1 $Y \sim \text{Poi}(2)$, so $p(y) = \frac{2^y}{y!}e^{-2}$. Thus

a. $P(Y = 4) = p(4) = \frac{2^4}{4!}e^{-4} = \frac{2}{3}e^{-2} \approx 0.09$
b. $P(Y < 4) = p(0) + p(1) + p(2) + p(3) = \frac{2^0}{0!}e^{-2} + \frac{2^1}{1!}e^{-2} + \frac{2^2}{2!}e^{-2} + \frac{2^3}{3!}e^{-2} \approx 0.135 + 0.271 + 0.271 + 0.180 \approx 0.857$
c. $P(Y \geq 4) = 1 - P(Y < 4) = 1 - P(Y < 4) \approx 1 - 0.857 = 0.143.$
d. $P(Y \geq 4 > Y \geq 2) = \frac{P(Y \geq 4 \cap Y \geq 2)}{P(Y \geq 2)} = \frac{P(Y \geq 4)}{P(Y \geq 2} = \frac{0.143}{1-(p(0)+p(1))} = \frac{0.143}{1-(0.135+0.271)} = \frac{0.143}{0.593} = 0.241$

2 Let X be the number of knots in the 10-cubic-foot block of wood. Then $X \sim \text{Poi}(1.5)$. We want to find $P(X \leq 1) = P(X = 0) + P(X = 1) = \frac{1.5^0}{0!}e^{-1.5} + \frac{1.5^1}{1!}e^{-1.5} = (1 + 1.5)e^{-1.5} \approx 0.558.$

3 Let X be the number of customers that arrive in the 1-hour period. Then $X \sim \text{Poi}(7)$, and $p(x) = \frac{7^x}{x!}e^{-7}$. Thus

a. $P(X \leq 3) = p(0) + p(1) + p(2) + p(3) = \frac{7^0}{0!}e^{-7} + \frac{7^1}{1!}e^{-7} + \frac{7^2}{2!}e^{-7} + \frac{7^3}{3!}e^{-7} \approx 0.0009 + 0.0064 + 0.022 + 0.052 = 0.0818$
b. $P(X \geq 2) = 1 - P(X \leq 1) \approx 1 - (0.0009 + 0.0064) = 0.993$
c. $P(X = 5) = p(5) = \frac{7^5}{5!}e^{-7} \approx 0.128.$
d. Letting $Y = 10X$ be the number of minutes servers spend serving customers, then $EY = E(10X) = 10EX = 10(7) = 70$ and $V(Y) = V(10X) = 10^2V(X) = 10^2(7) = 700.$
e. Then the standard deviation is $\sigma = \sqrt{700} \approx 26$. Then 150 minutes is approximately $\mu + 3\sigma = 70 + 3 * 26 = 148$. Since by the empirical rule almost all the measurements are within 3 standard deviations of the mean, this is unlikely.
f. The mean number of customers to come in this two hour period is 14, so the probability that exactly two come is $\frac{14^2}{2!}e^{-14}$.
g. Let X be the number that come in the first time period and Y be the number that come in the second time period. Both are Poi(7). We want to find $P(X + Y = 2)$. There are three ways this can happen: $(X = 0, Y = 2)$, $(X = 1, Y = 1)$, and $(X = 2, Y = 0)$. These are mutually exclusive so we'll add them up, and X and Y are independent so we can multiply probabilities. Then

$$\begin{aligned}
P(X + Y = 2) &= P((X = 0 \cap Y = 2) \cup (X = 1 \cap Y = 1) \cup (X = 2, Y = 0)) \\
&= P(X = 0)P(Y = 2) + P(X = 1)P(Y = 1) + P(X = 2)(Y = 0) \\
&= \left(\frac{7^0}{0!}e^{-7}\right)\left(\frac{7^2}{2!}e^{-7}\right) + \left(\frac{7^1}{1!}e^{-7}\right)\left(\frac{7^1}{1!}e^{-7}\right) + \left(\frac{7^2}{2!}e^{-7}\right)\left(\frac{7^0}{0!}e^{-7}\right) \\
&= \left(2\frac{7^2}{2!} + \frac{7^2}{1!}\right)e^{-7-7} \\
&= 2(7^2)e^{-14}
\end{aligned}$$

h. Notice that $(14^2/2!)e^{-14} = 2(7^2)e^{-14}$, so they are equal. The general rule is that if X and Y are each Poi λ, $X + Y \sim \text{Poi}(2\lambda)$. We'll study that more in Chapter 6.

4 From the table, the probabilities are 0.358, 0.378, 0.189, and 0.059.

Using the Poisson approximation, with $\lambda = np = 20(0.05) = 1$, they are $\frac{1^0}{0!}e^{-1} \approx 0.368$, $\frac{1^1}{1!}e^{-1} \approx 0.368$, $\frac{1^2}{2!}e^{-1} \approx 0.184$, and $\frac{1^3}{3!}e^{-1} \approx 0.061$.

For the exact calculation, the probability of no sales is the probability of 100 no sales in a row, or 0.97^{100}, so the probability of at least one sale is $1 - 0.97^{100} \approx 0.9524$. The number of sales the salesperson makes is approximately Poisson with mean $100(0.03) = 3$. The probability of no sales is then $\frac{3^0}{0!}e^{-3} \approx 0.0498$, so the probability of at least one sale is approximately 0.9502.

5

$$E[Y(Y-1)] = \sum_{y=0}^{\infty} y(y-1)\frac{\lambda^y e^{-\lambda}}{y!} = \sum_{y=2}^{\infty} y(y-1)\frac{\lambda^y e^{-\lambda}}{y!} = \sum_{y=2}^{\infty} \frac{\lambda^y e^{-\lambda}}{(y-2)!}$$
$$= \sum_{z=0}^{\infty} \frac{\lambda^(z+2) e^{-\lambda}}{z!} = \lambda^2 \sum_{z=0}^{\infty} \frac{\lambda^z e^{-\lambda}}{z!} = \lambda^2 \sum_{z=0}^{\infty} p(z) = \lambda^2$$

So $V(Y) = EY^2 - (EY)^2 = (EY^2 - EY) + EY - (EY)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$.

6 To find the moments of a random variable and to prove that a random variable follows a particular distribution.

7
a. $m(t) = E(e^t) = \sum_{y=0}^{n} e^{ty}\binom{n}{y}p^y q^{n-y} = \sum_{y=0}^{n}\binom{n}{y}(pe^t)^y q^{n-y} = (pe^t + q)^n$

b. $m'(t) = n(pe^t+q)^{n-1}(pe^t)$ and $m''(t) = n(n-1)(pe^t+q)^{n-2}(pe^t)^2 + n(pe^t+q)^{n-1}(pe^t)$.
Thus for $t = 0$, $pe^t + q = p + q = 1$, so $EY = m'(0) = np$ and $EY^2 = n(n-1)p^2 + np$.
So $V(Y) = EY^2 - (EY)^2 = n(n-1)p^2 + np - (np)^2 = -np^2 + np = np(1-p)$.

c. Since this moment-generating function looks exactly like that found in part a, with $p = 0.5$ and $n = 3$, it is a binomial distribution with $p = 0.5$ and $n = 3$.

8 $m'(t) = (1/6)e^t + (2/3)e^{2t} + (3/2)e^{3t}$ and $m''(t) = (1/6)e^t + (4/3)e^{2t} + (9/2)e^{3t}$. Then

a. $EY = m'(0) = 1/6 + 2/3 + 3/2 = 7/3$

b. $EY^2 = m''(0) = 1/6 + 4/3 + 9/2 = 6$, so $V(Y) = EY^2 - (EY)^2 = 6 - (7/3)^2 = 5/9$.

c. Since $m(t) = \sum_y e^{ty}p(y)$, we see that the coefficient of e^ty is the probability of y, for each term. So the probability function is

y	1	2	3
$p(y)$	1/6	1/3	1/2

.

9 $P(6 < Y < 16) = P(|Y - 11| < 5) = P(|Y - \mu| < \frac{5}{3}\sigma) \geq 1 - 1/(5/3)^2 = 1 - 9/25 = 16/25 = 0.64$.

Let $k = .3$