5-53

a) $F_{X+Y}(z) = P(X+Y \le z) = \int_0^z \int_0^{z-y} 2\,dx\,dy = z^2, 0 < z < 1$.

b) First, we find $\text{Var}(X+Y)$ directly using $\text{Var}(X+Y) = E[(X+Y)^2] - [E(X+Y)]^2$.

$$E[(X+Y)^2] = \int_0^1 z^2 f_Z(z)\,dz = \int_0^1 z^2(2x)\,dz = 1/2,$$

and from 5-43c, $E(X+Y) = 2/3$, so

$$\text{Var}(X+Y) = 1/2 - (2/3)^2 = 1/2 - 4/9 = \frac{1}{18}.$$

Then, we show this equals $\text{Var}(X)+\text{Var}(Y)+2\,\text{Cov}(X,Y)$. Now $\text{Var}(X) = \text{Var}(Y)$ by symmetry, so to find the variances, we first find

$$EX^2 = \int_0^1 2x^2(1-x)dx = 1/6.$$

So $\text{Var}(X) = \text{Var}(Y) = 1/6 - (1/3)^2 = 1/18$. From 5-43b, $EXY = 1/12$, so $\text{Cov}(X,Y) = 1/12 - (1/3)^2 = -1/36$. Thus

$$\text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X+Y) = \frac{1}{18} = \frac{1}{18} - \frac{2}{36} = \frac{1}{18}.$$

c) $\rho = \dfrac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\,\text{Var}(Y)}} = \dfrac{-1/36}{1/18} = -\dfrac{1}{2}$.

5-59 We know a uniform (0,1) has mean 1/2 and variance 1/12 (ex. 5.6a).

a) $E(X+Y+Z) = EX + EY + EZ = 1/2 + 1/2 + 1/2 = 3/2$.

b) Since they are independent, all covariances are 0, so

$$\text{Var}(X+Y+Z) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(Z) = 3/12 = 1/4.$$

c) $\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(-Y) + 2Cov(X,-Y) = \text{Var}(X) + (-1)^2\,\text{Var}(Y) - 2Cov(X,Y) = 2/12 = 1/6$.

d) Since they are independent, all covariances between different variables are 0, so

$$\begin{aligned}\text{Cov}(X+Y-Z, 2X-Y+3Z) &= \text{Cov}(X,2X) + \text{Cov}(Y,-Y) + \text{Cov}(-Z,3Z)\\ &= 2\,\text{Var}(X) - \text{Var}(Y) - 3\,\text{Var}(Z) = -2/12.\end{aligned}$$

5-60

a) $\displaystyle\int_0^\infty 2e^{-2x}e^{-y}dy = 2e^{-2x}, x > 0, \quad \int_0^\infty 2e^{-2x}e^{-y}dx = e^{-y}, y > 0$

b) They are independent, the product of the marginals equals to joint.

c) $E(X+Y) = EX + EY = 1/2 + 1 = 3/2$. Recognizing these as exponentials, we don't need to do any integration.

d) By independence, $\text{Var}(X-Y) = \text{Var}(X) + \text{Var}(Y) = 1 + 1/4 = 5/4$.