Assignment # 5, Stat 8311, Fall 2006

Reading

Chapter 5 and 6 of the notes.

Announcements

The first exam will be given as scheduled in the syllabus on Friday, November 3, in class. The exam is closed-book, but you may have one sheet of standard sized paper of notes (front and back OK). The exam will cover material through this homework assignment.

There will be NO CLASS on Friday, October 27. The class on Monday, October 30 will be longer than usual to make up for the missed class. We will meet at 2:30 in 313 Vincent, take a break and then meet in 213 Vincent at 3:35.

Problems

These problems are due in class on October 30.

1. Artificial heart valves are made of a ceramic material. To test the durability of the ceramic, an accelerated life test is performed on the valve in which the valve is cycled rapidly in a flowing solution that simulates the human blood system. The valve is removed from the system periodically, and the maximum wear is measured. A reasonable model for this experiment is

$$y_i = \beta x_i + (\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i), i = 1, 2, ..., n$$

where y_i is the wear at the i-th observation, β is an unknown constant corresponding the wear rate, and x_i is the known number of cycles (or the equivalent number of years of wear) up to the i-th observation. The ε_i are assumed independent random variables with mean 0 and variance var$(\varepsilon_i) = (x_i - x_{i-1})\sigma^2$ (set $x_0 = 0$).

(a) Compute the BLUE $\hat{\beta}$ of β, and find its variance, and also find the estimate of σ^2. To complete this problem, you will probably need the following matrix result:

$$\begin{pmatrix} a_1 & a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_p \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} \frac{1}{a_1} + \frac{1}{a_2 - a_1} & -\frac{1}{a_2 - a_1} & 0 & 0 & \cdots & 0 \\ -\frac{1}{a_2 - a_1} & \frac{1}{a_3 - a_2} + \frac{1}{a_2 - a_1} & -\frac{1}{a_3 - a_2} & 0 & \cdots & 0 \\ 0 & -\frac{1}{a_3 - a_2} & \frac{1}{a_4 - a_3} + \frac{1}{a_3 - a_2} & -\frac{1}{a_4 - a_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{a_p - a_{p-1}} \end{pmatrix}$$

This last matrix is called an *symmetric band matrix* as only the diagonal and first off-diagonals are nonzero, and all other elements are zero.

Solution. The covariance matrix is equal to σ^2 times the form shown above with $a_i = x_i$. Multiplication shows $(X'\Sigma^{-1}X)^{-1} = 1/x_n$, and $X'\Sigma^{-1}y = y_n$ and so the BLUE of β is $\hat{\beta} = y_n/x_n$, and var$\hat{\beta} = \sigma^2/x_n$. Using Theorem 4.23 $\hat{\sigma}^2 = (y'\Sigma^{-1}y - y_n^2/x_n)/(n-1)$. This can be simplified for this problem to

$$\hat{\sigma}^2 = \frac{1}{n-1}\left\{\sum_{i=1}^{n}\left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right)^2 - \frac{y_n^2}{x_n}\right\}$$

where we set $y_0 = x_0 = 0$. This formula is most easily obtained by transforming the problem by defining $y_i^* = (y_i - y_{i-1})/\sqrt{x_i - x_{i-1}}$, and $x_i^* = \sqrt{x_i - x_{i-1}}$, and then E$(y^*|x^*) = x^*\beta$ and var$(y^*|x^*) = \sigma^2$. ∎

(b) Derive a test of the hypothesis:

NH: $\beta = \beta_0$
AH: $\beta \neq \beta_0$

where β_0 is a fixed, predetermined number. Give the test statistic, and its distribution under the assumption of normality, for both the null and alternative hypotheses. **Solution.** One do either do an F test, replacing the response by $y - \beta_0 x$, or else do a t-test. In either case, we need to estimate $\sigma*^2$. The t-test is

$$t = \frac{\hat{\beta} - \beta_0}{\hat{\sigma}/\sqrt{x_n}}$$

and $t^2 \sim F(1, n-1, \delta^2)$, with $\delta^2 = n(\beta_0 - \beta)^2/\sigma^2)$. ∎

2. Consider the linear model $\mathrm{E}(y) = \mu, \mu \in \mathcal{E} \subset \Re^n$, with $\mathrm{var}(y) = \sigma^2\Sigma$, with $\Sigma = \Sigma' > 0$ known. Suppose further that we have parameterized $\mu = X\beta$ with X less than full rank. Characterize the set of all estimable functions.

   Solution. As long as the covariance matrix is of full rank, the definition of estimability and the conditions for it are unchanged. ∎

3. Prove Theorem 5.6 that the n-vector z and the m-vector w are independent if and only if (a, z) and (b, w) are independent for all $a \in \Re^n$ and all $b \in \Re^m$.

   Solution. The argument requires characteristic functions. Suppose z and w are independent. Then

$$\phi_{[z,w]}\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \phi_w(t_1)\phi_z(t_2) \tag{1}$$

   For any a and b, the characteristic function of $a'z$ and $b'w$ is

$$\begin{aligned} \phi_{[a'z,b'w]}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} &= \mathrm{E}(\exp(i(s_1 a'z + s_2 b'w))) \\ &= \phi_{[z,w]}\begin{pmatrix} s_1 a \\ s_2 b \end{pmatrix} \\ &= \phi_w(s_1 a)\phi_z(s_2 b) \end{aligned}$$

   In the other direction, start with (1), and set $t_1 = a$ and $t_2 = b$, and then apply (5.7) in the notes. ∎

4. Suppose that A is an $m \times n$ matrix and B is a $p \times q$ matrix. We can then define the *Kronecker product* $A \otimes B$ to be the $pm \times qn$ matrix,

$$A \otimes B = (a_{ij}B)$$

   Some useful properties of the Kronecker product are:

$$\begin{aligned} A \otimes (B \otimes C) &= (A \otimes B) \otimes C \\ (A \otimes B)(C \otimes D) &= (AC \otimes BD) \\ (A \otimes B)' &= (A' \otimes B') \\ (A + B) \otimes (C + D) &= A \otimes C + A \otimes D + B \otimes C + B \otimes D \\ \alpha(A \otimes B) &= (\alpha A \otimes B) = (A \otimes \alpha B) \\ (A \otimes B)^{-1} &= = A^{-1} \otimes B^{-1} \\ \mathrm{tr}(A \otimes B) &= \mathrm{tr}(A) \times \mathrm{tr}(B) \end{aligned}$$

   In this problem, I_p is the $p \times p$ identity matrix, and J_n is the $n \times 1$ vector of ones; consequently, $J_n J_n'$ is an $n \times n$ matrix of all ones. It is convenient further to define $P_n = J_n J_n'/n$ and $Q_n = I_n - P_n$.

   Consider the balanced one-way model with p groups and m observations per group. Suppose $\mathrm{E}(y_{ij}) = \gamma_i$, and write $y = (y_{11}, y_{12}, \ldots, y_{pm})'$ and $\mu = \mathrm{E}(y)$.

(a) A basis for $\mathcal{E}$ is given by the columns of $(I_p \otimes J_m)$. Find an expression for $P_{\mathcal{E}}$ using Kronecker products.

Solution. $(I_p \otimes P_m)$, where P_m is the projection on J_m.∎

(b) Consider testing the null hypothesis $\mu = \eta J_{mp} = \eta(J_p \otimes J_m)$ for some constant η against the alternative that $\mu \in \mathcal{E}$. Again using Kronecker products, find expressions for $P_{\mathcal{E}-\mathcal{E}_0}$ and for $P_{\mathcal{E}_0}$, and derive the appropriate F-test.

Solution. $\mathcal{E}_0 = \mathrm{R}(J_p \otimes J_m)$, and so $P_{\mathcal{E}_0} = (P_p \otimes P_m)$ and $P_{\mathcal{E}-\mathcal{E}_0} = (Q_p \otimes P_m)$, where $Q = I - P$.∎

(c) Citing the relevant theorems, and showing that they apply, find the distributions of $q_1 = \| P_{\mathcal{E}-\mathcal{E}_0} y \|^2$, $q_2 = \| Q_{\mathcal{E}} y \|^2$, and

$$\frac{q_2/r_2}{q_1/r_1} \tag{2}$$

where r_1 and r_2 are appropriate constants under both the null and alternative hypotheses. Also, find computing formulas for q_1 and q_2 based on Kronecker products.

Solution. $q_1 \sim \sigma^2\chi^2(p-1)$ and $q_2 \sim \sigma^2\chi^2(mp-p)$. The two sums of squares are independent because the subspaces are independent. The ratio is therefore a (non-central) $F(p-1, mp-p, \| P_{\mathcal{E}-\mathcal{E}_0}\mu \|^2/\sigma^2)$.∎

(d) Now suppose we modify the model to be $y \sim \mathrm{N}(\eta(J_p \otimes J_m), \sigma^2 V)$,where η is an unknown constant, and the $mp \times mp$ matrix V is of the form

$$\begin{aligned} V &= I + (\phi_1/\sigma^2)V_1 \\ &= I + (\phi_1/\sigma^2)(I_p \otimes J_m J_m') \\ &= (I_p \otimes I_m) + (\phi_1/\sigma^2)(Ip \otimes J_m J_m') \end{aligned}$$

where $\sigma^2 + m\phi_1 > 0$. Consider the test of the null hypothesis that $\phi_1 = 0$ against the alternative that it is positive. Show that the test given by (2) is appropriate for this hypothesis, and give its distribution.

Solution. This is the *one-way random effects* model. The covariance matrix is no longer a multiple of the identify matrix, so at least in principle we need to use the inner product $[a, b] = (a, \Sigma^{-1}b)$. Now according to Theorem 4.20 in the notes, the ols and gls estimates will be the same if $\mathrm{R}(VX) = \mathrm{R}(X)$, where in this problem $X = J_p \otimes J_m$. We compute:

$$\begin{aligned} VX &= [(I_p \otimes I_m) + (\phi_1/\sigma^2)(I_p \otimes J_m J_m')](J_p \otimes J_m) \\ &= (J_p \otimes J_m) + (J_p \otimes (\phi_1/\sigma^2)J_m J_m' J_m) \\ &= (1 + m(\phi_i/\sigma^2))(J_p \otimes J_m) \\ &= (1 + m\phi_1/\sigma^2)(J_p \otimes J_m) \end{aligned}$$

Thus ols and gls give the same *estimates*. Now

$$P_{\mathcal{E}-\mathcal{E}_0} y \sim \mathrm{N}(0, \sigma^2 P_{\mathcal{E}-\mathcal{E}_0} V P_{\mathcal{E}-\mathcal{E}_0})$$

and since $P_m = J_m J_m'/m$,

$$\begin{aligned} \sigma^2 P_{\mathcal{E}-\mathcal{E}_0} V P_{\mathcal{E}-\mathcal{E}_0} &= (Q_p \otimes P_m)[\sigma^2 I + (m\phi_1/\sigma^2)P_m](Q_p \otimes P_m) \\ &= (Q_p \otimes P_m)(\sigma^2 I)(Q_p \otimes P_m) + (Q_p \otimes P_m)((m\phi_1/\sigma^2)P_m)(Q_p \otimes P_m) \\ &= \sigma^2(Q_p \otimes P_m) + (m\phi_1/\sigma^2)(Q_p \otimes P_m) \\ &= (\sigma^2 + m\phi_1)(Q_p \otimes P_m) \end{aligned}$$

from which it follows immediately that $\| P_{\mathcal{E}-\mathcal{E}_0} y \|^2 \sim (\sigma^2 + m\phi_1)\chi^2(m-1)$. Thus in the random effects case, the null distribution of (2) is central $F(p-1, n-p)$, as in the fixed effects case, but the distribution under the alternative is $(\sigma^2 + m\phi_1)/\sigma^2$ times a central F, since the numerator and denominator are different multiples of chi-squared random variables. ∎

5. Use R in this problem. We consider a one-way model with $p = 4$ groups and $m = 2$ observations per group. Generate the following computer output:

```
set.seed(112)
x <- c(1,1,2,2,3,3,4,4)
x4 <- c(0,0,0,0,0,0,1,1)
y <- rnorm(8)
(m1 <- lm(y ~ x4))
(m2 <- lm(y ~ factor(x)))
(m3 <- lm(y ~ x4 + factor(x)))
(m4 <- lm(y ~ factor(x) + x4))
(m5 <- lm(y ~ factor(x) -1))
```

Describe the spaces ($\mathcal{E}$ and $\mathcal{E}_\prime$) for each of these models, and the tests performed by each of the anovas.

If you have access to another program, such as SPSS, SAS, S-Plus, Arc or Minitab, repeat this problem to see the difference in solutions.

6. Consider the one-way model, $y_{ij} \sim \mathrm{N}(\mu+\alpha_i, \sigma^2)$ for $i = 1, \ldots, 4$ and $j = 1, 2$, with all the y_{ij} independent. Suppose we wanted to test the hypothesis that simultaneously $\alpha_1+\alpha_3-2\alpha_2 = 0$ and $\alpha_1+\alpha_2+\alpha_3-3\alpha_4 = 0$.

   (a) Write this as a general parametric hypothesis, as in the notes; that is, if $\psi_1 = A_1\beta$, find A_1.
   Solution. If $\beta = (\mu, \alpha_1, \alpha_2, \alpha_3, \alpha_4)'$ then

   $$A_1 = \begin{pmatrix} 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 \end{pmatrix}$$

   ∎

   (b) Verify that the quantities to be tested are estimable.
   Solution. $\psi_1 = A_1\beta$ is estimable if the rows of A_1 are linear combinations of the rows of $X = (J_8, I_4 \otimes J_2)$, and this is easily verified. ∎

   (c) Find expressions for $\hat{\psi}_1$, Var($\hat{\psi}_1$), and for the F-test.
   Solution. The straightforward way to look at this problem requires using generalized inverses, and $\hat{\psi}_1 = A_1\hat{\beta}$, where $\hat{\beta}$ is any ols solution. Then var($\hat{\psi}_1$) $= \sigma^2 A_1(X'X)^- A_1'$, and the F-test is $\hat{\psi}_1'(A_1(X'X)^- A_1')^{-1}\hat{\psi}_1/2\hat{\sigma}^2$. ∎