. See, for example, Nowak [14], and Parthasarathy and Sinha [16]. Of course, stationary equilibria are, in particular, subgame perfect.

Since subgame-perfect equilibria do not always exist, it is natural to look for subgame-perfect correlated equilibria. Their existence has been established in different contexts by Harris et al. [3], Nowak and Raghavan [15], Nowak [13], and Solan and Vieille [20] among others.

Outline. In the next section, we prove an abstract existence result to the effect that an SPE exists for $\Gamma(f, \cdot)$ when the payoff f is DS continuous and can be uniformly approximated by a sufficiently nice function g such that $\Gamma(g, \cdot)$ has an SPE. Borel finitary functions are introduced in §3 and seen to provide an appropriate class of nice approximating functions for the proof of Theorem 1.1. Theorem 1.2 follows easily from Theorem 1.1 in §4. The abstract result of §2 is applied again in the last section to prove Theorem 1.3.

2. An adaptation of a proof of E. Solan.

Solan [19] gave a nice proof of the existence of SPEs for discounted stochastic games. In this section, we adapt his methods to prove a technical result, which will be the key to our proofs of Theorems 1.1–1.3.

We make the following assumptions throughout this section.

Assumption 2.1. *The payoff function $f = (f^1, f^2, \dots, f^n)$ is Borel and DS continuous from H into $[-R, R]^n$, where R is a fixed positive real number.*

Assumption 2.2. *There exist Borel functions $g_m\colon H \mapsto [-R, R]^n$, $m \in \mathbb{N}$ with the following properties*:

(i) *$\|g_m - f\| \to 0$ as $m \to \infty$, where, for a function $\phi\colon H \mapsto [-R, R]^n$, $\|\phi\| = \sup_h |\phi(h)|$ and $|\cdot|$ is the usual norm on Euclidean n-space.*

(ii) *For each m, there is an SPE σ_m in the game $\Gamma(g_m, \cdot)$ with corresponding equilibrium payoff $V_m\colon H^* \mapsto [-R, R]^n$; that is, for each $p \in H^*$,*

$$V_m(p) = E_{\sigma_m[p]}(g_m p). \tag{3}$$

Furthermore, for each $p \in H^*$, $\sigma_m(p)$ is an equilibrium profile in the one-move game with payoff $\int V_m(pat)\, q(dt \mid l(p), a)$ and equilibrium payoff

$$V_m(p) = \iint V_m(pat)\, q(dt \mid l(p), a) \sigma_m(p)\,(da), \tag{4}$$

and the family of functions $\{\int V_m(pat)\, q(dt \mid l(p), a)\colon m \in \mathbb{N}\}$ is equicontinuous in a.

Remark 2.1 (Notation).

(i) For $p \in H^*$, $a \in A$, and $t \in S$, the notation pat used above denotes the partial history that consists of the coordinates of p followed by a and t.

(ii) Recall that we use $\sigma_m(p)$ to denote the product measure $\sigma_m^1(p) \times \sigma_m^2(p) \times \cdots \times \sigma_m^n(p)$ in $\Delta(A)$ and also for the n-tuple $(\sigma_m^1(p), \sigma_m^2(p), \ldots, \sigma_m^n(p))$.

Here is the technical result we will need.

Theorem 2.1. *Under Assumptions* 1 *and* 2, *the game* $\Gamma(f, \cdot)$ *has an SPE.*

The proof will use certain properties of multifunctions, which we present in four lemmas.

Let X be a Borel subset of a Polish space, let Y be a Polish space, and let $\{\psi_m\}$ be a sequence of Borel functions from X into Y. The multifunction $Ls(\psi_m)$ from X to subsets of Y assigns to each $x \in X$ the set $Ls(\psi_m)(x)$ of all $y \in Y$ such that, for every open subset U of Y containing y, $\psi_m(x) \in U$ for infinitely many m. We write $\mathrm{Gr}(Ls(\psi_m))$ for the graph of $Ls(\psi_m)$; namely, the set

$$\mathrm{Gr}(Ls(\psi_m)) = \{(x, y): y \in Ls(\psi_m)(x)\}.$$

Lemma 2.1. $\mathrm{Gr}(Ls(\psi_m))$ *is a Borel subset of* $X \times Y$ *and, for each* x, $Ls(\psi_m)(x)$ *is a closed subset of* Y.

Proof. Let $\{U_m\}$ be a countable base for the topology of Y. Then,

$$\begin{aligned}(x, y) \notin \mathrm{Gr}(Ls(\psi_m)) \quad &\text{iff } (\exists m)\ [y \in U_m \ \&\ (\exists k)\ (\forall\, l \geq k)\ (\psi_l(x) \notin U_m)]\\ &\text{iff } (\exists m)\ \Big[y \in U_m \ \&\ x \in \liminf_l \psi_l^{-1}(Y - U_m)\Big].\end{aligned}$$

Hence

$$(X \times Y) - \mathrm{Gr}(Ls(\psi_m)) = \bigcup_m \Big[\liminf_l \psi_l^{-1}(Y - U_m) \times U_m\Big],$$

which is clearly Borel. The proof that $Ls(\psi_m)(x)$ is closed is completely straightforward. □

A multifunction F from X to Y is defined to be *Borel measurable* if, for every open subset U of Y, the set $\{x \in X: F(x) \cap U \neq \varnothing\}$ is a Borel subset of X.

Lemma 2.2. *Assume that, for each* x, *the set* $\{\psi_m(x): m \geq 1\}$ *is precompact in* Y. *Then,* (a) $Ls(\psi_m)$ *is a Borel measurable multifunction with nonempty compact values and* (b) $Ls(\psi_m)$ *admits a Borel measurable selector.*

Proof. Part (a) follows from the previous lemma, the precompactness assumption, and the Kunugui-Novikov theorem. (See 4F.12 in Moschovakis [11] or 4.7.11 in Srivastava [21].) Part (b) is a consequence of (a) and the selection theorem of Kuratowski and Ryll-Nardzewski [5]. (Or see Corollary 5.2.5 in Srivastava [21].) □

The next lemma records, for ease of reference, a part of Lemma 3, in Hildenbrand [4, p. 69].

Lemma 2.3. *Let* $Y = [-R, R]^n$ *and let* μ *be a probability measure on the Borel subsets of* Y. *Suppose* $\lim_m \int \psi_m\, d\mu$ *exists. Then, there is a Borel selector* ψ *of* $Ls(\psi_m)$ *such that* $\int \psi\, d\mu = \lim_m \int \psi_m\, d\mu$.

Our last lemma specializes a deep result of Mertens [8] to a Borel setting.

Lemma 2.4. *Suppose that* Y *and* Z *are Borel subsets of Polish spaces and* F *is a Borel measurable multifunction on* $Y \times Z$ *with nonempty compact subsets of* $[-R, R]^n$ *as values. Let* $q(\cdot \mid y)$ *be a Borel measurable transition function from* Y *to* Z. *Define a multifunction* G *on* Y *as follows*:

$$G(y) = \left\{\int f(y, z)\, q(dz \mid y): f \text{ is a Borel selector of } F\right\}.$$

Then,

(i) *G is a Borel measurable multifunction with nonempty compact values,*

(ii) *there is a Borel measurable function* g: $\mathrm{Gr}(G) \times Z \mapsto [-R, R]^n$ *such that, for every* $(y, x) \in \mathrm{Gr}(G)$, $g(y, x, \cdot)$ *is a selector for the multifunction* $F(y, \cdot)$ *and*

$$x = \int g(y, x, z)\, q(dz \mid y).$$

Remark 2.2. The multifunction G in Lemma 2.4 can also be described, for each $y^* \in Y$, as $G(y^*) = \{\int g(z)\, q(dz \mid y^*)$: g is a Borel selector of $F(y^*, \cdot)\}$. To see this, fix a Borel selector ϕ of F and define

$$f(y, z) = \begin{cases} g(z) & \text{if } y = y^*, \\ \phi(y, z) & \text{if } y \neq y^*. \end{cases}$$

The proof of Theorem 2.1 now proceeds in a number of steps.

Step 1. Define the multifunction D on H^* by $D(p) = Ls(V_m)(p)$. By Lemma 2.1, $\mathrm{Gr}(D)$ is a Borel subset of $H^* \times [-R, R]^n$. In addition, for each $p \in S^*$, $\{V_m(p)\colon m \geq 1\}$ is precompact since $[-R, R]^n$ is compact. So, by Lemma 2.2, D is Borel measurable with nonempty compact values.

Step 2. Define $u_m\colon H^* \times A \mapsto [-R, R]^n$ by

$$u_m(p, a) = \int V_m(pat)\, q(dt \mid l(p), a).$$

By Assumption 2.2, u_m is a Caratheodory function (Borel in p and continuous in a). Also, the set $\{u_m(p, \cdot)\colon m \geq 1\}$ is an equicontinuous subset of the space $C = C(A, [-R, R]^n)$ of continuous functions from A to $[-R, R]^n$. Hence, by the Arzela-Ascoli theorem, this set is precompact in the topology of uniform convergence on C.

Step 3. Next, define $\phi_m\colon H^* \mapsto [-R, R]^n \times C \times \Delta(A)$ by setting

$$\phi_m(p) = (V_m(p), u_m(p, \cdot), \sigma_m(p)).$$

Plainly, ϕ_m is Borel measurable. Also, note that, for each $p \in H^*$, the set $\{\sigma_m(p)\colon m \geq 1\}$ is precompact in the topology of weak convergence on $\Delta(A)$. Observe that a limit point of $\{\sigma_m(p)\colon m \geq 1\}$ is again a product measure on A.

Step 4. Define another multifunction G on $\mathrm{Gr}(D)$ by

$$G(p, x) = \{(v, \nu) \in C \times \Delta(A)\colon (x, v, \nu) \in Ls(\phi_m)(p)\}.$$

It is easy to see that $\mathrm{Gr}(G)$ is a Borel subset of $H^* \times [-R, R]^n \times C \times \Delta(A)$. Also, for each $(p, x) \in \mathrm{Gr}(D)$, $G(p, x)$ is the x-section of the nonempty compact set $Ls(\phi_m)(p)$, and so is itself compact. Hence, by Lemma 2.2, G is a Borel measurable multifunction on the Borel set $\mathrm{Gr}(D)$ with nonempty compact values, and there are Borel functions v^* and ν^* on $\mathrm{Gr}(D)$ into $C = C(A, [-R, R]^n)$ and $\Delta(A)$, respectively, such that $(v^*(p, x), \nu^*(p, x)) \in G(p, x)$ for every $(p, x) \in \mathrm{Gr}(D)$. (We remind the reader that, as mentioned at the end of Step 3, $\nu^*(p, x)$ is a product measure on A.)

Step 5. We define a third multifunction Ω on $H^* \times A$ by

$$\Omega(p, a) = \left\{\int g(t)\, q(dt \mid l(p), a)\colon g \text{ is a Borel selector of } D(pa\cdot)\right\}.$$

By Lemma 2.4 and Remark 2.2, Ω is a Borel measurable multifunction with nonempty compact values (so that $\mathrm{Gr}(\Omega)$ is a Borel subset of $H^* \times A \times [-R, R]^n$) and there exists a Borel function $\psi\colon \mathrm{Gr}(\Omega) \times S \mapsto [-R, R]^n$ such that

(i) $\psi(p, a, x, t) \in D(pat)$ for all $t \in S$ and

(ii) $\int \psi(p, a, x, t)\, q(dt \mid l(p), a) = x$, for every $(p, a, x) \in \mathrm{Gr}(\Omega)$.

Step 6. Claim: If $(p, x) \in \mathrm{Gr}(D)$, then $(\forall a)$ $[v^*(p, x)(a) \in \Omega(p, a)]$.

To verify this claim, let $(p, x) \in \mathrm{Gr}(D)$ and fix $a \in A$. Choose a subsequence $\{u_{m_i}(p, \cdot)\}$ of $\{u_m(p, \cdot)\}$ such that $u_{m_i}(p, \cdot)$ converges uniformly to $v^*(p, x)$. So, in particular, $u_{m_i}(p, a)$ converges to $v^*(p, x)(a)$. The last statement can be written as

$$\lim_i \int V_{m_i}(pat)\, q(dt \mid l(p), a) = v^*(p, x)(a). \tag{5}$$

So, by Lemma 2.3, there is a Borel function $g\colon S \mapsto [-R, R]^n$ such that g is a selector of $D(pa\cdot)$ and

$$\lim_i \int V_{m_i}(pat)\, q(dt \mid l(p), a) = \int g(t)\, q(dt \mid l(p), a). \tag{6}$$

It follows from (5) and (6) that $v^*(p, x)(a) \in \Omega(p, a)$.

Step 7. Let $\bar{D} = \{(p, a, x, t) \in H^* \times A \times [-R, R]^n \times S\colon x \in D(p)\}$ and define $\psi^*\colon \bar{D} \mapsto [-R, R]^n$ by setting

$$\psi^*(p, a, x, t) = \psi(p, a, v^*(p, x)(a), t),$$

where ψ is the function introduced in Step 5. Then, ψ^* is well defined by Step 6. Plainly, ψ^* is a Borel function. Note that it follows from (ii) of Step 5 that

$$v^*(p, x)(a) = \int \psi^*(p, a, x, t)\, q(dt \mid l(p), a), \tag{7}$$

for every $(p, a, x, t) \in \bar{D}$.

Step 8. Claim: For $(p, x) \in \mathrm{Gr}(D)$, $\nu^*(p, x)$ is an equilibrium profile in the one-move game with payoff $v^*(p, x)(a)$ and the corresponding equilibrium payoff is x.

To see this, choose a subsequence $\{(V_{m_i}(p), u_{m_i}(p, \cdot), \sigma_{m_i}(p))\}$ of $\{(V_m(p), u_m(p, \cdot), \sigma_m(p))\}$ such that

$$\lim_i \big(V_{m_i}(p), u_{m_i}(p, \cdot), \sigma_{m_i}(p)\big) = (x, v^*(p, x), \nu^*(p, x)). \tag{8}$$

Since $\sigma_{m_i}(p)$ is an equilibrium profile in the one-move game with payoff $u_{m_i}(p, \cdot)$ by Assumption 2.2(ii) and $V_{m_i}(p)$ is the corresponding equilibrium payoff by virtue of (4), the claim follows from (8).

Step 9. We are now in a position to define a profile τ for the game $\Gamma(f, \cdot)$. (It will turn out that τ is an SPE.) First, define a function $\pi\colon H^* \mapsto [-R, R]^n$ by recursion as follows: for $s \in S$, set $\pi(s) = \xi(s)$, where ξ is a fixed, but arbitrary, Borel selector for $D(s)$; and, for $p \in H^*$, $a \in A$, and $t \in S$, let

$$\pi(pat) = \psi^*(p, a, \pi(p), t).$$

Using Step 5 and induction on the length of p, one proves easily that $\pi(p) \in D(p)$ and that π is well defined. Clearly, π is a Borel function. The profile τ can now be defined on H^* by

$$\tau(p) = \nu^*(p, \pi(p)).$$

So, τ is also clearly Borel.

Step 10. Let $V(p) = E_{\tau[p]}(fp)$, $p \in H^*$. (Recall that $\tau[p]$ is the conditional profile given p.)

Claim: $V(p) = \pi(p)$.

We prove the claim for $p = s \in S$. (The proof for other p's is similar and is omitted.) Let $\epsilon > 0$ and fix

$$h = (s_1, (a_1, s_2), (a_2, s_3), \ldots) \in H.$$

By the DS continuity of f (Assumption 2.1), we can choose k so large that with

$$p = p_k(h) = (s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k)),$$

we have $\|f(ph') - f(h)\| < \epsilon/2$ for all $h' \in (A \times S)^{\mathbb{N}}$. (Here, ph' is the history that consists of the coordinates of p followed by those of h'.) Next, choose M so large that $\|g_m - f\| < \epsilon/2$ for all $m \geq M$. So, if $q \supseteq p$, $m \geq M$, and $x \in D(q)$, then $\|x - f(h)\| \leq \epsilon$. To see this, observe that

$$\|g_m(qh') - f(h)\| \leq \|g_m(qh') - f(qh')\| + \|f(qh') - f(h)\| < \epsilon.$$

It follows from (3) that $\|V_m(q) - f(h)\| \leq \epsilon$ for all $m \geq M$, so that $\|x - f(h)\| \leq \epsilon$. Consequently, if $l \geq k$ and $x \in D(p_l(h))$, then $\|x - f(h)\| \leq \epsilon$. Hence, by Step 9, $\|\pi(p_l(h)) - f(h)\| \leq \epsilon$, and therefore

$$\lim_{m\to\infty} \pi(p_m(h)) = f(h). \tag{9}$$

Denote the history generated by the probability measure P_τ with initial state s by

$$(s, Y_1, Y_2, \ldots, Y_m, \ldots).$$

(Thus $Y_m = (a_m, s_{m+1})$, $m \geq 1$.) Then, the process $\pi(s), \pi(sY_1), \ldots, \pi(sY_1 \cdots Y_m), \ldots$ is a martingale. Indeed,

$$\begin{aligned} E_\tau[\pi(sY_1 \cdots Y_m Y_{m+1}) \mid sY_1 \cdots Y_m = p] &= \iint \psi^*(p, a, \pi(p), t)\, q(dt \mid l(p), a)\nu^*(p, \pi(p))\,(da) \\ &= \int v^*(p, \pi(p))(a)\nu^*(p, \pi(p))\,(da) \\ &= \pi(p), \end{aligned}$$

where the last two equalities are by (7) and Step 8, respectively.

By (9), $\pi(sY_1\cdots Y_m) \to f(sY_1\cdots Y_m Y_{m+1}\cdots)$, so that by the dominated convergence theorem,

$$\lim_m E_\tau[\pi(sY_1\cdots Y_m)] = E_\tau(f) = V(s).$$

But, by the martingale property,

$$E_\tau[\pi(sY_1\cdots Y_m)] = \pi(s)$$

for all $m \geq 1$. Hence $\pi(s) = V(s)$ and the proof of the claim is complete for $p = s$.

Step 11. The last step in the proof of Theorem 2.1 is to show that τ is an SPE for $\Gamma(f,\cdot)$. Let $\tau = (\tau^1, \tau^2, \ldots, \tau^n)$ and fix a strategy σ for player i. Let $\hat{\tau} = (\tau^{-i}, \sigma)$. Note that $E_{\tau[p]}(f^i p)$ is just the ith coordinate $V(p)^i$ of $V(p)$ as defined in Step 10.

Again, we denote the history generated by $P_{\hat{\tau}}$ with initial state s by $(s, Y_1, Y_2, \ldots, Y_m, \ldots)$.

Claim: The process $V(s)^i, V(sY_1)^i, \ldots, V(sY_1Y_2\cdots Y_m)^i, \ldots$ is a supermartingale under $P_{\hat{\tau}}$.

To verify this, let $\lambda(p)$ be the product measure

$$\lambda(p) = \tau^1(p) \times \cdots \times \sigma(p) \times \cdots \times \tau^n(p),$$

for $p \in H^*$. Then, calculate

$$\begin{aligned}
E_{\hat{\tau}}[V(sY_1Y_2\cdots Y_mY_{m+1})^i \mid sY_1Y_2\cdots Y_m = p] &= E_{\hat{\tau}}[\pi(sY_1Y_2\cdots Y_mY_{m+1})^i \mid sY_1Y_2\cdots Y_m = p] \\
&= \iint \psi^*(p, a, \pi(p), t)^i\, q(dt \mid l(p), a)\, \lambda(da) \\
&\leq \iint \psi^*(p, a, \pi(p), t)^i\, q(dt \mid l(p), a)\nu^*(p, \pi(p))\,(da) \\
&= \pi(p)^i \\
&= V(p)^i.
\end{aligned}$$

Here, the inequality is by virtue of the fact that $\tau(p) = \nu^*(p, \pi(p))$ is an equilibrium profile in the one-move game with payoff $\int \psi^*(p, a, \pi(p), t)\, q(dt \mid l(p), a) = v^*(p, \pi(p))(a)$ by Step 8. Since $V(sY_1Y_2\cdots Y_m) \to f$, it follows from the supermartingale property that

$$E_{\hat{\tau}}(f^i) = \lim_m E_{\hat{\tau}}[V(sY_1Y_2\cdots Y_m)^i] \leq V(s)_i = E_\tau(f^i).$$

This proves that τ is an equilibrium profile in the game $\Gamma(f, s)$. The proof that, for $p \in H^*$, $\tau[p]$ is an equilibrium profile in the game $\Gamma(f, p)$ is similar and is omitted.

The proof of Theorem 2.1 is now complete.

3. Finitary games and the proof of Theorem 1.1. To deduce Theorem 1.1 from Theorem 2.1, we first identify the class of functions that will be used to approximate the payoff f as in Assumption 2.2.

Let t be a Borel function from $S^{\mathbb{N}}$ to $\{0, 1, \ldots\} \cup \{\infty\}$. We say that t is a *Borel stopping time* if, given elements $x = (s_1, s_2, \ldots)$ and $y = (r_1, r_2, \ldots)$ in $S^{\mathbb{N}}$ such that $t(x) < \infty$ and y agrees with x in the first $t(x)$ coordinates, then $t(x) = t(y)$. If, in addition, $t(x) < \infty$ for *all* $x \in S^{\mathbb{N}}$, then t is called a *stop rule*. A Borel function g: $S^{\mathbb{N}} \mapsto \mathbb{R}$ is a *Borel finitary function* if there exists a Borel stop rule t such that $g(x) = g(y)$ whenever y agrees with x in the first $t(x)$ coordinates. In this case, the function g is said to be determined by time t.

Borel finitary functions will play the role of the functions g_m in Assumption 2.2. First, we establish that they can be used to uniformly approximate the payoff functions of Theorem 1.1. As in the previous section, R denotes a fixed positive real number.

Lemma 3.1. *Suppose that ϕ: $S^{\mathbb{N}} \mapsto [-R, R]$ is a Borel DS-continuous function. Then, for every $\epsilon > 0$, there is a Borel finitary function ψ: $S^{\mathbb{N}} \mapsto [-R, R]$ such that* $\sup\{|\phi(x) - \psi(x)| \mid x \in S^{\mathbb{N}}\} \leq \epsilon$.

Proof. Let $\{c_1, c_2, \ldots, c_m\}$ be an ϵ-net in $[-R, R]$ and set $U_i = \phi^{-1}((c_i - \epsilon, c_i + \epsilon))$, $1 \leq i \leq m$. The sets U_i are Borel because ϕ is Borel; they are open in the product of discrete topologies on $S^{\mathbb{N}}$ because ϕ is DS continuous. It follows from Corollary 2.4 in Maitra et al. [7] that there is, for each i, a Borel stopping time t_i such that $[t_i < \infty] = U_i$. Set $t = \min\{t_i \mid 1 \leq i \leq m\}$. Clearly, t is a Borel stopping time. Moreover, since $\bigcup_{1\leq i\leq m} U_i = S^{\mathbb{N}}$, t is a stop rule. For $1 \leq i \leq m$, define

$$W_i = \{x \in S^{\mathbb{N}} \mid t(x) = t_i(x) \;\&\; t_j(x) > t(x),\ 1 \leq j < i\}.$$

It is now easy to check that $W_i \subseteq U_i$, $W_i \cap W_j = \varnothing$, for $i \neq j$, $\bigcup_i W_i = S^{\mathbb{N}}$, and each W_i is Borel. Set $\psi = c_i$ on W_i, $1 \leq i \leq m$. Then, ψ is obviously Borel. It is determined by time t and is therefore finitary. By construction, $|\phi(x) - \psi(x)| \leq \epsilon$ for all x. □

We need a bit of notation. Define Φ: $H \mapsto S^{\mathbb{N}}$ by

$$\Phi((s_1, (a_1, s_2), (a_2, s_3), \ldots)) = (s_1, s_2, s_3, \ldots).$$

Recall that $H^* = S \cup \bigcup_{k\geq 1}[S \times (A \times S)^k]$ and let $S^* = \bigcup_{k\geq 1} S^k$. Next, define Ψ: $H^* \mapsto S^*$ by

$$\Psi((s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k)) = (s_1, s_2, \ldots, s_k).$$

Suppose that $f = (f^1, f^2, \ldots, f^n)$ is the Borel, DS-continuous payoff function of Theorem 1.1. Since the f^i are bounded, we can assume that f: $H \mapsto [-R, R]^n$. Also, since f depends only on the sequence of states, we can find $\bar{f}$: $S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $f = \bar{f} \circ \Phi$ and $\bar{f}$ is Borel DS continuous on $S^{\mathbb{N}}$.

By Lemma 3.1, there is, for each $\epsilon > 0$, a Borel finitary function $\bar{g}$: $S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $\|\bar{g} - \bar{f}\| < \epsilon$. Define g on H by $g(h) = (\bar{g} \circ \Phi)(h)$. Then, g approximates f on H uniformly within ϵ. Most of the remainder of this section is devoted to the study of the game $\Gamma(g, \cdot)$, which we call a *finitary* game. At the end of the section, we will deduce Theorem 1.1 from Theorem 2.2 and the properties of finitary games.

For our treatment of finitary games, we will not need the full strength of Condition 1 and we will replace it by the weaker condition below.

Condition 3 (Feller Continuity). *For fixed $s \in S$, the law of motion $q(\cdot \mid s, a)$ is Feller continuous in a; that is, for every bounded, Borel measurable, real-valued function ϕ on S, $\int \phi(t)\, q(dt \mid s, a)$ is continuous in a.*

Theorem 3.1. *Assume Condition 3. Let $\bar{g}$: $S^{\mathbb{N}} \mapsto [-R, R]^n$ be a Borel finitary function and let g: $H \mapsto [-R, R]^n$ be the function $(\bar{g} \circ \Phi)$. Then, there exist Borel functions $\bar{\sigma}$: $S^* \mapsto \Delta(A)$ and $\overline{V}$: $S^* \mapsto [-R, R]^n$ such that*

(i) *$\sigma = \bar{\sigma} \circ \Psi$ is an SPE in the finitary game $\Gamma(g, \cdot)$ and the corresponding equilibrium payoff is $V = \overline{V} \circ \Psi$;*

(ii) *for each $p \in H^*$, the one-move game with payoff $\int V(pat)\, q(dt \mid l(p), a)$ has equilibrium profile $\sigma(p)$ with corresponding equilibrium payoff $V(p)$.*

Proof. Let t be a Borel stop rule on $S^{\mathbb{N}}$ such that $\bar{g}$ is determined by time t. The stop rule t defines a tree on S as follows:

$$\begin{aligned}(s_1, s_2, \ldots, s_m) \in T \quad &\text{iff } (\exists h \in S^{\mathbb{N}})\ \ (t(s_1, s_2, \ldots, s_m h) \geq m)\\ &\text{iff } (\forall\, h \in S^{\mathbb{N}})\ \ (t(s_1, s_2, \ldots, s_m h) \geq m).\end{aligned}$$

Thus T is both analytic and coanalytic. Hence, by Suslin's theorem (Moschovakis [11], 2E.2), T is Borel.

Furthermore, T is a Borel tree on S, which means

(i) T is a Borel subset of S^* and

(ii) T is closed under initial segments; that is, $(s_1, s_2, \ldots, s_m) \in T$ implies $(s_1, s_2, \ldots, s_k) \in T$, for $1 \leq k \leq m$.

Observe that, since t is everywhere finite, T is a Borel well-founded relation (i.e., T has no infinite branches). It now follows, courtesy of a result of Moschovakis [11, §4C.14], that there is a coanalytic, non-Borel subset C of a Polish space Z, a function η on C onto ω_1, the first uncountable ordinal, and a Borel function ξ on T into C such that

(a) η is a coanalytic norm on C and

(b) $(s_1, s_2, \ldots, s_l) \in T$ & $m < l$ implies $\eta(\xi((s_1, s_2, \ldots, s_l))) < \eta(\xi((s_1, s_2, \ldots, s_m)))$.

For the definition of a coanalytic norm, see Moschovakis [11, pp. 200–201]. We will use only the following two properties of coanalytic norms:

(c) For every ordinal $\alpha < \omega_1$, the set $\{z \in C\colon \eta(z) = \alpha\}$ is a Borel subset of Z (Moschovakis [11], 4C.7).

(d) If K is an analytic subset of C, then there is $\alpha^* < \omega_1$ such that $\eta(z) \leq \alpha^*$, for every $z \in K$.

Define a function i: $T \mapsto \omega_1$ by $i((s_1, s_2, \ldots, s_m)) = \eta(\xi((s_1, s_2, \ldots, s_m)))$. For every $\alpha < \omega_1$, let

$$T_\alpha = \{(s_1, s_2, \ldots, s_m) \in T\colon i((s_1, s_2, \ldots, s_m)) = \alpha\}.$$

Then, each T_α is a Borel subset of S^* by virtue of (c). Also, since $\xi(T)$ is an analytic subset of C, it follows from (d) that there is $\bar{\alpha} < \omega_1$ such that $T = \bigcup_{\alpha \leq \bar{\alpha}} T_\alpha$.

The following lemma will be needed. In the lemma and the sequel, we use the notation $\mathscr{B}(X)$ to denote the Borel σ-field of a topological space X.

LEMMA 3.2. *Assume Condition 3. Suppose Y is a Borel subset of a Polish space and let $\mathscr{D}$ be a countably generated sub-σ-field of $\mathscr{B}(Y)$. Let ζ: $Y \times S \mapsto [-R, R]^n$ be a $\mathscr{D} \times \mathscr{B}(S)$-measurable function. Let $G(y, s)$ be the one-move game with payoff $\int \zeta(y, t)\, q(dt \mid s, a)$. Then, there exist $\mathscr{D} \times \mathscr{B}(S)$-measurable functions ρ: $Y \times S \mapsto \Delta(A)$ and W: $Y \times S \mapsto [-R, R]^n$ such that $\rho(y, s)$ is an equilibrium profile in the game $G(y, s)$ and $W(y, s)$ is the corresponding equilibrium payoff, i.e.,*

$$W(y, s) = \iint \zeta(y, t)\, q(dt \mid s, a)\rho(y, s)\,(da).$$

PROOF. Let $F(y, s)$ be the set of all equilibrium profiles $(\mu^1, \mu^2, \dots, \mu^n)$ for the game $G(y, s)$. It is easy to verify that $F(y, s)$ is a nonempty, compact subset of the set of all profiles $\mathscr{P} = \Delta(A^1) \times \Delta(A^2) \times \cdots \times \Delta(A^n)$. Also, the graph of F belongs to $\mathscr{D} \times \mathscr{B}(S) \times \mathscr{B}(\mathscr{P})$. To see this, imitate the proof of Lemma 2.1 in Maitra and Sudderth [6] and use the fact that the function

$$(y, s, a) \mapsto \int \zeta(y, t)\, q(dt \mid s, a),$$

being $\mathscr{D} \times \mathscr{B}(S)$-measurable for fixed a and continuous in a for fixed (y, s) is, in fact, $\mathscr{D} \times \mathscr{B}(S) \times \mathscr{B}(A)$-measurable. (See Theorem 3.1.30 in Srivastava [21].) An application of Theorem 5.7.1 of Srivastava [21] will now yield a $\mathscr{D} \times \mathscr{B}(S)$-measurable selector ρ: $Y \times S \mapsto \mathscr{P}$ for F. Finally, it is easily checked that W is $\mathscr{D} \times \mathscr{B}(S)$-measurable. □

Now, let $\mathscr{C}$ be the smallest σ-field on H^*, which makes Ψ measurable, i.e., $\mathscr{C} = \Psi^{-1}(\mathscr{B}(\tilde{S}))$. Fix $\nu^i \in \Delta(A^i)$, $i = 1, 2, \dots, n$, $s^* \in S$, and let $x^* = (s^*, s^*, \dots)$ be that point in S^* all of whose coordinates are s^*. Now, for $p \in \Psi^{-1}(S^* - T)$, the function $x \mapsto \bar{g}(\Psi(p)x)$ is constant for $x \in S^*$. So, we define

$$V(p) = \bar{g}(\Psi(p)x^*), \qquad \sigma(p) = \nu^1 \times \nu^2 \times \cdots \times \nu^n.$$

Then,

(e) V and σ are defined on the set $\Psi^{-1}(S^* - T) \in \mathscr{C}$ and are measurable with respect to the restriction of $\mathscr{C}$ to $\Psi^{-1}(S^* - T)$.

Also, for $p \in \Psi^{-1}(S^* - T)$,

(f) $\sigma(p)$ is an equilibrium profile in the one-move game with payoff $\int V(pat)\, q(dt \mid l(p), a)$ and

(g) $V(p) = \iint V(pat)\, q(dt \mid l(p), a)\, \sigma(p)\,(da)$.

We will now extend the definitions of the functions V and σ to all of H^*, so that they are $\mathscr{C}$-measurable and properties (f) and (g) continue to hold. Since $\Psi^{-1}(T) = \bigcup_{\alpha \le \bar{\alpha}} \Psi^{-1}(T_\alpha)$, the definitions of V and σ will proceed by transfinite induction.

So, suppose that $\alpha \le \bar{\alpha}$ and V, σ have been defined for all $p \in \bigcup_{\beta<\alpha} \Psi^{-1}(T_\beta)$, so that (f) and (g) still hold and also that

(h) V and σ are measurable with respect to the restriction of $\mathscr{C}$ to $(\bigcup_{\beta<\alpha} \Psi^{-1}(T_\beta)) \cup \Psi^{-1}(S^* - T)$.

We will now define V and σ on the $\mathscr{C}$-set $\Psi^{-1}(T_\alpha)$. For each $p \in \Psi^{-1}(T_\alpha)$, note that, by property (b) above, $pat \in (\bigcup_{\beta<\alpha} \Psi^{-1}(T_\beta)) \cup \Psi^{-1}(S^* - T)$, so that $V(pat)$ and $\sigma(pat)$ are defined for all $a \in A$, $t \in S$. We next apply Lemma 3.2 with $Y = \Psi^{-1}(T_\alpha)$, $\zeta(p, t) = V(pa^*t)$, where a^* is a fixed element of A, and the σ-field $\mathscr{D}$ is equal to the restriction of $\mathscr{C}$ to $\Psi^{-1}(T_\alpha)$. Let ρ and W be the functions whose existence is asserted in Lemma 3.2. We now define, for $p \in \Psi^{-1}(T_\alpha)$,

$$\sigma(p) = \rho(p, l(p)), \qquad V(p) = W(p, l(p)).$$

It is straightforward to check that σ and V satisfy (f)–(h). Since $\bar{\alpha}$ is countable, this completes the extension of σ and V to H^*, so that (f)–(h) are satisfied.

It remains to be verified that, for all $p \in H^*$, the conditional profile $\sigma[p]$ is an equilibrium in the game $\Gamma(g, p)$ and that $V(p) = E_{\sigma[p]}(gp)$ is the corresponding equilibrium payoff. This is trivially true for $p \in \Psi^{-1}(S^* - T)$ since the function gp is constant for such p. For $p \in \Psi^{-1}(T)$, we will prove the assertion by another transfinite induction. So, suppose the assertion is true for all $p \in \bigcup_{\beta<\alpha} \Psi^{-1}(T_\beta)$ and let $p \in \Psi^{-1}(T_\alpha)$. Then, by property (b), $pat \in (\bigcup_{\beta<\alpha} \Psi^{-1}(T_\beta)) \cup \Psi^{-1}(\tilde{S} - T)$. Let player i deviate by using $\tau[p]$ at p and let $\bar{\sigma} = (\sigma^{-i}, \tau)$ be the resulting profile. Recall that g^i, the ith coordinate of g, is the payoff function for player i. Recall also that $g^i p$ is the section of g^i by the partial history p as in (2). Now, calculate as follows:

$$\begin{aligned} E_{\sigma[p]}(g^i p) &= \iint E_{\sigma[pat]}(g^i pat)\, q(dt \mid l(p), a)\sigma(p)\,(da) \\ &= \iint V(pat)^i\, q(dt \mid l(p), a)\sigma(p)\,(da) \end{aligned}$$

$$\begin{aligned}
&\geq \iint V(pat)^i \, q(dt \mid l(p), a)\bar{\sigma}(p)\,(da) \\
&= \iint E_{\sigma[pat]}(g^i pat)\, q(dt \mid l(p), a)\bar{\sigma}(p)\,(da) \\
&\geq \iint E_{\bar{\sigma}[pat]}(g^i pat)\, q(dt \mid l(p), a)\bar{\sigma}(p)\,(da) \\
&= E_{\bar{\sigma}[p]}(g^i p).
\end{aligned}$$

Here, the second and fourth equalities and the second inequality are by virtue of the inductive hypothesis, while the first inequality holds because of (f). Finally,

$$\begin{aligned}
E_{\sigma[p]}(gp) &= \iint E_{\sigma[pat]}(gpat)\, q(dt \mid l(p), a)\sigma(p)\,(da) \\
&= \iint V(pat)\, q(dt \mid l(p), a)\sigma(p)\,(da) \\
&= V(p),
\end{aligned}$$

where the second equality is by virtue of the inductive hypothesis and the third is by (g). □

We are now ready to complete the proof of Theorem 1.1. So, assume Condition 1 of the introduction and note that it implies the weaker Condition 3 of this section. The payoff function $f\colon H \mapsto [-R, R]^n$ is assumed to be Borel DS continuous, and to depend only on the sequence of states. Hence there is a Borel DS-continuous function $\bar{f}\colon S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $f = \bar{f} \circ \Phi$. Use Lemma 3.1 to choose Borel, finitary functions $\bar{g}_m\colon S^{\mathbb{N}} \mapsto [-R, R]^n$ such that $\|\bar{g}_m - \bar{f}\| \to 0$ as $m \to \infty$. Let $g_m = \bar{g}_m \circ \Phi$. Then, also $\|g_m - f\| \to 0$ as $m \to \infty$.

For each m, one can choose, by virtue of Theorem 3.1, Borel functions $\bar{\sigma}_m\colon S^* \mapsto \Delta(A)$ and $\overline{V}_m\colon S^* \mapsto [-R, R]^n$ such that, if $\sigma_m = \bar{\sigma}_m \circ \Psi$ and $V_m = \overline{V}_m \circ \Psi$, then

(i) σ_m is an SPE in the game $\Gamma(g_m, \cdot)$ and the corresponding equilibrium payoff is V_m;

(ii) for each $p \in H^*$, the one-move game with payoff $\int V_m(pat)\, q\,(dt \mid l(p), a)$ has equilibrium profile $\sigma_m(p)$ with corresponding equilibrium payoff $V_m(p)$.

Finally, it follows from Condition 1 and Lemma 3.6 in Solan [19] that, for each $p \in H^*$, the family

$$\left\{\int V_m(pat)\, q(dt \mid l(p), a)\colon m \in \mathbb{N}\right\} = \left\{\int \overline{V}_m(\Psi(p)t)\, q(dt \mid l(p), a)\colon m \in \mathbb{N}\right\}$$

is equicontinuous in a. (The fact that $V_m(pat)$ does not depend on a is crucial for the application of Solan's [19] lemma.) So Assumption 2.2 of §2 is verified; Assumption 2.1 is true by hypothesis. Theorem 1.1 now follows from Theorem 2.1.

4. Finite action sets and the proof of Theorem 1.2. Assume the hypotheses of Theorem 1.2; that is, the actions sets $A^1, A^2, \ldots, A^n$ are finite and the payoff functions $f^1, f^2, \ldots, f^n$ are bounded, Borel, and DS continuous from H to $\mathbb{R}$. Note that the payoffs may now depend on actions as well as states.

We use the *partial history trick* to deduce Theorem 1.2 from Theorem 1.1. We take H^* to be our state space and $\bar{q}$ to be our law of motion, where

$$\bar{q}(pat \mid p, a) = q(t \mid l(p), a).$$

We will define $\bar{f}$ on the new history space $\overline{H} = H^* \times (A \times H^*)^{\mathbb{N}}$ as follows: Let $\bar{h} = (p_1, (b_1, p_2), (b_2, p_3), \ldots) \in \overline{H}$. If there exists an $h = (s_1, (a_1, s_2), (a_2, s_3), \ldots) \in H$ such that

$$p_k = (s_1, (a_1, s_2), \ldots, (a_{k-1}, s_k)), \tag{10}$$

for all $k = 1, 2, \ldots$, we set $\bar{f}(\bar{h}) = f(h)$. If there is no such $h \in H$, let $m = m(\bar{h})$ be the largest integer, possibly zero, such that (10) holds for all $k = 1, 2, \ldots, m$ for some $h = (s_1, (a_1, s_2), (a_2, s_3), \ldots)$ and let

$$\bar{f}(\bar{h}) = \begin{cases} f(s^*, (a^*, s^*), (a^*, s^*), \ldots) & \text{if } m = 0, \\[2ex] f(s_1, (a_1, s_2), \ldots, (a_{m-1}, s_m), (a^*, s^*), (a^*, s^*), \ldots) & \text{if } m \geq 1, \end{cases}$$

where a^* and s^* are fixed elements of A and S, respectively. (Note that $m(\bar{h})$ could also be described as the largest m such that, for all $k = 1, 2, \ldots, m$, $lh(p_k) = k - 1$ and p_k extends p_{k-1}.) Then, $\bar{f}$ is bounded, Borel, DS continuous, and depends only on states.

Since A is finite, $\bar{q}$ satisfies Condition 1. Hence, by Theorem 1.1, $\Gamma(\bar{f}, \cdot)$ has an SPE. It is now immediate that $\Gamma(f, \cdot)$ has an SPE.

5. Additive payoffs and the proof of Theorem 1.3. In this section, we assume both Conditions 1 and 2 of the introduction. We write r_k for the profile $(r_k^1, r_k^2, \ldots, r_k^n)$ of the player's reward functions at stage k, $k \geq 1$. Let

$$g_m(h) = \sum_{k=1}^{m} r_k(s_k, a_k)$$

for all $h = (s_1, (a_1, s_2), (a_2, s_3), \ldots) \in H$ and $m \geq 1$. For each m, the function g_m is a bounded, Borel, finitary function on H. By Condition 2, $\|g_m - f\| \to 0$ as $m \to \infty$. Thus f is bounded, Borel, and DS continuous. Assume, without loss of generality, that the range of f and the range of g_m, $m \geq 1$, is contained in $[-R, R]^n$.

First, we fix m and analyze the game $\Gamma(g_m, \cdot)$. Let $W_0^m(t) = 0$ for $t \in S$, and, for $1 \leq k \leq m$, define W_k^m on S by induction as follows:

$$W_k^m(s) = \iint [r_{m-k+1}(s, a) + W_{k-1}^m(t)]\, q(dt \mid s, a) \rho_k^m(s)\,(da),$$

where $\rho_k^m(s)$ is an equilibrium profile in the one-move game with payoff

$$r_{m-k+1}(s, a) + \int W_{k-1}^m(t)\, q(dt \mid s, a).$$

Condition 3, which follows from Condition 1, ensures that the functions ρ_k^m can be chosen to be Borel, so that the functions W_k^m are also Borel. (See the proof of Lemma 2.1 in Maitra and Sudderth [6].)

We now define V_m and σ_m on H^*. Let $p = (s_1, (a_1, s_2), \ldots, (a_{l-1}, s_l)) \in H^*$ and fix $\nu^i \in \Delta(A_i)$, $i = 1, 2, \ldots, n$.

Case 1. If $l > m$, set

$$V_m(p) = \sum_{k=1}^{m} r_k(s_k, a_k), \qquad \sigma_m(p) = \nu^1 \times \nu^2 \times \cdots \times \nu^n.$$

Case 2. If $1 \leq l \leq m$, set

$$V_m(p) = \sum_{k=1}^{l-1} r_k(s_k, a_k) + W_{m-l+1}^m(s_l), \qquad \sigma_m(p) = \rho_{m-l+1}^m(s_l).$$

Then, V_m and σ_m are Borel functions on H^*. It is easy to check that, for each $p \in H^*$, $\sigma_m(p)$ is an equilibrium profile in the one-move game with payoff $\int V_m(pat)\, q(dt \mid l(p), a)$ and that $V_m(p)$ is the corresponding equilibrium payoff. This is obvious if $l > m$. For $1 \leq l \leq m$, the assertion follows by induction from the equality

$$\int V_m(pat)\, q(dt \mid l(p), a) = \sum_{k=1}^{l-1} r_k(s_k, a_k) + \left[r(s_l, a) + \int W_{m-l}^m(t)\, q(dt \mid s_l, a) \right]. \tag{11}$$

Finally, by repeating the argument in the last paragraph of the proof of Theorem 3.1, it is not hard to see that σ_m is an SPE in the game $\Gamma(g_m, \cdot)$ with corresponding equilibrium payoff V_m. (The argument is essentially a standard backward induction as in Rieder [17] or Lemma 2.1 of Maitra and Sudderth [6].)

Next, Condition 1 and (11) imply, by another application of Lemma 3.6 in Solan [19], that, for fixed $p \in H^*$, the family

$$\left\{ \int V_m(pat)\, q(dt \mid l(p), a) \colon m \in \mathbb{N} \right\}$$

is equicontinuous in a. Thus Assumptions 1 and 2 of §2 hold. So, Theorem 1.3 now follows from Theorem 2.1.

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