De Finetti Coherence and Logical Consistency

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The talk is based on joint work with David Heath, David Lane, James Dickey, and Morris Eaton.
Interpretations of Probability

Relative Frequency (Kolmogorov)

\[ p(A) = \lim_{n} \frac{n(A)}{n} \]

Infinite number of trials
Existence of limit
Circular?

Degree of Belief

Objective - epistemic (Laplace, Jeffreys)
Subjective - coherent opinions (de Finetti, Savage)
Logical consistency and coherence

Example. Basketball game: Minnesota vs Columbia.

\[ A = \text{"Minnesota wins"}, \quad B = \text{"Columbia wins"} \]

Inconsistent truth values.

\[ A \text{ is true and } B \text{ is true} \]

Incoherent probability values.

\[ p(A) = \frac{2}{3}, \quad p(B) = \frac{3}{4} \]
The elementary theory of de Finetti

A bookie (or a broker) buys and sells tickets on a nonempty collection $\mathcal{D}$ of subsets of a sample space $\Omega$. For $A \in \mathcal{D}$

$$p(A) = \text{price of a ticket worth $1$ if } \omega \in A.$$ 

The payoff from a tickets on $A$ is

$$a \cdot [1_A(\omega) - p(A)].$$

The bookie or the price function $p$ is **coherent** if there do not exist events $A_1, \ldots, A_n$ in $\mathcal{D}$ and real numbers $a_1, \ldots, a_n$ such that

$$f(\omega) = \sum a_i \cdot [1_{A_i}(\omega) - p(A_i)] > 0,$$

for all $\omega \in \Omega$. That is, there is no **sure win** for a gambler.
Theorem 1 (de Finetti). Suppose $D$ is an algebra. Then $p$ is coherent iff $p$ is a finitely additive probability measure.

Proof. Suppose $p$ is a finitely additive probability measure. Then for any payoff

$$f(\omega) = \sum a_i \cdot [1_{A_i}(\omega) - p(A_i)],$$

we have

$$\int f \, dp = \sum a_i \cdot [\int 1_{A_i} \, dp - p(A_i)] = 0.$$

So $f$ cannot have a positive infimum.
Proof of the other direction. Suppose \( p \) is coherent and that
\[ p(A \cup B) > p(A) + p(B) \]
for \( A \) and \( B \) in \( \mathcal{D} \) with \( A \cap B = \emptyset \).

Buy tickets on \( A \) and \( B \), and sell a ticket on \( A \cup B \). Your payoff is

\[
- [1_{A \cup B} - p(A \cup B)] + [1_A - p(A)] + [1_B - p(B)] \\
= p(A \cup B) - p(A) - p(B) \\
> 0,
\]
a sure win.

If \( p(A \cup B) < p(A) + p(B) \), buy a ticket on \( A \cup B \) and sell tickets on \( A \) and \( B \) to get a sure win.

It’s also easy to see \( p(\Omega) = 1 \) and \( 0 \leq p(A) \leq 1 \) for all \( A \).
Blue graph plots price of $1 ticket on Obama and red graph plots price of $1 ticket on McCain.
A generalization. Suppose now that $p$ is defined on an arbitrary collection $\mathcal{D}$ of subsets of $\Omega$.

Theorem 2 (de Finetti, Heath & S). The price function $p$ is coherent iff there is a finitely additive probability measure $\mu$ defined on all subsets of $\Omega$ which agrees with $p$ on $\mathcal{D}$.

Proof. Suppose the probability measure $\mu$ agrees with $p$ on $\mathcal{D}$. Then, as in Theorem 1, every payoff function $f$ has $\int f \, d\mu = 0$ and cannot be a sure win.
Now assume that $p$ is coherent. Let $L$ be the linear space of all payoff functions $f$, and let $B$ be the space of all bounded real-valued functions on $\Omega$ with the sup norm topology. Set

$$N = \{ g \in B : \inf g > 0 \}.$$

By coherence,

$$L \cap N = \emptyset.$$

By a separation theorem, $\exists T \in B^*$ such that

$$Tg \geq 0 \text{ for } g \in N, \text{ and } Tg \leq 0 \text{ for } g \in L.$$

We can assume that $T(1) = 1$ and can easily show that

$$Tg = \int g \, d\mu \quad \forall g \in B,$$

where $\mu$ is a finitely additive probability. Now $Tf \leq 0$ for $f \in L$, but also $-f \in L$. So $Tf = 0$ for all payoffs $f$. In particular,

$$T(1_A - p(A)) = \mu(A) - p(A) = 0$$

for $A \in \mathcal{D}$. 
Digression on Finite Additivity

1. Finitely additive integrals have many of the usual properties such as linearity and positivity. Most of the standard inequalities and identities still hold.

2. They can always be extended to all bounded functions.

3. Results about convergence in probability or distribution usually hold but many almost sure convergence theorems fail.

4. What about allowing countably many transactions and proving countable additivity?
Conditional Bets

Let $p(B|A)$ be the price of a $1 ticket on $B$ conditional on the occurrence of $A$; that is, the transaction is cancelled if $A$ does not occur. The payoff is

$$1_A(\omega)[1_B(\omega) - p(B|A)].$$

**Theorem 3 (de Finetti).** If $p$ is coherent, then $p(A \cap B) = p(A)p(B|A)$.

**Proof.** If $p(A \cap B) < p(A)p(B|A)$, buy a ticket on $A \cap B$, sell one on $B$ given $A$ and sell $p(B|A)$ tickets on $A$. Your payoff is

$$1_{A \cap B} - p(A \cap B) - p(B|A) \cdot [1_A - p(A)]$$
$$- 1_A \cdot [1_B - p(B|A)]$$
$$= - p(A \cap B) + p(A)p(B|A) > 0.$$  

If $p(A \cap B) > p(A)p(B|A)$, sell $A \cap B$, and buy $B$ given $A$ and $p(B|A)$ tickets on $A$. 

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Generalized conditioning

Suppose $\Omega = X \times Y$ and $x \in X$ is observed first and then $y \in Y$ is observed. The bookie’s price for a ticket on a subset $A \subseteq X \times Y$ is $p(A)$ and the payoff for $a$ such tickets is

$$f(x, y) = a[1_A(x, y) - p(A)].$$

For $B \subseteq X \times Y$ and $x \in X$, let

$$B_x = \{y \in Y : (x, y) \in B\}.$$

The price for a ticket on $B_x$ given $x$ is written $p(B|x)$. The payoff for $b$ such tickets conditional on $x \in S \subseteq X$ is

$$g(x, y) = b1_S(x)[1_B(x, y) - p(B|x)].$$

The pair $p(\cdot), p(\cdot|x)$ is **coherent** if there do not exist payoffs $f_1, \ldots, f_m, g_1, \ldots, g_n$ such that

$$\inf\{\sum_i f_i + \sum_j g_j\} > 0.$$
**Theorem 4 (Lane & S).** The pair \( p(\cdot), p(\cdot|x) \) is coherent iff \( p(\cdot) \) is a finitely additive probability measure and, for sets \( B \subseteq X \times Y \),

\[
p(B) = \int p(B|x) p_0(dx), \tag{1}
\]

where \( p_0 \) is the marginal of \( p \) on \( X \).

**Sketch of proof.** If (1) holds, it’s easy to check that for payoff functions \( f \) and \( g \),

\[
\int f \, dp = \int g \, dp = 0
\]

and coherence follows.

For the converse, assume coherence. By Theorem 2, \( p \) is a finitely additive probability. To get a contradiction, assume that (1) fails - say

\[
p(B) < \int p(B|x) p_0(dx)
\]

for some \( B \).
Lemma. There is a set $S \subseteq X$ such that
\[ p(B \cap (S \times Y)) < \left\{ \inf_{x \in S} p(B|x) \right\} \cdot p_0(S). \]

Let $p_* = \{\inf_{x \in S} p(B|x)\}$. Buy a ticket on $B \cap (S \times Y)$, sell $p_*$ tickets on $S$ and a ticket on $B$ conditional on $x \in S$. The payoff is
\[
 f(x, y) = \left[ 1_B(x, y)1_S(x) - p(B \cap (S \times Y)) \right] \\
 - p_*[1_S(x) - p_0(S)] - 1_S(x)[1_B(x, y) - p(B|x)] \\
 = -p(B \cap (S \times Y)) - p_*[1_S(x) - p_0(S)] + 1_S(x)p(B|x).
\]

For $(x, y) \in B \cap (S \times Y),
\[
 f(x, y) = \left[ 1 - p(B \cap (S \times Y)) \right] - p_*[1 - p_0(S)] - [1 - p(B|x)] \\
 = [p_*p_0(S) - p(B \cap (S \times Y))] + [p(B|x) - p_*] \\
 > 0.
\]

It’s also easy to check that $f(x, y) > 0$ for $(x, y) \in B^c \cap (S \times Y)$ and for $x \in S^c$. 
Prices are martingales

Suppose \( \Omega = X_1 \times X_2 \times \cdots \) is a product space and \( A \subseteq \Omega \). Let \( Y_1 = p(A) \) and let the conditional prices \( Y_{n+1} = p(A| x_1, \ldots, x_n) \) and corresponding payoffs be defined similarly to the two-stage case. Theorem 2 says coherence requires

\[
E[E[Y_2|X_1]] = \int p(A|x_1) p_0(dx_1) = p(A) = Y_1.
\]

This generalizes to

\[
E[E[Y_{n+1}|X_1, \ldots, X_n]] = \int p(A|x_1, \ldots, x_n) p(dx_{n+1}|x_1, \ldots, x_n)
= p(A|x_1, \ldots, x_n) = Y_n.
\]

So coherent prices form a finitely additive martingale.
Coherence and statistical inference

$\Theta = \text{set of states of nature.}$
$X = \text{set of possible observations.}$

**Statistical model:** $\{p(\cdot|\theta) : \theta \in \Theta\}$

**Inference:** $\{q(\cdot|x) : x \in X\}$

For $A \subseteq \Theta$, $q(A|x)$ is the price for a $1$ ticket on $A$ after observing $x$. 
Coherent inference

Let

\[ A \subset X \times \Theta. \]

A gambler who buys \( b(x) \) tickets on \( A_x \) after observing \( x \) has payoff

\[ g(x, \theta) = b(x) \cdot [1_A(x, \theta) - q(A|x)] \]

and expected payoff

\[ E_\theta g = \int g(x, \theta) p(dx|\theta). \]

The inference \( q \) is coherent if there do not exist \( g_1, \ldots, g_n \) such that

\[ \inf_\theta [E_\theta g_1 + \cdots E_\theta g_n] > 0. \]
**Theorem 5 (Heath & S).** An inference $q$ is coherent iff there is a finitely additive probability $\pi$ on $\Theta$ such that

$$\int_\Theta \int_X f(x, \theta) p(dx|\theta) \pi(d\theta) = \int_X \int_\Theta f(x, \theta) q(d\theta|x) m(dx)$$

for all bounded $f$, where $m$ is the marginal on $X$ of the measure $p(dx|\theta)\pi(d\theta)$.

So coherent inferences are the same as Bayes posteriors from priors that may be only finitely additive.

The proof is another separation argument.
Example. $\Theta = X = \mathbb{R}$. The model is

$$p(\cdot|\theta) \sim N(\theta, 1).$$

The usual inference is

$$q_1(\cdot|x) \sim N(x, 1).$$

It is coherent and is Bayes for an invariant finitely additive prior. A proper (countably additive) Bayes inference is

$$q_2(\cdot|x) \sim N(x/2, 1/2).$$

It is coherent by the theorem. The inference

$$q_3(\cdot|x) \sim N(x + 1, 1)$$

is improper Bayes and is incoherent.
Coherence and logical consistency

**Truth function.** $v: \mathcal{D} \mapsto \{0, 1\}$. $v(A) = 1$ means $A$ is "true" and $v(A) = 0$ means $A$ is "false."

$v$ is consistent if it is not possible to deduce, for some $B \subseteq \Omega$ both $v(B) = 1$ and $v(B) = 0$ using the

**Rules of inference:**

\[
\begin{align*}
v(A \cup B) &= \max\{v(A), v(B)\} \\
v(A \cap B) &= \min\{v(A), v(B)\} \\
v(A^c) &= 1 - v(A).
\end{align*}
\]
Extensions of $v$

$v$ can always be extended to the algebra $\mathcal{A}$ generated by $\mathcal{D}$ using the rules of inference. The extension, also written $v$, may not be unique.

**Trivial example.**

$\Omega = \{a, b, c, d\}, \mathcal{D} = \{A_1, A_2, A_3\}$

where $A_1 = \{a, b\}, A_2 = \{b, c\}, A_3 = \{b, d\}$.

Suppose $v(A_1) = 1, v(A_2) = 1, v(A_3) = 0$.

Then

$v(A_1 \cap A_2) = \min\{v(A_1), v(A_2)\} = 1$,

but also

$A_1 \cap A_2 = \{b\} = A_1 \cap A_3$

and

$v(A_1 \cap A_3) = \min\{v(A_1), v(A_3)\} = 0$. 

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Theorem 6. The extension of $v$ to $\mathcal{A}$ is unique iff $v$ is consistent.

**Trivial proof.** If $v(A) = 1$ and $v(A) = 0$, then $v(A^c) = 1 - v(A) = 1$. So $v$ is inconsistent.

If $v(A) = v(A^c) = 1$, then $v(A) = 1 - v(A^c) = 0$. So $v$ is not unique.
Theorem 7 (Dickey, Eaton & S). Regard the truth function $v$ as a price function. Then $v$ is coherent iff $v$ is consistent.

Proof. If $v$ is coherent, then $v$ agrees with a finitely additive probability $\mu$ on $\mathcal{D}$. But $\mu$ satisfies the rules of inference and must agree with $v$ on the algebra $\mathcal{A}$. Since $\mu$ is finitely additive, we cannot have $\mu(A) = \mu(A^c) = 1$.

Now assume $v$ is consistent and let $v$ also denote the unique extension to $\mathcal{A}$. For $A \in \mathcal{A}$, either $v(A) = 1$ and $v(A^c) = 0$ or vice versa. In either case, $v(\Omega) = \max\{v(A), v(A^c)\} = 1$. It's also easy to check that

$$v(A \cup B) = v(A) + v(B)$$

by considering cases.
Some references


