Assignment # 3, Stat 8311, Fall 2006

Reading

Chapter 3 of the notes

Problems

The following problems are due on Wednesday, October 4, 2006, in class. There is no class on Monday, October 2.

1. Prove that if $P$ is a projection and $\|Px\| \leq \|x\|$ for all $x \in V$, then $P$ is an orthogonal projection. (Draw a picture for the $\mathbb{R}^2$ case.)

2. Let $A$ be a symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Let $x$ be $n \times 1$ and define $D = \{x | x \in \mathbb{R}^n, \|x\| = 1\}$. Show that:

$$\max_D (x^tAx) = \lambda_n$$

and

$$\min_D (x^tAx) = \lambda_1.$$ 

Find vectors that achieve the minimum and the maximum.

3. Let $A$ be an $n \times p$ matrix with linearly independent columns. Show that $A$ can be written uniquely in the form $A = Q_1R$, where $Q_1$ is $n \times p$ with orthonormal columns and $R$ is upper triangular with positive diagonal elements. (Hint: recall the Gram-Schmidt method.)

4. Let $A$ be an $n \times p$ matrix. Show that:

(a) $\rho(A) = \rho(A') = \rho(A'A) = \rho(AA')$.

(b) $N(A) = N(A'A)$.

(c) $R(A) = R(AA')$ and $R(A') = R(A'A)$.

5. In $\mathbb{R}^3$, consider the set of linear equations

$$\beta_1 + \beta_2 = 2$$

$$\beta_1 + \beta_3 = 4$$

$$2\beta_1 + \beta_2 + \beta_3 = 6$$

What is $X$? What is $y$? What is $R(X)$? What is $R(X)^\perp$? What is the set of all possible solutions to this set of equations? Give the dimensions of all spaces.

6. $V$ is an $n$-dimensional space with inner product $(\cdot, \cdot)$. Suppose that $M$ is a proper linear subspace of $V$ (that is, $M \neq V$, and $M \neq \{0\}$). Define $A$ to be a reflector across $M$ if for all $z \in V$, we can write $z = x + y$ with $x \in M$ and $y \in M^\perp$, then $Az = x - y$.

(a) Is $A$ symmetric? Why or why not?

(b) Is $A$ an orthogonal projection? Why or why not?

(c) Is $A$ orthogonal? Why or why not?

(d) Find the rank of $A$. If $A$ has rank $n$, find $A^{-1}$.

(e) If $\{v_1, \ldots, v_n\}$ is an orthonormal basis for $V$ whose first $p$ elements span $M$, what is the matrix of $A$ with respect to the $v$ basis?

7. Show that for any matrix $A$, $(A'A)^+ = A^+(A')^+$, where $A^+$ is the Moore-Penrose generalized inverse.