1. *What is the distribution of X (or (X, Y)) in each of the following situations? If the distribution has a name, please state it, and if it has parameters, please state those also. If the distribution has a shorthand notation, you may use that, for example, $X \sim \text{Bin}(5, 0.4)$. (In some situations, the distribution may not be exact. In that case, choose the distribution that best fits the situation.)*

   a) *A machine produces parts one at at a time, and each part is judged to be either good or defective. The probability of a defective part is 0.1. The operator needs to produce 100 good parts to fill his quota for the day. Let X be the total number of parts he makes in order to fill his quota.*
   Solution: $X \sim \text{NegBin}(100, 0.1)$

   b) *Same machine and operator as above. The boss stops by to watch, and stays while 10 parts are made. Let X be the number of bad parts the boss sees made.*
   Solution: $X \sim \text{Bin}(10, 0.1)$

   c) *Flaws occur in a rope at random, but on average are five feet apart. Let X be the number of flaws in a fifty foot piece of rope.*
   Solution: This is a Poisson process, with $\lambda = 1/5$, and $t = 50$, so $m = (1/5)(50) = 10$, so $X \sim \text{Poi}(10)$.

   d) *A box of 100 nails has 5 that are defective. A carpenter takes 10 out of the box at random to make his next project. Let X be the number of defective nails that he took.*
   Solution: X is Hypergeometric; $N = 100$, $M = 5$, $n = 10$.

   e) *Customers come to a certain post office either to send a package or pick up their mail (but never both). On average, someone comes in to mail a package every ten minutes, and to pick up their mail every five minutes. Let X be the total number of customers that enter the post office between noon and 1pm.*
   Solution: Let X_1 be the number sending packages and X_2 be the number picking up mail, so $X = X_1 + X_2$. Since $X_1 \sim \text{Poi}(6)$ and $X_2 \sim \text{Poi}(12)$, X is also a Poisson, with mean parameter being the sum of the mean parameters of X_1 and X_2, so $X \sim \text{Poi}(18)$.

   f) *A person drops a ball into a box that is one foot long and one foot wide. Assume the ball is equally likely to be anywhere in the box, and let (X, Y) denote the position of the ball in the box.*
   Added for clarity: *That is, if the box is directly in front of you, find X by measuring (in feet) how far the ball fell from the left side of the box, and find Y by measuring (in feet) how far the ball fell from the side of the box closest to you.*
   Solution: (X, Y) has a uniform distribution over the unit square.

   g) *John says meteors hit the earth on average every 100 years. Assuming John is right, let X be the number of years until the next meteor hits earth.*
   Solution: $X \sim \text{Exp}(100)$

   h) *49 Democrats, 49 Republicans, and 2 Independents are in the Senate. We choose a senator at random. Let X be the party of this senator.*
   Solution: X is multinomial with three possible states; the probability of a Democrat or a Republican is 49/100 and the probability of an Independent is 2/100.

2. *Let $X \sim \text{Bin}(10, 0.4)$. Find $P(X = 2)$. Find $P(X \le 2)$. Find $E(X)$. (Copies of p. 648 and p. 650 will be included in the exam if a problem like this is included.)*
   Solution: Using the table on p. 649 with $n = 10$, $p = 0.4$, and $k = 2$, $P(X = 2) = .1209$. Using the table on p. 651 with $n = 10$, $p = 0.4$, and $k = 3$, $P(X \ge 3) = .8327$, so $P(X \le 2) = 1 - .8327 = .1673$. $EX = np = 10 \times 0.4 = 4$.

3. *Let $X \sim \text{Geo}(p)$ and $Y \sim \text{NegBin}(3, p)$, where X and Y are independent. Find $P(X + Y = 0)$.* **Doesn't make sense; how about $P(X + Y = 5)$ instead!)** *Find the mean and variance of $X + Y$.*

   Solution: We can also write $X \sim \text{NegBin}(1, p)$, and we know that adding two negative binomials with the same p results in a negative binomial with the r's added, so $X + Y \sim \text{NegBin}(4, p)$. Then to find $P(X + Y = 5)$, use the probability function for the negative binomial with $r = 4$, $p = p$, and $k = 5$, which is

$$\binom{4}{3} p^4 q^1 = 4p^4(1 - p).$$

   Since it's a negative binomial, the mean is $r/p = 4/p$ and the variance is $rq/p^2 = 4(1 - p)/p^2$.

4. *Suppose that some event occurs according to a Poisson process, with 2 events per minute on average. What's the average length of time until the next event? What's the probability of no events in the next minute?*

Solution: The length of time until the next event is an Exp(2), with has mean 1/2. The probability of no events in the next minute can be calculated in two ways: first, by using the exponential and calculating the probability of the first event taking place after one minute (Let T be the time of the first event.)

$$P(T > 1) = 1 - F(1) = 1 - (1 - e^{-2}) = e^{-2}.$$

Or we can count the number of events in the first minute, which is a Poi(2), and calculating the probability of no events. Let X be the number of events, then

$$P(X = 0) = (2^0/0!)e^{-2} = e^{-2}.$$

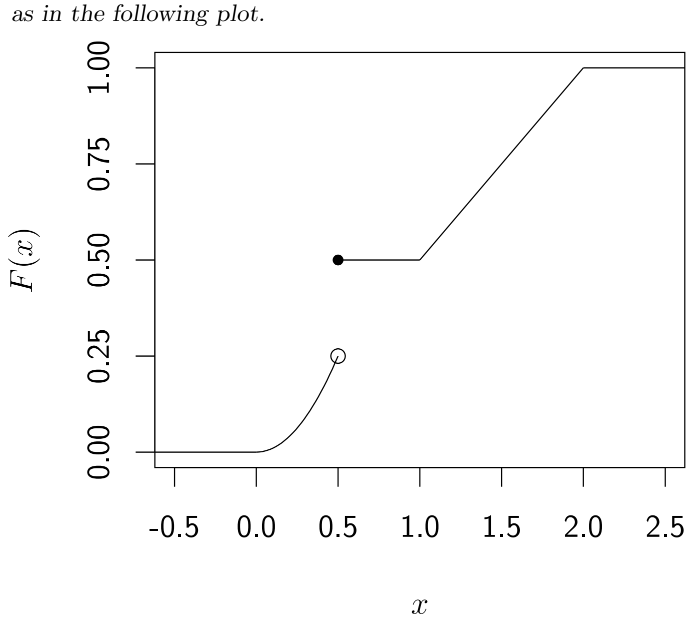
5. *Suppose calls come into a service center following a Poisson process, average 10 calls an hour. Let X be the number of calls an operator answers in a given hour, so $X \sim \text{Poi}(10)$. With probability 0.1, the operator cannot answer the question and must forward it to the supervisor. What's the expected number of calls an operator must forward to the supervisor in a given hour? Use the following steps to help you.*

   a) *Assume we already know how many total calls were received (X), let Y be the number of calls that were forwarded. What's the distribution of Y, given X known?*
   Solution: $Y \sim \text{Bin}(X, 0.1)$
   b) *What's the expectation of Y given X known?*
   Solution: $E(Y|X) = X \times 0.1 = 0.1X$
   c) *Now find $E(Y)$. Remember $E(Y) = E(E(Y|X))$.*
   Solution: $E(Y) = E(E(Y|X)) = E(0.1X) = 0.1EX = 0.1 \times 10 = 1.$

6. *Let the cumulative distribution function of X be*

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ x^2 & \text{for } 0 \le x < 1/2 \\ 1/2 & \text{for } 1/2 \le x < 1 \\ x/2 & \text{for } 1 \le x < 2 \\ 1 & \text{for } x > 2, \end{cases}$$

as in the following plot.



a) *Find $P(X < 1/4)$.*
Solution: $P(X < 1/4) = F(1/4) - P(X = 1/4) = (1/4)^2 - 0 = 1/16$.

b) *Find $P(X = 1/2)$.*
Solution: It jumps from 0.25 to 0.5 at $X = 1/2$, so $P(X = 1/2) = 0.5 - 0.25 = 0.25$.

c) *Find $P(X > 1|X > 1/2)$.*
Solution:

$$\begin{aligned} P(X > 1|X > 1/2) &= P(X > 1, X > 1/2)/P(X > 1/2) = P(X > 1)/P(X > 1/2) \\ &= (1 - F(1))/(1 - F(1/2)) = (1 - 1/2)/(1 - 1/2) = 1 \end{aligned}$$

d) *Let $Y = 2X$. What's the cumulative distribution function of Y?*
Solution: $F_Y(y) = P(Y \le y) = P(2X \le y) = P(X \le y/2) = F_X(y/2)$, which is

$$F_Y(y) = F_X(y/2) = \begin{cases} 0 & \text{for} \quad y/2 < 0 \\ (y/2)^2 & \text{for} \quad 0 \le y/2 < 1/2 \\ 1/2 & \text{for} \quad 1/2 \le y/2 < 1 \\ (y/2)/2 & \text{for} \quad 1 \le y/2 < 2 \\ 1 & \text{for} \quad y/2 > 2 \end{cases} = \begin{cases} 0 & \text{for} \quad y < 0 \\ y^2/4 & \text{for} \quad 0 \le y < 1 \\ 1/2 & \text{for} \quad 1 \le y < 2 \\ y/4 & \text{for} \quad 2 \le y < 4 \\ 1 & \text{for} \quad y > 4 \end{cases}$$

7. *The probability density function for X is $f(x) = c(1 - x^2)$ for $-1 < x < 1$ and 0 otherwise.*

a) *Show $c = 3/4$.*
Solution: The integral over the support must equal 1, so

$$1 = \int_{-1}^{1} c(1 - x^2)dx = c(x - x^3/3)|_{-1}^{1} = c[(1 - 1/3) - (-1 + 1/3)] = \frac{4}{3}c,$$

so $c = 3/4$.

b) *Find the mean and median of X.*
Solution: The pdf is symmetric around 0, so both are 0.

c) *Find $P(X = 1/2)$.*
Solution: This is a continuous random variable, so the probability of equalling any number is 0.

d) *Find F and use it to find $P(0 < X < 1/2)$.*
Solution:

$$F(x) = \int_{-1}^{x} \frac{3}{4}(1 - t^2)dt = \frac{3}{4}(t - t^3/3)|_{-1}^{x} = \frac{3}{4}\left(x - \frac{x^3}{3} + \frac{2}{3}\right) \quad \text{for} \quad -1 < x < 1.$$

$F(x) = 0$ for $x < -1$ and $F(x) = 1$ for $x > 1$.

$$\begin{aligned} P(0 < X < 1/2) = F(1/2) - F(0) &= \frac{3}{4}(1/2 - (1/2)^3/3 + 2/3) - \frac{3}{4}(0 + 2/3) \\ &= \frac{3}{4}(1/2 - 1/24) = \frac{3}{4} \cdot \frac{11}{24} = \frac{11}{32} \end{aligned}$$

e) *Find the density of $2X$.*
Solution: Let $Y = 2X$. Then

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(2X \le y) = P(X \le y/2) \\ &= F_X(y/2) = \frac{3}{4}\left((y/2) - \frac{(y/2)^3}{3}\right) = \frac{3}{8}y - \frac{1}{32}y^3 \end{aligned}$$

for $-1 < y/2 < 1$, or $-2 < y < 2$. $F_Y(y)$ is 0 for $y < -2$ and 1 for $y > 2$. Then the density

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{3}{8} - \frac{3}{32} y^2$$

for $-2 < y < 2$ and 0 otherwise.

8. *Let $X \sim \text{Unif}(0,1)$ and $Y \sim \text{Exp}(1)$, independently.*

   a) *What's the joint density of X and Y? Remember to state the support.*
   Solution: $f_X(x) = 1$ for $0 < x < 1$ and $f_Y(y) = e^{-y}$ for $y > 0$. Since they're independent, the joint is the product of the marginals, so $f_{XY}(x,y) = 1 \cdot e^{-y}$ for $0 < x < 1$ and $y > 0$ and 0 otherwise.

   b) *Find $E(XY)$.*
   Solution: By independence, $E(XY) = EXEY = (1/2) \cdot 1 = 1/2$.

   c) *Find $P(Y < X)$.*
   Solution: We need to integrate the density over the region where $0 < y < x < 1$:

$$\int_0^1 \int_0^x e^{-y}\, dy\, dx = \int_0^1 (-e^{-y})\Big|_{y=0}^{y=x} dx = \int_0^1 (1 - e^{-x}) dx$$
$$= x + e^{-x}\Big|_0^1 = (1 + e^{-1}) - (0 + 1) = e^{-1}$$

9. *Let the joint density of X and Y be uniform over the unit circle.*
Are X and Y independent? Why or why not?
Solution: No. The support isn't rectangular.

10. *$P(X = 0) = 1/2$, $P(X = 1) = 1/4$, and $P(X = 2) = 1/4$. Use the pgf to find $\text{Var}(X)$.*
Solution:

The pgf is $\eta_X(t) = E(t^X) = \frac{1}{2} + \frac{1}{4}t + \frac{1}{4}t^2.$

The first two derivatives are $\eta_X'(t) = \frac{1}{4} + \frac{2}{4}t$ and $\eta_X''(t) = \frac{1}{2}.$

Now $EX = \eta_X'(1) = 1/4 + 1/2 = 3/4$ and $E(X(X-1)) = E(X^2 - X) = \eta_X''(1) = 1/2,$

so $\text{Var}(X) = E(X^2 - X) + EX - (EX)^2 = 1/2 + 3/4 - (3/4)^2 = 5/4 - 9/16 = 11/16.$