Stat 8311, Fall 2006, Homework 6

This handout is to help think about parameterizations for two factor models. It is shown in the notes that

$$\begin{aligned} P_E y &= \hat{\mu} \\ &= \text{vec}((\bar{y})') \otimes J_m \\ &= X\hat{\beta} \\ &= (J_r \otimes J_c \otimes J_m, G_r \otimes J_c \otimes J_m, J_r \otimes G_c \otimes J_m, G_r \otimes G_c \otimes J_m)\hat{\beta} \\ &= (J_r \otimes J_c, G_r \otimes J_c, J_r \otimes G_c, G_r \otimes G_c)\beta \otimes J_m \end{aligned}$$

so we must therefore have that

$$\begin{aligned} \text{vec}((\bar{y})') &= (J_r \otimes J_c, G_r \otimes J_c, J_r \otimes G_c, G_r \otimes G_c)\hat{\beta} \\ &= D\hat{\beta} \end{aligned}$$

Since D is of full rank, $\hat{\beta} = D^{-1}\text{vec}((\bar{y})')$, and $\text{var}(\hat{\beta}) = (\sigma^2/m)D^{-1}(D^{-1})'$. Here are some special cases:

```
> get.D <- function(r = 2, c = 3) {
+     R <- factor(rep(letters[1:r], c))
+     C <- factor(rep(LETTERS[1:c], rep(r, c)))
+     mm <- model.matrix(~R + C + R:C)[, ]
+     cat("Basis\n")
+     print(round(mm, 2))
+     cat("\nInverse\n")
+     print(round(inv <- solve(mm), 2))
+     cat("\nVariance\n")
+     print(round(inv %*% t(inv), 2))
+ }
> opt <- options(contrasts = c("contr.treatment", "contr.poly"))
> get.D(2, 4)

Basis
  (Intercept) Rb CB CC CD Rb:CB Rb:CC Rb:CD
1           1  0  0  0  0     0     0     0
2           1  1  0  0  0     0     0     0
3           1  0  1  0  0     0     0     0
4           1  1  1  0  0     1     0     0
5           1  0  0  1  0     0     0     0
6           1  1  0  1  0     0     1     0
7           1  0  0  0  1     0     0     0
8           1  1  0  0  1     0     0     1

Inverse
             1  2  3 4  5 6  7 8
(Intercept)  1  0  0 0  0 0  0 0
Rb          -1  1  0 0  0 0  0 0
CB          -1  0  1 0  0 0  0 0
CC          -1  0  0 0  1 0  0 0
CD          -1  0  0 0  0 0  1 0
Rb:CB        1 -1 -1 1  0 0  0 0
Rb:CC        1 -1  0 0 -1 1  0 0
Rb:CD        1 -1  0 0  0 0 -1 1

Variance
            (Intercept) Rb CB CC CD Rb:CB Rb:CC Rb:CD
```

```
(Intercept)           1 -1 -1 -1 -1     1     1     1
Rb                   -1  2  1  1  1    -2    -2    -2
CB                   -1  1  2  1  1    -2    -1    -1
CC                   -1  1  1  2  1    -1    -2    -1
CD                   -1  1  1  1  2    -1    -1    -2
Rb:CB                 1 -2 -2 -1 -1     4     2     2
Rb:CC                 1 -2 -1 -2 -1     2     4     2
Rb:CD                 1 -2 -1 -1 -2     2     2     4
```

Here is what you get with Helmert contrasts.

```
> options(contrasts = c("contr.helmert", "contr.poly"))
> get.D(2, 4)

Basis
  (Intercept) R1 C1 C2 C3 R1:C1 R1:C2 R1:C3
1           1 -1 -1 -1 -1     1     1     1
2           1  1 -1 -1 -1    -1    -1    -1
3           1 -1  1 -1 -1    -1     1     1
4           1  1  1 -1 -1     1    -1    -1
5           1 -1  0  2 -1     0    -2     1
6           1  1  0  2 -1     0     2    -1
7           1 -1  0  0  3     0     0    -3
8           1  1  0  0  3     0     0     3

Inverse
                1     2     3     4     5     6     7    8
(Intercept)  0.12  0.12  0.12  0.12  0.12  0.12  0.12 0.12
R1          -0.12  0.12 -0.12  0.12 -0.12  0.12 -0.12 0.12
C1          -0.25 -0.25  0.25  0.25  0.00  0.00  0.00 0.00
C2          -0.08 -0.08 -0.08 -0.08  0.17  0.17  0.00 0.00
C3          -0.04 -0.04 -0.04 -0.04 -0.04 -0.04  0.12 0.12
R1:C1        0.25 -0.25 -0.25  0.25  0.00  0.00  0.00 0.00
R1:C2        0.08 -0.08  0.08 -0.08 -0.17  0.17  0.00 0.00
R1:C3        0.04 -0.04  0.04 -0.04  0.04 -0.04 -0.12 0.12

Variance
            (Intercept)   R1   C1   C2   C3 R1:C1 R1:C2 R1:C3
(Intercept)        0.12 0.00 0.00 0.00 0.00  0.00  0.00  0.00
R1                 0.00 0.12 0.00 0.00 0.00  0.00  0.00  0.00
C1                 0.00 0.00 0.25 0.00 0.00  0.00  0.00  0.00
C2                 0.00 0.00 0.00 0.08 0.00  0.00  0.00  0.00
C3                 0.00 0.00 0.00 0.00 0.04  0.00  0.00  0.00
R1:C1              0.00 0.00 0.00 0.00 0.00  0.25  0.00  0.00
R1:C2              0.00 0.00 0.00 0.00 0.00  0.00  0.08  0.00
R1:C3              0.00 0.00 0.00 0.00 0.00  0.00  0.00  0.04

> get.D(3, 3)

Basis
  (Intercept) R1 R2 C1 C2 R1:C1 R2:C1 R1:C2 R2:C2
1           1 -1 -1 -1 -1     1     1     1     1
2           1  1 -1 -1 -1    -1     1    -1     1
3           1  0  2 -1 -1     0    -2     0    -2
4           1 -1 -1  1 -1    -1    -1     1     1
5           1  1 -1  1 -1     1    -1    -1     1
6           1  0  2  1 -1     0     2     0    -2
```

```
7           1 -1 -1  0  2     0     0    -2    -2
8           1  1 -1  0  2     0     0     2    -2
9           1  0  2  0  2     0     0     0     4

Inverse
               1     2     3     4     5     6     7     8    9
(Intercept)  0.11  0.11  0.11  0.11  0.11  0.11  0.11  0.11 0.11
R1          -0.17  0.17  0.00 -0.17  0.17  0.00 -0.17  0.17 0.00
R2          -0.06 -0.06  0.11 -0.06 -0.06  0.11 -0.06 -0.06 0.11
C1          -0.17 -0.17 -0.17  0.17  0.17  0.17  0.00  0.00 0.00
C2          -0.06 -0.06 -0.06 -0.06 -0.06 -0.06  0.11  0.11 0.11
R1:C1        0.25 -0.25  0.00 -0.25  0.25  0.00  0.00  0.00 0.00
R2:C1        0.08  0.08 -0.17 -0.08 -0.08  0.17  0.00  0.00 0.00
R1:C2        0.08 -0.08  0.00  0.08 -0.08  0.00 -0.17  0.17 0.00
R2:C2        0.03  0.03 -0.06  0.03  0.03 -0.06 -0.06 -0.06 0.11

Variance
            (Intercept)   R1   R2   C1   C2 R1:C1 R2:C1 R1:C2 R2:C2
(Intercept)        0.11 0.00 0.00 0.00 0.00  0.00  0.00  0.00  0.00
R1                 0.00 0.17 0.00 0.00 0.00  0.00  0.00  0.00  0.00
R2                 0.00 0.00 0.06 0.00 0.00  0.00  0.00  0.00  0.00
C1                 0.00 0.00 0.00 0.17 0.00  0.00  0.00  0.00  0.00
C2                 0.00 0.00 0.00 0.00 0.06  0.00  0.00  0.00  0.00
R1:C1              0.00 0.00 0.00 0.00 0.00  0.25  0.00  0.00  0.00
R2:C1              0.00 0.00 0.00 0.00 0.00  0.00  0.08  0.00  0.00
R1:C2              0.00 0.00 0.00 0.00 0.00  0.00  0.00  0.08  0.00
R2:C2              0.00 0.00 0.00 0.00 0.00  0.00  0.00  0.00  0.03
```

What if we fit without the interaction? Then the matrix D above is $rc \times (r + c - 1)$, but we have by definition that $\mathrm{vec}((\bar{y})') \in \mathrm{R}(D)$. Thus, from page 59 of the notes, $\hat{\beta} = D^{-}\mathrm{vec}((\bar{y})')$ for any generalized inverse:

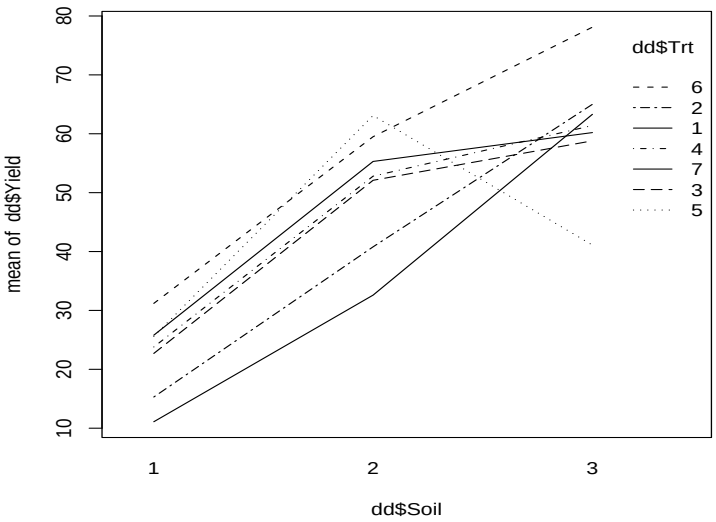
```
> get.D1 <- function(r = 2, c = 3) {
+     R <- factor(rep(letters[1:r], c))
+     C <- factor(rep(LETTERS[1:c], rep(r, c)))
+     mm <- model.matrix(~R + C)[, ]
+     cat("Basis\n")
+     print(round(mm, 2))
+     SVD <- svd(mm)
+     ginv <- t(SVD$u %*% diag(1/SVD$d) %*% t(SVD$v))
+     cat("\nG-Inverse\n")
+     print(round(ginv, 2))
+     cat("\nVariance\n")
+     print(round(ginv %*% t(ginv), 2))
+ }
> opt <- options(contrasts = c("contr.treatment", "contr.poly"))
> get.D(2, 4)

Basis
  (Intercept) Rb CB CC CD Rb:CB Rb:CC Rb:CD
1           1  0  0  0  0     0     0     0
2           1  1  0  0  0     0     0     0
3           1  0  1  0  0     0     0     0
4           1  1  1  0  0     1     0     0
5           1  0  0  1  0     0     0     0
6           1  1  0  1  0     0     1     0
```

```
7           1  0  0  0  1     0     0     0
8           1  1  0  0  1     0     0     1

Inverse
             1  2  3 4  5 6  7 8
(Intercept)  1  0  0 0  0 0  0 0
Rb          -1  1  0 0  0 0  0 0
CB          -1  0  1 0  0 0  0 0
CC          -1  0  0 0  1 0  0 0
CD          -1  0  0 0  0 0  1 0
Rb:CB        1 -1 -1 1  0 0  0 0
Rb:CC        1 -1  0 0 -1 1  0 0
Rb:CD        1 -1  0 0  0 0 -1 1

Variance
            (Intercept) Rb CB CC CD Rb:CB Rb:CC Rb:CD
(Intercept)           1 -1 -1 -1 -1     1     1     1
Rb                   -1  2  1  1  1    -2    -2    -2
CB                   -1  1  2  1  1    -2    -1    -1
CC                   -1  1  1  2  1    -1    -2    -1
CD                   -1  1  1  1  2    -1    -1    -2
Rb:CB                 1 -2 -2 -1 -1     4     2     2
Rb:CC                 1 -2 -1 -2 -1     2     4     2
Rb:CD                 1 -2 -1 -1 -2     2     2     4

> options(contrasts = c("contr.helmert", "contr.poly"))
> get.D(2, 4)

Basis
  (Intercept) R1 C1 C2 C3 R1:C1 R1:C2 R1:C3
1           1 -1 -1 -1 -1     1     1     1
2           1  1 -1 -1 -1    -1    -1    -1
3           1 -1  1 -1 -1    -1     1     1
4           1  1  1 -1 -1     1    -1    -1
5           1 -1  0  2 -1     0    -2     1
6           1  1  0  2 -1     0     2    -1
7           1 -1  0  0  3     0     0    -3
8           1  1  0  0  3     0     0     3

Inverse
                1     2     3     4     5     6     7    8
(Intercept)  0.12  0.12  0.12  0.12  0.12  0.12  0.12 0.12
R1          -0.12  0.12 -0.12  0.12 -0.12  0.12 -0.12 0.12
C1          -0.25 -0.25  0.25  0.25  0.00  0.00  0.00 0.00
C2          -0.08 -0.08 -0.08 -0.08  0.17  0.17  0.00 0.00
C3          -0.04 -0.04 -0.04 -0.04 -0.04 -0.04  0.12 0.12
R1:C1        0.25 -0.25 -0.25  0.25  0.00  0.00  0.00 0.00
R1:C2        0.08 -0.08  0.08 -0.08 -0.17  0.17  0.00 0.00
R1:C3        0.04 -0.04  0.04 -0.04  0.04 -0.04 -0.12 0.12

Variance
            (Intercept)   R1   C1   C2   C3 R1:C1 R1:C2 R1:C3
(Intercept)        0.12 0.00 0.00 0.00 0.00  0.00  0.00  0.00
R1                 0.00 0.12 0.00 0.00 0.00  0.00  0.00  0.00
C1                 0.00 0.00 0.25 0.00 0.00  0.00  0.00  0.00
C2                 0.00 0.00 0.00 0.08 0.00  0.00  0.00  0.00
```

```
C3                0.00 0.00 0.00 0.00 0.04  0.00  0.00  0.00
R1:C1             0.00 0.00 0.00 0.00 0.00  0.25  0.00  0.00
R1:C2             0.00 0.00 0.00 0.00 0.00  0.00  0.08  0.00
R1:C3             0.00 0.00 0.00 0.00 0.00  0.00  0.00  0.04
```

Problem 5:

```
> file <- "http://www.stat.umn.edu/~sandy/courses/8311/rir096.txt"
> dd <- read.table(url(file), header = TRUE)
> dd$Soil <- factor(dd$Soil)
> dd$Trt <- factor(dd$Trt)
> interaction.plot(dd$Soil, dd$Trt, dd$Yield)
```




```
> m1 <- lm(Yield ~ Soil + Trt, data = dd)
> anova(m2 <- update(m1, ~. + I(predict(m1)^2)))

Analysis of Variance Table

Response: Yield
                Df Sum Sq Mean Sq F value    Pr(>F)
Soil             2 5696.3  2848.2 33.1078 2.215e-05
Trt              6  731.1   121.8  1.4163    0.2916
I(predict(m1)^2) 1    1.1     1.1  0.0132    0.9106
Residuals       11  946.3    86.0

> soilbetas <- rep(c(0, coef(m1)[2:3]), 7)
> anova(m3 <- update(m2, ~. + soilbetas:Trt))

Analysis of Variance Table

Response: Yield
                Df Sum Sq Mean Sq F value   Pr(>F)
Soil             2 5696.3  2848.2 16.0156 0.006699
Trt              6  731.1   121.8  0.6851 0.673515
I(predict(m1)^2) 1    1.1     1.1  0.0064 0.939423
Trt:soilbetas    6   57.1     9.5  0.0535 0.998654
Residuals        5  889.2   177.8
```