Assignment # 7, Stat 8311, Fall 2006

Reading

Pinherio and Bates, Sections 2.1-2.2 and all of Chapter 1.

Problems

The following problems are due on Wednesday, November 29, 2006, in class.

1. In the balanced one-way random-effects model with p groups and m observations per group, the log-likelihood function can be written as:

$$\ell = -\frac{pm}{2}\log(2\pi) - \frac{p(m-1)}{2}\log(\sigma^2) - \frac{p\log(\lambda)}{2} - \frac{\| Q_{\mathcal{E}}y \|^2}{2\sigma^2} - \frac{\| P_{\mathcal{E}}y \|^2}{2\lambda} - \frac{pm(\bar{y}_{++}-\mu)^2}{2\lambda} \tag{1}$$

   where $\lambda = \sigma^2 + m\sigma_b^2$, and $\mathcal{E} = \mathrm{R}(Q_p \otimes J_m)$ is the usual estimation space for one-way models, orthogonal to the column of ones. Ignoring the possibility of negative estimates, verify that the maximum likelihood estimates are given by

$$\begin{array}{rcl} \hat{\mu} & = & \bar{y}_{++} \\ \hat{\sigma}^2 & = & = \| Q_{\mathcal{E}}y \|^2/(p(m-1)) \\ \hat{\sigma}_b^2 & = & = \dfrac{\hat{\lambda}-\hat{\sigma}^2}{m} = \dfrac{\| P_{\mathcal{E}}y \|^2/p - \| Q_{\mathcal{E}}y \|^2/(p(m-1))}{m} \end{array}$$

2. Ignoring possible negative estimates, we can estimate the large-sample variance of $\theta = (\mu, \sigma^2, \lambda)'$ by computing the expected Fisher information, $\mathrm{E}(\partial\ell/\partial\theta(\partial\theta'))^{-1}$. Verify that the expected Fisher information is given by:

$$\mathrm{E}\left(\frac{\partial\ell}{\partial\theta(\partial\theta')}\right)^{-1} = \begin{pmatrix} \frac{\lambda}{np} & 0 & 0 \\ 0 & \frac{2\sigma^4}{p(m-1)} & 0 \\ 0 & 0 & \frac{2\lambda^2}{p} \end{pmatrix}$$

   Use the delta method to get an expression for the variance of $\hat{\sigma}_b^2$. (According to the delta method, if $g(\theta)$ is a function of θ, then the estimated variance of $g(\theta)$ is $g(\theta)'\mathrm{var}(\theta)g(\theta)$).

3. The likelihood function $L(\mu, \sigma^2, \sigma_b^2|y)$ for the one-way balanced random effects model is found by exponentiating (1). This likelihood can be factored into two pieces, one of which is independent of μ:

$$L(\mu,\sigma^2,\sigma_b^2|y) = L_1(\mu,\sigma^2,\sigma_b^2|\bar{y}_{++}) \times L_2(\sigma^2,\sigma_b^2|(\| P_{\mathcal{E}}y \|^2, \| Q_{\mathcal{E}}y \|^2)) \tag{2}$$

   The term L_2 in (2) is also a *marginal likelihood* because

$$L_2(\sigma^2,\sigma_b^2|(\| P_{\mathcal{E}}y \|^2, \| Q_{\mathcal{E}}y \|^2)) = \int L(\mu,\sigma^2,\sigma_b^2|y)d\mu \tag{3}$$

   Thompson and Patterson (1971) suggested that to estimate the variance parameters we should use the marginal likelihood (3) rather than L. This separates estimation of the fixed parameters and the variance parameters. Estimation of the fixed parameters can be based on L_1, with the variance parameters assumed known to be equal to their estimates based on L_2, and so these are then standard least squares estimates with a known inner product. These are called REML estimates, short for restricted ML, or residual ML, estimates, or can also be called marginal ML estimates. The properties of these estimates are quite similar to the usual ML estimates.

   Let $\delta = (\| P_{\mathcal{E}}y \|^2/(p-1) - \hat{\sigma}^2)/m$. Show that the REML estimates for the balanced one-way random effects model are given by $\hat{\sigma}_b^2 = \max(\delta, 0)$ and

$$\hat{\sigma}^2 = \begin{cases} \| Q_{\mathcal{E}}y \|^2/[p(m-1)] & \text{if } \delta \geq 0 \\ \| y - \bar{y}_{++}J_{pm} \|^2/(pm-1) & \text{if } \delta < 0 \end{cases}$$

4. For the rails data, draw a contour plot of the restricted log-likelihood for (σ^2, σ_b^2).