Assignment #2, Stat 8311, Fall 2006

Reading
Finish Chapter 2.

Reminder
No class on Monday, October 2.

Problems
Due: Monday, September 25, 2006

1. Consider the following linear transformations of \((\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3\). In each case, show whether it is or is not a projection. When it is a projection, what is it on and what is it along?

   (a) \[ A \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_1 - \eta_3 \\ \eta_2 - \eta_3 \\ 0 \end{pmatrix} \]

   (b) \[ B \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_1 + \eta_2 \\ \eta_1 + \eta_2 + \eta_3 \end{pmatrix} \]

2. For each of the following functions of two arguments in \(\mathbb{R}^2\), show whether or not it is an inner product.

   (a) \( x_1y_1 - x_2y_2 \)

   (b) \( 2x_1y_1 + x_1x_2 + 3x_2y_2 \)

   (c) \( 3x_1y_1 + 5x_2y_2 \)

   (d) \( 2x_1y_1 + x_1y_2 + 3x_2y_2 \)

3. Show that for any inner product, and the Euclidean norm \(\|x\|^2 = (x, x)\),

   \[
   \| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2)
   \]

   \[
   (x, y) = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2).
   \]

   Thus if we know lengths, we know the inner product. Does this relationship hold for other norms?

4. Consider the linear model with response \(y \in \mathbb{R}^3\), and \(E(y) = \mu\), with \(\mu \in C\), with

   \[ C = \mathbb{R} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]
Define:

\[ D = \{ y | (x, y) = 0, x \in C \}, \]
where \((x, y)\) is the usual inner product, and

\[ E = \{ x | x = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R} \}. \]

Verify that \( C + D = C + E = \mathbb{R}^3 \) and \( C \cap D = C \cap E = \{ 0 \} \). Find the projections \( P_{C|D} \) and \( P_{C|E} \).

5. Consider \( P_3 \), the vector space of polynomials of degree \( \leq 3 \). We know a basis for \( P_3 \) is \( \{ 1, t, t^2, t^3 \} \). Also, \( \int_0^1 p(t)q(t)dt \) is an inner product on \( P_3 \). Apply the Gram-Schmidt method to find an orthonormal basis for \( P_3 \).

6. Let \( y_1, \ldots, y_n \) be random variables with \( \text{var}(y_i) = \sigma_{ii} > 0, \text{cov}(y_i, y_j) = \sigma_{ij} \). Find new random variables \( z_1, \ldots, z_n \) as linear combinations of the \( y_i \) so that: \( \text{var}(z_j) = 1 \) and \( \text{cov}(z_i, z_j) = 0, i \neq j \). (Hint: Apply the Gram-Schmidt method to random variables rather than vectors. Give explicit formulas for \( z_1 \) and \( z_2 \), and a general formula for \( z_j, j > 2 \).)