Homework 5 Solution

9–27a, then for 2/50 and 25/50 successes, compute a 95% CI using both the traditional way and the way described in part b, using the answer in the back.

The Z-score is

$$\frac{\hat{p}-p}{\sqrt{p(1-p)/n}},$$

which is a function of the data and the parameter, and is approximately $N(0,1)$, so it's independent of any population parameters.

The traditional 95% confidence interval is $\hat{p} \pm 1.96\sqrt{\hat{p}(1-\hat{p})/n}$. For 2/50 successes, $\hat{p} = 2/50$ and $n = 50$, so it's $(-0.143, 0.094)$. For 25/50 successes, $\hat{p} = 25/50$ and $n = 50$, so it's $(0.361, 0.636)$.

The new confidence interval is described in part b and is given in the back of the book. The confidence intervals are $(0.037, 0.108)$ and $(.376, 0.624)$.

9–29: Blacks constitute 15% of the population in a given city. Find the probability that the proportions of blacks in two independent random samples, each of size 500, will differ by more than 2 percentage points (either way).

We use the fact that $\hat{p}_1 - \hat{p}_2 \sim N(p_1 - p_2, p_1 q_1/n_1 + p_1 q_1/n_2) = N(0, 0.0226^2)$, using $p_1 = p_2 = 0.15$ and $n_1 = n_2 = 500$. Then

$$P(|\hat{p}_1 - \hat{p}_2| > .02) = 2P(\hat{p}_1 - \hat{p}_2 < -.02) = 2P\left(Z < \frac{-.02 - 0}{0.0226}\right) = 0.376.$$

9–31: An experiment compared the number of cavities for flouride and non-flouride toothpaste users. For flouride, $n = 295$, $\bar{x} = 10.88$, and $s = 6.36$. For non-flouride, $n = 284$, $\bar{x} = 13.41$, and $s = 7.20$. Give an estimate for the difference in mean number of cavities, with a standard error.

The estimate of the mean difference is $13.41 - 10.88 = 2.53$.

The standard error is $\sqrt{s_1^2/n_1 + s_2^2/n_2} = \sqrt{6.36^2/295 + 7.2^2/284} = 0.565$.

9–35: Prior to the 1984 election, 1549 adults were sampled; 638 Democrats and 441 Independents. 43% of Democrats and 28% of Independents preferred Mondale. Give a 95% CI for the population proportion of Democrats, and a 95% CI for the difference in proportion between Democrats and Independents preference of Mondale.

For proportion of Democrats, $\hat{p} = 638/1549 \approx 0.412$, so

$$\hat{p} \pm 1.96\sqrt{\hat{p}(1-\hat{p})/n} = (.387, .436).$$

For difference of proportion between Democrats and Independence preference of Mondale, $\hat{p}_1 = .43$, $\hat{p}_2 = .28$, $n_1 = 638$ and $n_2 = 441$. Following equation 6 on p. 393, the confidence interval is $(0.093, 0.207)$.

9–37: We know that $X_i \sim N(0, \sigma^2)$. Then $X_i/\sigma \sim N(0,1)$. Since we're interested in σ^2, we need a function including that; since a standard normal squared is a chi-squared with one degree of freedom, we know that $(X_i/\sigma)^2 \sim \chi^2_1$. Then $\sum(X_i^2/\sigma^2) \sim \chi^2_n$. This is a pivot statistic because it's a function of the data and the parameter, and its distribution is independent of any population parameter. Using the table,

$$P(6.26 < \chi^2_{15} < 27.5) = 0.95.$$

Then

$$P(6.26 < \sum(X_i^2)/\sigma^2 < 27.5) = 0.95.$$

Simplifying to put σ^2 in the middle, we get

$$P\left(\frac{\sum X_i^2}{27.5} < \sigma^2 < \frac{\sum X_i^2}{6.26}\right) = .95.$$

This is our 95% confidence interval.

Using data from 9–R11, calculate an ordinary 95% CI for θ, and then Bayesian 95% credible intervals for θ for the two priors shown. How do you interpret each of these intervals? Finally, assuming the true θ equals 4, what's the corresponding p-value? What if the true θ is 1?

The ordinary 95% CI for θ is $2.41 \pm 1.96\sqrt{1.252} = (0.217, 4.603)$.

To find the posterior, we can use the calculations as already done in the book, since our prior and our likelihood are both normally distributed. We'll use the notation from p. 407, which uses the precision $\pi = 1/\sigma^2$. The posterior precision is the sum of the prior precision and the data precision, that is,

$$\pi_{\text{po}} = \pi_{\text{pr}} + \pi_{\text{data}}.$$

The posterior mean is a weighted average of the sample and prior means, with weights proportional to the precisions of the prior and data:

$$\mu_{\text{po}} = \frac{\pi_{\text{pr}}}{\pi_{\text{po}}}\mu_{\text{pr}} + \frac{\pi_{\text{data}}}{\pi_{\text{po}}}\bar{x}$$

Then the posterior distribution of θ is $N(\mu_{\text{po}}, 1/\pi_{\text{po}})$.

We are given that $\bar{x} = 2.41$ and $\pi_{\text{data}} = 1/1.252 = 0.799$.

For prior of $N(4,4)$, $\mu_{\text{pr}} = 4$ and $\pi_{\text{pr}} = 1/4 = 0.25$, so

$$\pi_{\text{po}} = 0.25 + 0.799 = 1.049$$
$$\mu_{\text{po}} = (0.25/1.049)(4) + (0.799/1.049)(2.41) = 2.789$$

So a 95% credible interval is

$$2.789 \pm 1.96\sqrt{1/1.049} = (0.875, 4.703).$$

For prior of $N(1,16)$, $\mu_{\text{pr}} = 1$ and $\pi_{\text{pr}} = 1/16 = 0.0625$, so

$$\pi_{\text{po}} = 0.0625 + 0.799 = 0.8615$$
$$\mu_{\text{po}} = (0.0625/0.8615)(4) + (0.799/0.8615)(2.41) = 2.525$$

So a 95% credible interval is

$$2.525 \pm 1.96\sqrt{1/0.8615} = (0.413, 4.637).$$

Interpretation of these three intervals?

For the confidence interval, we know that if we repeated the procedure of taking data and making an interval, 95% of those intervals would include the true value of θ. Note that it's the interval that is random, and the true value that is fixed, so we are careful to talk about the interval including the true value, not the true value falling into the interval.

For the two credible intervals, we believe that there is a 95% probability that the true value of θ falls in the interval. Since this is a Bayesian analysis, we treat the true value of θ as random, so it is now appropriate to talk about the true value falling into the interval.

The final question isn't totally clear; I'll assume it means to use θ equals the given value as the null hypothesis, and to use θ not equal to the given value as the alternate; that is, that it should be a two-sided test.

For $\theta = 4$, the p-value is

$$2P(\bar{x} < 2.41|\theta = 4) = 2P\left(Z < (2.41 - 4)/\sqrt{1.252}\right) = 0.155.$$

For $\theta = 1$, the p-value is

$$2P(\bar{x} > 2.41|\theta = 1) = 2P\left(Z > (2.41 - 1)/\sqrt{1.252}\right) = 0.207.$$

These p-values give the probability of getting a sample mean as extreme or more extreme than the one we actually got (that is, 2.41). Since these values are fairly large, we see that our data is somewhat likely to be seen for each of the given values of θ, so we have no evidence for rejecting either of the null hypotheses.

Comparing these conclusions to that of our confidence intervals, we see that each of the intervals include 1 and 4, so each of the intervals gives us the same conclusion. This isn't too surprising, because the two priors we used had 1 and 4 as their means, so there was some prior expectation that θ would be near those values. This is in some sense analogous to using a null hypothesis of θ equal to 1 and 4; in both cases, we're looking for evidence to disprove this claim. The frequentist method is to find the probability of getting our data, given that the claim is true; the Bayesian method is to make a compromise between our prior claim and the information from the data.