STAT4101 Fall 2007 Practice Exam 2 Solution.

1. (a) We know that $P(-1 < X < 1) = 1$, so we set

$$1 = \int_{-1}^{1} c(1-x^2)dx = c(x - x^3/3)|_{-1}^{1}$$
$$= c[(1-1/3) - (-1+1/3)] = 4/3c,$$

so $c = 3/4$.

(b) $F(x) = P(X < x)$, so between -1 and 1,

$$F(x) = \int_{-1}^{x} \frac{3}{4}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{x}$$
$$= \frac{3}{4}\left[\left(x - \frac{x^3}{3}\right) - \left(-1 + \frac{1}{3}\right)\right] = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right),$$

and

$$F(x) = \begin{cases} 0 & \text{for } x < -1 \\ \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) & \text{for } -1 < x < 1 \\ 1 & \text{for } x > 1. \end{cases}$$

(c) $P(X > 0) = \int_0^1 \frac{3}{4}(1-x^2)dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{3}{4}\left(1 - \frac{1}{3}\right) = \frac{1}{2}$.

(d) $E\left(\frac{1}{1-x^2}\right) = \int_{-1}^{1}\left(\frac{1}{1-x^2}\right)\left(\frac{3}{4}(1-x^2)\right)dx = \int_{-1}^{1}\frac{3}{4}dx = \frac{3}{4}x\,|_{-1}^{1} = \frac{3}{4}(1+1) = \frac{3}{2}$.

2. a) 75.8% b) $0.5/0.758 = 0.660$ c) 81.75

3. (a) Write as $e^{3t + \frac{1}{2}4t^2}$: Normal with mean 3, variance 2.

(b) $m_Y(t) = e^{-3t}e^{3(t/2)+2(t/2)^2} = e^{-(3/2)t + t^2/2}$

(c) Normal with mean $-3/2$, variance 1.

4. (a) 0.18

(b)

x	1	2
$p(x)$	0.4	0.6

(c)

x	1	2
$p(x\|Y=1)$	$\frac{0.08}{0.2} = 0.4$	$\frac{0.12}{0.2} = 0.6$

(d) $E(X|Y=1) = 0.4(1) + 0.6(2) = 1.6$

(e) No. Even though $p(x) = p(x|Y=1)$, $p_X(1)p_Y(2) = 0.4 \cdot 0.3 = 0.12 \neq 0.1 = p(1,2)$.

5. (a) $P(Y < 1) = 1 - P(Y > 1) = 1 -$ volume between Y=1 and Y=2 $= 1 - \frac{1}{2}1 \cdot 1 \cdot \frac{1}{2} = \frac{3}{4}$

(b) $P(Y < 1|X = 1) = 1$, as when $X = 1$, Y is only between 0 and 1.

(c) $f_X(x)$ is the area above $X = x$, which is $\frac{1}{2}(2-x)$ for $0 < x < 2$.

(d) $Y|X$ is Uniform between 0 and $2-X$, so $f(y|x) = \frac{1}{2-x}$ for $0 < y < 2-x$.

(e) $E(Y|X)$ is therefore $\frac{2-x}{2}$.

(f) X and Y are not independent because the support is not rectangular.

6. (a) $f_Y(y) = \int_0^1 c\frac{x}{y+1}dx = \frac{c}{y+1}\frac{x^2}{2}\Big|_{x=0}^{x=1} = \frac{c}{2(y+1)}$ for $0 < y < 1$.

   (b) $f_(x|y) = \left(\frac{cx}{y+1}\right) / \left(\frac{c}{2(y+1)}\right) = 2x$ for $0 < x < 1$.

   (c) $E(Y+1) = \int_0^1 \int_0^1 (y+1)\frac{cx}{y+1}\,dx\,dy = \int_0^1 \frac{cx^2}{2}\Big|_{x=0}^{x=1} dy = \frac{c}{2}y\Big|_0^1 = \frac{c}{2}$.

   (d) $P(X < Y) = \int_0^1 \int_0^y c\frac{x}{y+1}\,dx\,dy$

   (e) Yes. The support is rectangular and the joint density can be factored: $(cx)\left(\frac{1}{y+1}\right)$.

7. (a) $\mathrm{Cor}(X,Y) = \mathrm{Cov}(X,Y)/\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} = (E(XY) - EX\,EY)/\sqrt{4\cdot 1}$
   $= (16 - 5\cdot 3)/2 = 1/2$.

   (b) $E(2X - 3Y + 4) = 2EX - 3EY + 4 = 2(5) + 3(3) + 4 = 5$

   (c) $\mathrm{Var}(2X - 3Y + 4) = \mathrm{Var}(2X) + \mathrm{Var}(-3Y) + 2\,\mathrm{Cov}(2X, -3Y)$
   $= 4\,\mathrm{Var}(X) + 9\,\mathrm{Var}(Y) + 2(2)(-3)\,\mathrm{Cov}(X,Y) = 4(4) + 9(1) - 12(1) = 13$.

8. $E(X) = E(E(X|Y)) = E(100Y) = 100EY = 100\frac{99}{99+1} = 99$.