

Spectral Radii of Large Non-Hermitian Random Matrices

Tiefeng Jiang¹ · Yongcheng Qi²

Received: 28 March 2015 / Revised: 10 July 2015 / Published online: 21 August 2015
© Springer Science+Business Media New York 2015

Abstract By using the independence structure of points following a determinantal point process, we study the radii of the spherical ensemble, the truncation of the circular unitary ensemble and the product ensemble with parameters n and k . The limiting distributions of the three radii are obtained. They are not the Tracy–Widom distribution. In particular, for the product ensemble, we show that the limiting distribution has a transition phenomenon: When $k/n \rightarrow 0$, $k/n \rightarrow \alpha \in (0, \infty)$ and $k/n \rightarrow \infty$, the limiting distribution is the Gumbel distribution, a new distribution μ and the logarithmic normal distribution, respectively. The cumulative distribution function (cdf) of μ is the infinite product of some normal distribution functions. Another new distribution ν is also obtained for the spherical ensemble such that the cdf of ν is the infinite product of the cdfs of some Poisson-distributed random variables.

Keywords Spectral radius · Determinantal point process · Eigenvalue · Independence · Non-Hermitian random matrix · Extreme value

Mathematics Subject Classification (2010) 15B52 · 60F99 · 60G55 · 60G70

The research of Tiefeng Jiang was supported in part by NSF Grant DMS-1209166 and DMS-1406279. The research of Yongcheng Qi was supported in part by NSF Grant DMS-1005345.

✉ Tiefeng Jiang
tjiang@stat.umn.edu; jiang040@umn.edu
Yongcheng Qi
yqi@d.umn.edu

¹ School of Statistics, University of Minnesota, 224 Church Street, S. E., Minneapolis, MN 55455, USA

² Department of Mathematics and Statistics, University of Minnesota Duluth, Duluth, MN 55812, USA

1 Introduction

The largest eigenvalues of the three Hermitian matrices (Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble) are proved to converge to the Tracy–Widom laws by [49,50]. Since then, there have been very active researches in this direction. For example, [6] established a connection between the longest increasing subsequence problem and the Tracy–Widom law. The relationships between the largest eigenvalues, combinatorics, growth processes, random tilings and the determinantal point processes are found (see, e.g., [51] and [29] and the references therein). In the studies of the high-dimensional statistics [27,31,32] prove that the largest eigenvalues of Wishart and Jacobi matrices converge to the Tracy–Widom law. Ramírez et al. [42] obtained the asymptotic distribution of the largest eigenvalues of beta-Hermite ensemble. Recently, a research interest is the universality of the largest eigenvalues of matrices with non-Gaussian entries; see, for example, [20,48] and the references therein.

In this paper, we will study the largest absolute values of the eigenvalues of some non-Hermitian matrices. Initiated by Ginibre [23] for the study of Gaussian random matrices (real, complex and symplectic), the interest has continued and theoretical results are found to have many applications in quantum chromodynamics, chaotic quantum systems and growth processes; see more descriptions from the paper by Akemann et al. [2]. The applications also include dissipative quantum maps [25] and fractional quantum Hall effect [15]. We refer the readers to [33] for more details.

For a matrix \mathbf{M} with eigenvalues z_1, \dots, z_n , the quantity $\max_{1 \leq j \leq n} |z_j|$ is referred to as the spectral radius of \mathbf{M} . In the pioneer work by Rider [45,46] and Rider and Sinclair [47], the spectral radii of the real, complex and symplectic Ginibre ensembles are studied. For the complex Ginibre ensemble, it is shown that the spectral radius converges to the Gumbel distribution. This phenomenon is very different from the Tracy–Widom distribution. The key observation is that the absolute values of the eigenvalues of the complex Ginibre ensemble are independent random variables with the Gamma distributions; see [34]. Later, it is found that the independence phenomenon is true not only for the complex Ginibre ensemble, but also for other complex-valued determinantal point processes; see, for example, [26] for further details.

In this paper, we will study the largest radii of three rotation-invariant and non-Hermitian random matrices: the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$, where \mathbf{A} and \mathbf{B} are independent complex Ginibre ensembles, the truncation of circular unitary ensemble, and the product ensemble $\prod_{j=1}^k \mathbf{X}_j$, where $\mathbf{X}_1, \dots, \mathbf{X}_k$ are independent $n \times n$ complex Ginibre ensembles. The spectral radius of the first one converges to a new distribution ν , that of the second one converges to the Gumbel distribution, and that of the third one, depending on the ratio $\alpha := \lim_{n \rightarrow \infty} k_n/n$, converges to the Gumbel distribution when $\alpha = 0$, a new distribution μ when $\alpha \in (0, \infty)$, and the logarithmic normal distribution when $\alpha = \infty$.

Our analysis of the spectral radius is based on the following result. It is a special case of Theorem 1.2 from [11] which is another version of Theorem 4.7.1 from [26].

Lemma 1.1 (Independence of radius) *Assume that the density function of $(Z_1, \dots, Z_n) \in \mathbb{C}^n$ is proportional to $\prod_{1 \leq j < k \leq n} |z_j - z_k|^2 \cdot \prod_{j=1}^n \varphi(|z_j|)$, where $\varphi(x) \geq 0$ for all*

Plots of density curves for limiting spectral radii

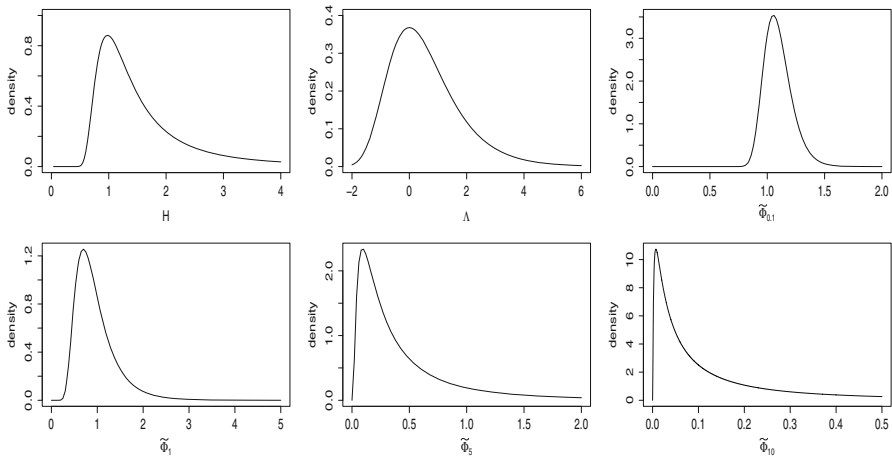


Fig. 1 Distribution $H(x)$ is given in Theorem 1, Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$ is given in Theorems 2 and 3, and distributions $\tilde{\Phi}_\alpha(x) := \Phi_\alpha(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x)$ for $\alpha = 0.1, 1, 5, 10$ are given in Theorem 3

$x \geq 0$. Let Y_1, \dots, Y_n be independent r.v.'s such that the density of Y_j is proportional to $y^{2j-1}\varphi(y)I(y \geq 0)$ for each $1 \leq j \leq n$. Then, $g(|Z_1|, \dots, |Z_n|)$ and $g(Y_1, \dots, Y_n)$ have the same distribution for any symmetric function $g(y_1, \dots, y_n)$.

In this paper, we will study the limiting distributions of $\max_{1 \leq j \leq n} |z_j|$ for three specific φ 's. In addition to Lemma 1.1, [11] also proves that the limiting distribution of $\max_{1 \leq j \leq n} |z_j|$ is the Gumbel distribution under certain assumptions on $\varphi(x)$. However, their results do not apply to our three ensembles because their assumptions are not satisfied. See further elaborations in the section “strategy of the proofs.”

Now, we present our results on the three ensembles in Sects. 1.1, 1.2 and 1.3, respectively. After this, the strategy of the proofs and some comments are given. We compare the limiting curves appearing in this paper in Fig. 1.

1.1 Spherical Ensemble

Let \mathbf{A} and \mathbf{B} be two $n \times n$ matrices, and all of the $2n^2$ entries of the matrices are i.i.d. $\mathcal{CN}(0, 1)$ -distributed random variables. Then, $\mathbf{A}^{-1}\mathbf{B}$ is called a spherical ensemble [26]. It has a connection to the matrix F distribution in the statistics literature; see, for instance, p. 331 from [18]. Let z_1, \dots, z_n be the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$. Then, their joint probability density function is given by

$$C \cdot \prod_{j < k} |z_j - z_k|^2 \cdot \prod_{k=1}^n \frac{1}{(1 + |z_k|^2)^{n+1}} \tag{1.1}$$

where C is a normalizing constant; see, for example, [35]. The joint density of z_1, \dots, z_n of the real analogue of the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$, where \mathbf{A} and \mathbf{B}

are i.i.d. real Ginibre ensembles, is given by Forrester and Mays [21], Forrester and Nagao [22].

The empirical distribution of the eigenvalues has an asymptotic distribution μ with density $\frac{1}{\pi(1+|z|^2)^2}$ [8]. When mapping the eigenvalues on the complex plane to the Riemann sphere through the stereographic projection, the induced (pushforward) measure of μ is the uniform distribution on the sphere. The spectral distribution of the singular values of $\mathbf{A}^{-1}\mathbf{B}$, which is the same as the eigenvalues of the F -matrix $(\mathbf{A}\mathbf{A}^*)^{-1}(\mathbf{B}\mathbf{B}^*)$, converges weakly to a non-random distribution; see, for instance, [5, 52].

In this paper, we say

X_n converges weakly to the cdf $F(x)$ or a random variable X

if the probability distribution of X_n converges weakly to that generated by the cumulative distribution function (cdf) $F(x)$ or X . Now, we study the spectral radius.

Theorem 1 *Let z_1, \dots, z_n have the density as in (1.1). Define $H_k(x) = e^{-x} \sum_{j=0}^{k-1} \frac{x^j}{j!}$ for $k \geq 1$. Then, $\frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} |z_j|$ converges weakly to probability distribution function $H(x) = \prod_{k=1}^{\infty} H_k(x^{-2})$ for $x > 0$ and $H(x) = 0$ for $x \leq 0$.*

Observe that $H_k(x)$ is the cdf $P(\text{Poi}(x) \leq k - 1)$ for each $k \geq 1$, where $\text{Poi}(x)$ is a Poisson random variable with parameter $x > 0$. So, $H(x)$ is the product of those cdfs evaluated at x^{-2} . Moreover, we have

$$1 - H(x) \sim \frac{1}{x^2} \tag{1.2}$$

as $x \rightarrow +\infty$. So, $H(x)$ is heavy-tailed. This property will be verified in Sect. 2.4. It will be very helpful to provide some other descriptions of $H(x)$ in terms of, say, its Taylor expansion or a differential equation.

Johnstone [31] proves that, under a trivial transformation, the largest singular value of the F matrix $(\mathbf{A}\mathbf{A}^*)^{-1}(\mathbf{B}\mathbf{B}^*)$ asymptotically follows a Tracy–Widom distribution. Here, the spectral radius converges weakly to the new distribution $H(x)$.

1.2 Truncation of Circular Unitary Ensemble

Now, we consider the truncation of the circular unitary ensemble. Let \mathbf{U} be an $n \times n$ Haar-invariant unitary matrix (see, e.g., [16, 27, 28]). For $n > p \geq 1$, write

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where \mathbf{A} , as a truncation of \mathbf{U} , is a $p \times p$ submatrix. Let z_1, \dots, z_p be the eigenvalues of \mathbf{A} . It is known from [53] that their density function is

$$C \cdot \prod_{1 \leq j < k \leq p} |z_j - z_k|^2 \prod_{j=1}^p (1 - |z_j|^2)^{n-p-1} \tag{1.3}$$

where C is a normalizing constant. Assuming $c = \lim \frac{p}{n}$, Życzkowski and Sommers [53] show that the empirical distribution of z_i 's converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \leq c$ if $c \in (0, 1)$. Dong et al. [17] prove that the empirical distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively.

Collins [13] proves that $\mathbf{A}^* \mathbf{A}$ forms a Jacobi ensemble. Jiang [27] and Johansson [30] show that a transform of the largest eigenvalue of $\mathbf{A}^* \mathbf{A}$ converges weakly to the Tracy–Widom distribution. For the spectral radius $\max_{1 \leq j \leq p} |z_j|$ of \mathbf{A} itself, we obtain the following result.

Theorem 2 Assume that z_1, \dots, z_p have density as in (1.3) and there exist constants $h_1, h_2 \in (0, 1)$ such that $h_1 < \frac{p}{n} < h_2$ for all $n \geq 2$. Then, $(\max_{1 \leq j \leq p} |z_j| - A_n) / B_n$ converges weakly to the cdf $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where $A_n = c_n + \frac{1}{2}(1 - c_n^2)^{1/2}(n - 1)^{-1/2} a_n$, $B_n = \frac{1}{2}(1 - c_n^2)^{1/2}(n - 1)^{-1/2} b_n$,

$$c_n = \left(\frac{p - 1}{n - 1}\right)^{1/2}, \quad b_n = b\left(\frac{nc_n^2}{1 - c_n^2}\right), \quad a_n = a\left(\frac{nc_n^2}{1 - c_n^2}\right)$$

with

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2} \log(\sqrt{2\pi} \log y) \quad \text{and} \quad b(y) = (\log y)^{-1/2}$$

for $y > 3$.

Trivially, in the above theorem, $\{A_n; n \geq 3\}$ is bounded and B_n has the scale of $(n \log n)^{-1/2}$.

1.3 Product Ensemble

Given integer $k \geq 1$. Assume that $\mathbf{X}_1, \dots, \mathbf{X}_k$ are i.i.d. $n \times n$ random matrices and the n^2 entries of \mathbf{X}_1 are i.i.d. with distribution $\mathbb{C}N(0, 1)$. Let z_1, \dots, z_n be the eigenvalues of the product $\prod_{j=1}^k \mathbf{X}_j$. It is known that their joint density function is

$$C \prod_{1 \leq j < l \leq n} |z_j - z_l|^2 \prod_{j=1}^n w_k(|z_j|) \tag{1.4}$$

where C is a normalizing constant and $w_k(z)$ has a recursive formula given by $w_1(z) = \exp(-|z|^2)$ and

$$w_k(z) = 2\pi \int_0^\infty w_{k-1}\left(\frac{z}{r}\right) \exp(-r^2) \frac{dr}{r}$$

for all integer $k \geq 2$; see, e.g., [3]. The function $w_k(z)$ also has a representation in terms of the so-called Meijer G-function; see the previous reference.

Götze and Tikhomirov [24] prove that the mean of the empirical distribution of $\{z_j/n^{k/2}, 1 \leq j \leq n\}$ converges to a distribution with density $\frac{1}{k\pi}|z|^{\frac{2}{k}-2}$ for $|z| \leq 1$. Later, Bordenave [8], O’Rourke and Soshnikov [38] and O’Rourke et al. [39] further generalize this result to the almost sure convergence. The Gaussian case was first considered by Burda [9] and Burda et al. [10] through investigating the limit of the kernel of a determinantal point process. Second, the empirical distribution of the singular values of $\mathbf{X}_1\mathbf{X}_2/n$ converges weakly to a non-random distribution, see, for instance, Theorem 2.10 from [4].

Now, we consider the largest radius, and the result is given below. We allow k changes with n in this paper. First, we need some notation. Let Φ denote the cumulative distribution function of $N(0, 1)$. For $\alpha \in (0, \infty)$, define

$$\Phi_\alpha(x) = \prod_{j=0}^\infty \Phi\left(x + j\alpha^{1/2}\right),$$

and $\Phi_\infty(x) = \Phi(x)$. The digamma function ψ is defined by

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.5}$$

where $\Gamma(z)$ is the Gamma function.

Theorem 3 *Let $k = k_n$ be a sequence of positive integers. The following holds.*

(a) *If $\lim_{n \rightarrow \infty} k_n/n = 0$, particularly for $k_n \equiv k$, then $\alpha_n(n^{-k_n/2} \max_{1 \leq j \leq n} |z_j| - 1) - \beta_n$ converges weakly to the cdf $\exp(-e^{-x})$, where*

$$\alpha_n = \left(\frac{n}{k_n} \log \frac{n}{k_n}\right)^{1/2} \quad \text{and} \quad \beta_n = \log \frac{n}{k_n} - \log \log \frac{n}{k_n} - \frac{1}{2} \log(2\pi).$$

(b) *If $\lim_{n \rightarrow \infty} k_n/n = \alpha \in (0, \infty)$, then*

$$\frac{\max_{1 \leq j \leq n} |z_j|}{n^{k_n/2}} \text{ converges weakly to the cdf } \Phi_\alpha\left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\right), \quad x > 0.$$

(c) *If $\lim_{n \rightarrow \infty} k_n/n = \infty$, then*

$$\frac{\max_{1 \leq j \leq n} \log |z_j| - k_n \psi(n)/2}{\sqrt{k_n/n}/2} \text{ converges weakly to } N(0, 1).$$

Taking $k = 1$ in (a) of Theorem 3, the corresponding limiting result is obtained by [45]. Here, we get the result not only for finite k , but for all possible range of k_n , which leads to the three transition zones: $k_n/n \rightarrow \alpha$ with $\alpha = 0$, $\alpha \in (0, \infty)$ and $\alpha = \infty$.

It is not known to the authors whether Theorem 3 still holds if the entries of matrices \mathbf{X}_i ’s are not Gaussian. In fact, it is not clear even for the largest radius of \mathbf{X}_1 .

Theorem 3 gives the asymptotic distribution of the largest radius of the product of independent (non-Hermitian) Ginibre ensembles. A natural question is what if the Ginibre ensembles X_j 's are replaced by (Hermitian) Wigner matrices $Y_j := (X_j + X_j^*)/2$? As stated in the Introduction, the largest eigenvalue of each Y_j asymptotically follows the Tracy–Widom law.

Second, let $\eta_{\max}(\mathbf{X}_i)$ be the largest singular value of \mathbf{X}_i for each i . It is proved that $\eta_{\max}(\mathbf{X}_1)/\sqrt{n} \rightarrow 2$ in probability (see, e.g., [4]). Let $\lambda_1(\mathbf{X}_i), \dots, \lambda_n(\mathbf{X}_i)$ be the eigenvalues of \mathbf{X}_i for each i . Obviously, $\max_{1 \leq j \leq n} |\lambda_j(\mathbf{X}_i)| \leq \eta_{\max}(\mathbf{X}_i)$ for each i . From [45] or (a) of Theorem 3, we know $\max_{1 \leq j \leq n} |\lambda_j(\mathbf{X}_i)|/\sqrt{n} \rightarrow 1$ in probability for each i . It is interesting to see that the first limit is 2 and the second is 1. Now, the assertion (b) of Theorem 3 says that, though $\max_{1 \leq j \leq n} |\lambda_j(\mathbf{X}_i)|/\sqrt{n} \rightarrow 1$ for each i , the radius of $\prod_{i=1}^k (\mathbf{X}_i/\sqrt{n})$ goes to a distribution with support $[0, \infty)$.

Finally, let us look at the tail behavior of the distribution in (b) of Theorem 3. In fact, we have

$$1 - \Phi_\alpha\left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log y\right) \sim \frac{C}{x \log x} e^{-2(\log x)^2/\alpha} \tag{1.6}$$

as $x \rightarrow +\infty$, where $C = \frac{\sqrt{\alpha}e^{-\alpha/8}}{2\sqrt{2\pi}}$. It is different from that of $e^{N(0,1)}$, the standard logarithmic normal distribution: $P(e^{N(0,1)} \geq x) \sim \frac{1}{\sqrt{2\pi} \log x} e^{-(\log x)^2/2}$ as $x \rightarrow +\infty$. This will be verified in Sect. 2.4.

Plots of the density curves of the limiting spectral radii To visually compare the limiting distributions of the spectral radii, we plot the density curves in Fig. 1 for the following distributions: distribution $H(x)$ in Theorem 1, Gumbel distribution $\Lambda(x) = \exp(-e^{-x})$ in Theorems 2 and 3 and distributions $\tilde{\Phi}_\alpha(x) := \Phi_\alpha\left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\right)$ in Theorem 3.

Strategy of the proofs By using Lemma 1.1, the absolute values of eigenvalues $|z_j|$ s are ‘‘independent.’’ So, we are dealing with the maxima of independent random variables with different distributions. The first step is to identify these distributions. In fact, we show that the largest radii in Theorems 1, 2 and 3 are essentially the maximum of independent random variables of density $y^{2j-1}(1+y^2)^{-(n+1)}$ for $1 \leq j \leq n$ in (2.2), the maximum of independent random variables with beta distributions in (2.18), and the maximum of products of i.i.d. Gamma-distributed random variables in Lemma 2.4, respectively. The next step is the analysis of their tail probabilities through moderate deviations. This procedure costs the major efforts.

Assuming that the function $\varphi(t)$ in Lemma 1.1 has the form of $e^{-nV(t)}$ ($V(t)$ not depending on n) with $V(t) = t^\alpha$ for some $\alpha \geq 1$ or $V(t)$ being a special type of convex functions, [11] show that the limiting distribution of $\max_{1 \leq j \leq n} |z_j|$ is always the Gumbel distribution. In our Theorems 1, 2 and 3, $V(t)$ either depends on n or is not the type of special convex functions required in their theorems. In particular, we see different limiting distributions in Theorems 1 and 3.

Comments

1. It is noteworthy to mention that, though the main idea is analyzing the maxima of independent random variables, the proofs are not trivial. In the classical study of the maxima of i.i.d. random variables, the limiting distributions are only of three types: Fréchet distribution, Gumbel distribution and Weibull distribution; (see, for example, [44]). However, the limiting distributions appeared in Theorems 1 and (b) of Theorem 3 are new.
2. The eigenvalues of the three random matrices investigated in this paper are rotation-invariant. This special property gives us the advantage of independence by Lemma 1.1. When the eigenvalues are not of the invariant property, it seems there have no good understanding on the largest radii. For example, there is no invariance property for z_1, \dots, z_n if they have the joint density

$$f(z_1, \dots, z_n) = C \cdot \prod_{1 \leq j < k \leq n} |z_j - z_k|^2 \cdot \exp \left\{ -n \sum_{j=1}^n \left(\frac{(\operatorname{Re} z_j)^2}{1 + \tau} + \frac{(\operatorname{Im} z_j)^2}{1 - \tau} \right) \right\}$$

where $\tau \in (-1, 1)$ is a parameter and C is a normalizing constant (Lemma 4 from [41]). See also similar examples on page 3403 from [45] and (1.1) from [36].

3. In this paper, we work on matrices with complex Gaussian entries. A similar study may be done for matrices with real and symplectic Gaussian random variables. For example, we know from [19,23,37] that the densities of the eigenvalues of real and symplectic Ginibre ensembles are also explicit. Rider [45] and Rider and Sinclair [47] obtain the limiting distributions of the largest radii for the real and symplectic cases. It is possible that our current work can be carried out to the three real and symplectic analogues: the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$, where \mathbf{A} and \mathbf{B} are real or symplectic Ginibre ensembles (further information can be seen from [22] and [21]), the truncation of Haar-invariant orthogonal or symplectic matrices (see, e.g., [28]) and $\prod_{j=1}^k \mathbf{X}_j$, where $\mathbf{X}_1, \dots, \mathbf{X}_k$ are i.i.d. real or symplectic Ginibre ensembles.
4. Tracy and Widom [49,50] prove that the largest eigenvalues of the Gaussian orthogonal, unitary and symplectic ensembles converge to the Tracy–Widom laws. Recently, there has been an active research on the universality of the eigenvalues of non-Gaussian matrices; (see, for example, [20,48] and the references therein). In particular, Erdős et al. generalize the results by Tracy–Widom to the matrices with non-Gaussian entries. Our Theorems 1, 2 and 3 consider the eigenvalues of matrices with Gaussian entries. It will be interesting to study the universality of the three results for the matrices with non-Gaussian entries.

Finally, the organization of the rest of paper is as follows. We will prove Theorems 1, 2 and 3 in Sects. 2.1, 2.2 and 2.3, respectively. The verifications of (1.2) and (1.6) are given in Sect. 2.4.

2 Proofs

In this section, we will prove Theorems 1, 2 and 3 in each subsection.

2.1 The Proof of Theorem 1

We start with a lemma.

Lemma 2.1 *Let $a_{ni} \in [0, 1)$ be constants for $i \geq 1, n \geq 1$ and $\sup_{n \geq 1, i \geq 1} a_{ni} < 1$. For each $i \geq 1, a_i := \lim_{n \rightarrow \infty} a_{ni}$. Assume $c_n := \sum_{i=1}^{\infty} a_{ni} < \infty$ for each $n \geq 1$ and $c := \sum_{i=1}^{\infty} a_i < \infty$, and $\lim_{n \rightarrow \infty} c_n = c$. Then,*

$$\lim_{n \rightarrow \infty} \prod_{i=1}^{\infty} (1 - a_{ni}) = \prod_{i=1}^{\infty} (1 - a_i).$$

Proof Note that $\prod_{i=1}^{\infty} (1 - a_{ni})$ and $\prod_{i=1}^{\infty} (1 - a_i)$ are well defined, and $\prod_{i=1}^{\infty} (1 - a_{ni}) > 0$ for each $n \geq 1$ and $\prod_{i=1}^{\infty} (1 - a_i) > 0$. It suffices to show that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \log(1 - a_{ni}) = \sum_{i=1}^{\infty} \log(1 - a_i). \tag{2.1}$$

Note that for each fixed $k > 1$,

$$\begin{aligned} \limsup_{n \rightarrow \infty} \sum_{i=1}^{\infty} |a_{ni} - a_i| &\leq \limsup_{n \rightarrow \infty} \sum_{i=1}^k |a_{ni} - a_i| + \limsup_{n \rightarrow \infty} \sum_{i=k+1}^{\infty} |a_{ni} - a_i| \\ &= \limsup_{n \rightarrow \infty} \sum_{i=k+1}^{\infty} |a_{ni} - a_i| \\ &\leq \limsup_{n \rightarrow \infty} \sum_{i=k+1}^{\infty} a_{ni} + \sum_{i=k+1}^{\infty} a_i \\ &= \limsup_{n \rightarrow \infty} \left(c_n - \sum_{i=1}^k a_{ni} \right) + c - \sum_{i=1}^k a_i \\ &= 2 \left(c - \sum_{i=1}^k a_i \right), \end{aligned}$$

which goes to zero as $k \rightarrow \infty$. Therefore, we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} |a_{ni} - a_i| = 0.$$

Set $a = \sup_{n \geq 1, i \geq 1} a_{ni}$. Then, $0 \leq a < 1$. It follows that

$$\begin{aligned} \left| \sum_{i=1}^{\infty} \log(1 - a_{ni}) - \sum_{i=1}^{\infty} \log(1 - a_i) \right| &\leq \sum_{i=1}^{\infty} |\log(1 - a_{ni}) - \log(1 - a_i)| \\ &= \sum_{i=1}^{\infty} \left| \int_{a_i}^{a_{ni}} \frac{1}{1-t} dt \right| \\ &\leq \frac{1}{1-a} \sum_{i=1}^{\infty} |a_{ni} - a_i| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, proving (2.1). □

Proof of Theorem 1 By Lemma 1.1 and (1.1), $\max_{1 \leq j \leq n} |z_j|$ and $\max_{1 \leq j \leq n} Y_{nj}$ have the same distribution, where Y_{n1}, \dots, Y_{nm} are independent such that Y_{nj} has the probability density function (pdf) proportional to $y^{2j-1}(1 + y^2)^{-(n+1)} I(y \geq 0)$ for $1 \leq j \leq n$. Thus, to prove the theorem, it suffices to show

$$\lim_{n \rightarrow \infty} P\left(\frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} Y_{nj} \leq x\right) = H(x) \tag{2.2}$$

for each $x > 0$.

Let $X_i, i \geq 1$ be a sequence of i.i.d. random variables with cumulative distribution function (cdf) F . Let $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ be the order statistics of X_1, X_2, \dots, X_n for each $n \geq 1$. Then, from page 14 on the book by Balakrishnan and Cohen [7], we know that the cdf of $X_{i:n}$ is given by

$$F_{i:n}(x) = \sum_{r=i}^n \binom{n}{r} F(x)^r (1 - F(x))^{n-r} = \frac{n!}{(i-1)!(n-i)!} \int_0^{F(x)} t^{i-1} (1-t)^{n-i} dt \tag{2.3}$$

for each $1 \leq i \leq n$. If F has a probability density function f , then the pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} F(x)^{i-1} (1 - F(x))^{n-i} f(x). \tag{2.4}$$

The monotonicity of the order statistics implies that $F_{i:n}(x)$ is non-increasing in i for each x , that is,

$$F_{1:n}(x) \geq F_{2:n}(x) \geq \dots \geq F_{n:n}(x). \tag{2.5}$$

Let $\{u_n, n \geq 1\}$ be a sequence of constants such that $\lim_{n \rightarrow \infty} n(1 - F(u_n)) =: \tau \in (0, \infty)$. Write $\tau_n = n(1 - F(u_n))$. Then, it follows from the first equality in equation (2.3) that

$$\begin{aligned}
 F_{n-i+1:n}(u_n) &= \sum_{r=n-i+1}^n \binom{n}{r} F(u_n)^r (1 - F(u_n))^{n-r} \\
 &= \sum_{j=0}^{i-1} \binom{n}{j} \left(1 - \frac{\tau_n}{n}\right)^{n-j} \left(\frac{\tau_n}{n}\right)^j \\
 &= \sum_{j=0}^{i-1} \frac{1}{j!} \cdot \prod_{l=1}^{j-1} \left(1 - \frac{l}{n}\right) \cdot \left(1 - \frac{\tau_n}{n}\right)^{n-j} (\tau_n)^j \\
 &\rightarrow e^{-\tau} \sum_{j=0}^{i-1} \frac{\tau^j}{j!} = H_i(\tau)
 \end{aligned} \tag{2.6}$$

as $n \rightarrow \infty$ for each fixed integer $i \geq 1$.

Now, we take $F(y) = \frac{y^2}{1+y^2}$ for $y > 0$. Fix $x > 0$, set $u_n = u_n(x) = \sqrt{nx}$. Then, $\lim_{n \rightarrow \infty} n(1 - F(u_n)) = x^{-2}$. Then, from (2.6)

$$\lim_{n \rightarrow \infty} F_{n-i+1:n}(u_n) = H_i(x^{-2}) \tag{2.7}$$

for each fixed integer $i \geq 1$. For each $n \geq 1$, define

$$a_{ni} = \begin{cases} 1 - F_{n-i+1:n}(u_n), & \text{if } 1 \leq i \leq n; \\ 0, & \text{if } i > n. \end{cases}$$

Then, it follows from (2.5) that $\sup_{n \geq 1, i \geq 1} a_{ni} = \sup_{n \geq 1} a_{n1}$. By the first identity in (2.3),

$$a_{n1} = 1 - \left(\frac{nx^2}{1 + nx^2}\right)^n = 1 - \left(1 + \frac{x^{-2}}{n}\right)^{-n}$$

is increasing in n . Hence,

$$\sup_{n \geq 1, i \geq 1} a_{ni} = \lim_{n \rightarrow \infty} a_{n1} = 1 - \exp(-x^{-2}) \in (0, 1).$$

From (2.7), we have $\lim_{n \rightarrow \infty} a_{ni} = 1 - H_i(x^{-2}) =: a_i$ for each $i \geq 1$. Moreover, we have

$$\begin{aligned}
 \sum_{i=1}^{\infty} a_{ni} &= E \left[\sum_{i=1}^n I(X_{n-i+1:n} > u_n) \right] \\
 &= E \left[\sum_{i=1}^n I(X_i > u_n) \right] \\
 &= n(1 - F(u_n)) \rightarrow x^{-2}
 \end{aligned}$$

and

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} (1 - H_i(x^{-2})) = \exp(-x^{-2}) \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \frac{(x^{-2})^k}{k!}.$$

By exchanging the ordering of the sums, we know $\sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \frac{(x^{-2})^k}{k!} = \sum_{k=1}^{\infty} \sum_{i=1}^k \frac{(x^{-2})^k}{k!} = \sum_{k=1}^{\infty} \frac{(x^{-2})^k}{(k-1)!} = x^{-2} \exp\{x^{-2}\}$. It follows that $\sum_{i=1}^{\infty} a_i = x^{-2}$. By Lemma 2.1, we have

$$\lim_{n \rightarrow \infty} \prod_{i=1}^n F_{i:n}(\sqrt{nx}) = \lim_{n \rightarrow \infty} \prod_{i=1}^{\infty} (1 - a_{ni}) = \prod_{i=1}^{\infty} H_i(x^{-2}) = H(x) \quad \text{for } x > 0.$$

From (2.4), we obtain the pdf of $X_{j:n}$ given by

$$f_{j:n}(y) = \frac{2n!}{(j-1)!(n-j)!} y^{2j-1} (1+y^2)^{-(n+1)}, \quad y > 0,$$

which is also the pdf of Y_{nj} . Therefore, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} P\left(\frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} Y_{nj} \leq x\right) &= \lim_{n \rightarrow \infty} \prod_{j=1}^n P(Y_{nj} \leq \sqrt{nx}) \\ &= \lim_{n \rightarrow \infty} \prod_{j=1}^n F_{j:n}(\sqrt{nx}) = H(x) \end{aligned}$$

for $x > 0$. This completes the proof of Theorem 1. □

2.2 The Proof of Theorem 2

In the rest of the paper, we will need the following notation. The symbol $C_n \sim D_n$ as $n \rightarrow \infty$ means that $\lim_{n \rightarrow \infty} \frac{C_n}{D_n} = 1$. Similarly, $C_n(t) \sim D_n(t)$ uniformly over $t \in T_n$ if $\lim_{n \rightarrow \infty} \sup_{t \in T_n} | \frac{C_n(t)}{D_n(t)} - 1 | = 0$. Also, $C_n(t) = O(D_n(t))$ uniformly over $t \in T_n$ if $\sup_{t \in T_n} | \frac{C_n(t)}{D_n(t)} |$ is bounded. We write $C_n(t) = o(D_n(t))$ uniformly over $t \in T_n$ if $\sup_{t \in T_n} | \frac{C_n(t)}{D_n(t)} |$ converges to zero as $n \rightarrow \infty$.

For random variables $\{X_n; n \geq 1\}$ and constants $\{a_n; n \geq 1\}$, we write $X_n = O_P(a_n)$ if $\lim_{x \rightarrow +\infty} \lim_{n \rightarrow \infty} P(|\frac{X_n}{a_n}| \geq x) = 0$. In particular, if $X_n = O_P(a_n)$ and $\{b_n; n \geq 1\}$ is a sequence of constants with $\lim_{n \rightarrow \infty} b_n = \infty$, then $\frac{X_n}{a_n b_n} \rightarrow 0$ in probability as $n \rightarrow \infty$.

Recall $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt$ for $x \in \mathbb{R}$. Let also $a(x)$ and $b(x)$ be as in Theorem 2.

Lemma 2.2 Let $\{j_n, n \geq 1\}$ and $\{x_n, n \geq 1\}$ be positive numbers with $\lim_{n \rightarrow \infty} x_n = \infty$ and $\lim_{n \rightarrow \infty} j_n x_n^{-1/2} (\log x_n)^{1/2} = \infty$. For fixed $y \in \mathbb{R}$, if $\{c_{n,j}, 1 \leq j \leq j_n, n \geq 1\}$ are real numbers such that $\lim_{n \rightarrow \infty} \max_{1 \leq j \leq j_n} |c_{n,j} x_n^{1/2} - 1| = 0$, then

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{j_n} (1 - \Phi((j - 1)c_{n,j} + a(x_n) + b(x_n)y)) = e^{-y}; \tag{2.8}$$

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{j_n} \frac{1}{(j - 1)c_{n,j} + a(x_n) + b(x_n)y} \phi((j - 1)c_{n,j} + a(x_n) + b(x_n)y) = e^{-y}. \tag{2.9}$$

Proof From the definition, it is easy to see that $\lim_{n \rightarrow \infty} a(x_n) = +\infty$ and $\lim_{n \rightarrow \infty} b(x_n) = 0$ and $\min_{1 \leq j \leq j_n} c_{n,j} > 0$ as n is large enough. Thus, $\min_{1 \leq j \leq j_n} [(j - 1)c_{n,j} + a(x_n) + b(x_n)y] \rightarrow +\infty$ as $n \rightarrow \infty$. By using the integration by parts, we see that $1 - \Phi(x) \sim \frac{\phi(x)}{x} (1 + O(x^{-2}))$ as $x \rightarrow \infty$. Therefore, (2.8) follows from (2.9). Now, let us prove (2.9).

It is easy to verify that for $y \in \mathbb{R}$,

$$\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right) \sim \frac{\sqrt{2\pi} \log x_n}{\sqrt{x_n}} e^{-y} \tag{2.10}$$

as $n \rightarrow \infty$. For large n , define

$$l_n = \text{the integer part of } \min \left\{ \frac{j_n^{1/2} x_n^{1/4}}{(\log x_n)^{1/4}}, \frac{x_n^{1/2}}{(\log x_n)^{1/4}} \right\}.$$

Then, as $n \rightarrow \infty$,

$$j_n > l_n \rightarrow \infty, \quad l_n x_n^{-1/2} (\log x_n)^{1/2} \rightarrow \infty \quad \text{and} \quad l_n x_n^{-1/2} \rightarrow 0. \tag{2.11}$$

Fix $y \in \mathbb{R}$, and set

$$u_{nj} = (j - 1)c_{n,j} + a(x_n) + b(x_n)y, \quad 1 \leq j \leq j_n.$$

Then, we conclude the following facts:

Fact 1: Uniformly over $1 \leq j \leq l_n$,

$$u_{nj} = (\log x_n)^{1/2} (1 + o(1)) \tag{2.12}$$

by using the third assertion in (2.11) and

$$u_{nj}^2 = (a(x_n) + b(x_n)y)^2 + 2(j - 1) \frac{(\log x_n)^{1/2}}{x_n^{1/2}} (1 + o(1)) + o(1); \tag{2.13}$$

Fact 2: Uniformly over $l_n < j \leq j_n$, which is different from the assumption on (2.12) and (2.13),

$$u_{nj} \geq (\log x_n)^{1/2}(1 + o(1)) \tag{2.14}$$

and

$$u_{nj}^2 \geq (a(x_n) + b(x_n)y)^2 + 2(j - 1) \frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1)) + o(1). \tag{2.15}$$

It then follows from (2.12), (2.13) and (2.10) that

$$\begin{aligned} \sum_{j=1}^{l_n} \frac{1}{u_{nj}} \phi(u_{nj}) &\sim \frac{\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right)}{(2\pi \log x_n)^{1/2}} \\ &\quad \sum_{j=1}^{l_n} \exp\left(- (j - 1) \frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right) \\ &\sim \frac{\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right)}{(2\pi \log x_n)^{1/2}} \frac{1}{1 - \exp\left(-\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right)} \end{aligned} \tag{2.16}$$

$$\sim \frac{\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right)}{(2\pi \log x_n)^{1/2}} \frac{x_n^{1/2}}{(\log x_n)^{1/2}} \sim e^{-y}, \tag{2.17}$$

where the middle limit in (2.11) is used in the second step. Similarly, it follows from (2.14), (2.15) and (2.10) that

$$\begin{aligned} &\sum_{j=l_n+1}^{j_n} \frac{1}{u_{nj}} \phi(u_{nj}) \\ &\leq \frac{\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right)}{(2\pi \log x_n)^{1/2}} \sum_{j=l_n+1}^{\infty} \exp\left(- (j - 1) \frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right) \\ &\leq \frac{\exp\left(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\right)}{(2\pi \log x_n)^{1/2}} \frac{1}{1 - \exp\left(-\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right)} \\ &\quad \times \exp\left(-l_n \frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right) \\ &= O\left(\exp\left(-l_n \frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\right)\right) \rightarrow 0 \end{aligned}$$

by using (2.16) and (2.17) in the equality and the middle assertion in (2.11) in the last step. By adding up the above equation and (2.17), we obtain (2.9). □

Let $\{U_i; i \geq 1\}$ be a sequence of i.i.d. random variables uniformly distributed over $(0, 1)$, and let $U_{1:n} \leq U_{2:n} \leq \dots \leq U_{n:n}$ be the order statistics of U_1, U_2, \dots, U_n for

each $n \geq 1$. Recall \mathcal{B} is the collection of all Borel sets on \mathbb{R} . The following lemma is a special case of Proposition 2.10 from [43].

Lemma 2.3 *There exists a constant $C > 0$ such that for all $r > k \geq 1$,*

$$\sup_{B \in \mathcal{B}} \left| P \left(\frac{r^{3/2}}{\sqrt{(r-k)k}} \left(U_{r-k+1:r} - \frac{r-k}{r} \right) \in B \right) - \int_B (1 + l_1(t) + l_2(t)) \phi(t) dt \right| \leq C \cdot \left(\frac{r}{(r-k)k} \right)^{3/2}$$

where for $i = 1, 2$, $l_i(t)$ is a polynomial in t of degree $\leq 3i$, depending on r and k , and all of its coefficients are of order $O\left(\left(\frac{r}{(r-k)k}\right)^{i/2}\right)$.

Proof of Theorem 2 Review the density formula in (1.3). Set $m_n = n - p$. For ease of notation, we sometimes write m for m_n . By assumption, $h'_2 < \frac{m_n}{n} < h'_1$ for all $n \geq 2$, where $h'_i = 1 - h_i \in (0, 1)$ for $i = 1, 2$. Then, we need to prove $(\max_{1 \leq j \leq p} |z_j| - A_n)/B_n$ converges weakly to the cdf $\exp(-e^{-x})$, where $A_n = c_n + \frac{1}{2}(1 - c_n^2)^{1/2}(n - 1)^{-1/2}a_n$, $B_n = \frac{1}{2}(1 - c_n^2)^{1/2}(n - 1)^{-1/2}b_n$,

$$c_n = \left(\frac{p-1}{n-1} \right)^{1/2}, \quad b_n = b \left(\frac{nc_n^2}{1 - c_n^2} \right), \quad a_n = a \left(\frac{nc_n^2}{1 - c_n^2} \right)$$

with

$$a(x) = (\log x)^{1/2} - (\log x)^{-1/2} \log(\sqrt{2\pi} \log x) \quad \text{and} \quad b(x) = (\log x)^{-1/2}$$

for $x > 3$. We proceed this through several steps.

Step 1 Reducing the problem to the maximum of independent random variables with beta distributions. Let $U_i, i \geq 1$, be a sequence of i.i.d. random variables uniformly distributed over $(0, 1)$ and $U_{1:n} \leq U_{2:n} \leq \dots \leq U_{n:n}$ be the order statistics of U_1, U_2, \dots, U_n for each $n \geq 1$. From (2.4), the density function of $U_{j:m_n+j-1}$ is

$$f_{j:m_n+j-1}(x) = \frac{(m_n + j - 1)!}{(j - 1)!(m_n - 1)!} x^{j-1} (1 - x)^{m_n-1}, \quad x \in (0, 1).$$

Denote the corresponding cdf as $F_{j:m_n+j-1}(x)$. Notice that the pdf of $(U_{j:m_n+j-1})^{1/2}$ is proportional to $x^{2j-1}(1 - x^2)^{m_n-1}$. For each $n \geq 2$, let $\{Y_{nj}; 1 \leq j \leq p\}$ be independent random variables such that Y_{nj} and $(U_{j:m_n+j-1})^{1/2}$ have the same distribution. By Lemma 1.1 and (1.1), $\max_{1 \leq j \leq p} |z_j|$ and $\max_{1 \leq j \leq p} Y_{nj}$ have the same distribution. We claim that, to prove the theorem, it suffices to show

$$\lim_{n \rightarrow \infty} P \left(\max_{1 \leq j \leq p} Y_{nj}^2 \leq \beta_n(x) \right) = \exp(-e^{-x}) \tag{2.18}$$

for every $x \in \mathbb{R}$, where $\beta_n(x) = c_n^2 + c_n(1 - c_n^2)^{1/2}(n - 1)^{-1/2}(a_n + b_n x)$. In fact, (2.18) implies that

$$W_n := \frac{1}{b_n} \left(c_n^{-1} (1 - c_n^2)^{-1/2} (n - 1)^{1/2} \left(\max_{1 \leq j \leq p} Y_{nj}^2 - c_n^2 \right) - a_n \right) \xrightarrow{d} \Lambda, \tag{2.19}$$

where Λ is a probability distribution with cdf $\exp(-e^{-x})$, $x \in \mathbb{R}$. From Taylor’s expansion

$$\begin{aligned} \max_{1 \leq j \leq p} Y_{nj} &= \left(c_n^2 + c_n(1 - c_n^2)^{1/2}(n - 1)^{-1/2}(a_n + b_n W_n) \right)^{1/2} \\ &= c_n \left(1 + c_n^{-1} (1 - c_n^2)^{1/2} \frac{a_n + b_n W_n}{(n - 1)^{1/2}} \right)^{1/2} \\ &= c_n \left(1 + \frac{1}{2} c_n^{-1} (1 - c_n^2)^{1/2} \frac{a_n + b_n W_n}{(n - 1)^{1/2}} + O_P \left(\frac{a_n^2}{n - 1} \right) \right) \\ &= c_n + \frac{1}{2} (1 - c_n^2)^{1/2} (n - 1)^{-1/2} a_n + \frac{1}{2} (1 - c_n^2)^{1/2} (n - 1)^{-1/2} b_n W_n \\ &\quad + O_P \left(\frac{\log n}{n} \right) \\ &= A_n + B_n W_n + O_p \left(\frac{\log n}{n} \right), \end{aligned}$$

where we use the facts $a_n \rightarrow \infty$, $b_n \rightarrow 0$ and $c_n \in (0, 1)$ in the above. Since B_n has the scale of $(n \log n)^{-1/2}$, by (2.19),

$$\frac{\max_{1 \leq j \leq p} Y_{nj} - A_n}{B_n} = W_n + O_p \left(\frac{(\log n)^{3/2}}{n^{1/2}} \right) \rightarrow \Lambda$$

weakly, which leads to the desired conclusion. Now, we proceed to show (2.18).

Step 2 A preparation. We claim that

$$1 - F_{1:m_n}(x) \leq 1 - F_{2:m_n+1}(x) \leq \dots \leq 1 - F_{p:m_n+p-1}(x) \tag{2.20}$$

for $x \in (0, 1)$. In fact, since for each $1 < j \leq p$,

$$U_{j:m_n+j-1} = \begin{cases} U_{j-1:m_n+j-2}, & \text{if } U_{m_n+j-1} \leq U_{j-1:m_n+j-2}; \\ \min(U_{j:m_n+j-2}, U_{m_n+j-1}), & \text{if } U_{m_n+j-1} > U_{j-1:m_n+j-2}, \end{cases}$$

which implies that $U_{j-1:m_n+j-2} \leq U_{j:m_n+j-1}$ for $1 < j \leq p$. This yields (2.20).

For each $n \geq 2$, set $a_{nj} = 1 - F_{p+1-j:m_n+p-j}(\beta_n(x)) = 1 - F_{p+1-j:n-j}(\beta_n(x))$ for $1 \leq j \leq p$. From (2.20), for each n , a_{ni} is non-increasing in i . Since Y_{nj}^2 and $U_{j:m_n+j-1}$ are identically distributed, we have

$$P\left(\max_{1 \leq j \leq p} Y_{nj}^2 \leq \beta_n(x)\right) = \prod_{j=1}^p P(Y_{nj}^2 \leq \beta_n(x)) = \prod_{j=1}^p (1 - a_{nj}). \tag{2.21}$$

It is easy to check that the following holds: Suppose $\{l_n; n \geq 1\}$ is a sequence of positive integers. Let $z_{ni} \in [0, 1)$ be constants for all $1 \leq i \leq l_n$ with $\max_{1 \leq i \leq l_n} z_{ni} \rightarrow 0$ and $\sum_{i=1}^{l_n} z_{ni} \rightarrow z \in [0, \infty)$. Then,

$$\lim_{n \rightarrow \infty} \prod_{i=1}^{l_n} (1 - z_{ni}) = e^{-z}. \tag{2.22}$$

Next, we will use (2.21) and (2.22) to prove (2.18). In fact, we only need to verify that

$$\sum_{j=1}^p a_{nj} \rightarrow e^{-x} \tag{2.23}$$

and

$$\max_{1 \leq j \leq p} a_{nj} = a_{n1} \rightarrow 0.$$

Step 3 The analysis of dominated terms in the maximum from (2.18). Fix $\delta \in (\frac{1}{2}, \frac{2}{3})$. Let $j_n = [n^\delta]$, the integer part of n^δ . For $1 \leq j \leq j_n$, define

$$u_{nj} = \frac{(n - j)^{3/2}}{((p - j)m_n)^{1/2}} \left(\beta_n(x) - \frac{p - j}{n - j} \right).$$

Meanwhile, we rewrite

$$\beta_n(x) = \frac{p - 1}{n - 1} + \frac{((p - 1)m_n)^{1/2}}{(n - 1)^{3/2}} (a_n + b_n x).$$

Then, we see that uniformly over $1 \leq j \leq j_n$,

$$\begin{aligned} u_{nj} &= \left(\frac{p - 1}{n - 1} - \frac{p - j}{n - j} \right) \cdot \frac{(n - j)^{3/2}}{((p - j)m_n)^{1/2}} \\ &\quad + \left(\frac{n - j}{n - 1} \right)^{3/2} \cdot \left(\frac{p - 1}{p - j} \right)^{1/2} (a_n + b_n x) \\ &= \left(\frac{p - j}{p - 1} \right)^{-1/2} \cdot \left(\frac{n - j}{n - 1} \right)^{1/2} \cdot \left(\frac{n - p}{p - 1} \right)^{1/2} \cdot \left(\frac{n - 1}{n} \right)^{-1/2} \cdot \frac{j - 1}{n^{1/2}} \\ &\quad + \left(\frac{n - j}{n - 1} \right)^{3/2} \cdot \left(\frac{p - j}{p - 1} \right)^{-1/2} (a_n + b_n x). \end{aligned}$$

Now, $\frac{n-p}{p-1} = \frac{1-c_n^2}{c_n^2}$. Also, given $\tau \in \mathbb{R}$, trivially $\left(\frac{p-j}{p-1}\right)^\tau = 1 + O\left(\frac{j}{n}\right)$ and $\left(\frac{n-j}{n-1}\right)^\tau = 1 + O\left(\frac{j}{n}\right)$ uniformly for all $1 \leq j \leq j_n$. Since $a_n \sim (\log n)^{1/2}$ and $b_n = o(1)$, we have

$$\begin{aligned} u_{nj} &= \frac{(1 - c_n^2)^{1/2}}{n^{1/2}c_n} (j - 1)(1 + o(1)) + a_n + b_n x + O\left(\frac{j \log n}{n}\right) \\ &= \frac{(1 - c_n^2)^{1/2}}{n^{1/2}c_n} (j - 1)(1 + o(1)) + a_n + b_n x \end{aligned} \tag{2.24}$$

uniformly for all $1 \leq j \leq j_n$.

In Lemma 2.3, take $r = n - j$ and $k = n - p$ to have

$$\sup_{B \in \mathcal{B}} \left| P(V_{p-j+1:n-j} \in B) - \int_B (1 + l_1(t) + l_2(t))\phi(t)dt \right| = O(n^{-3/2})$$

uniformly over $1 \leq j \leq j_n$ as $n \rightarrow \infty$, where

$$V_{p-j+1:n-j} = \frac{(n - j)^{3/2}}{((n - j)(n - p))^{1/2}} \left(U_{p-j+1:n-j} - \frac{p - j}{n - j} \right)$$

and where, for $i = 1, 2$, $l_i(t)$ is a polynomial in t of degree $\leq 3i$, depending on n , and all of its coefficients are of order $O(n^{-i/2})$ by the assumption $h_1 < \frac{p}{n} < h_2$ for all $n \geq 2$. Now, by taking $B = (u_{nj}, \infty)$, we obtain

$$a_{nj} = P(V_{p-j+1:n-j} > u_{nj}) = \int_{u_{nj}}^\infty (1 + l_1(t) + l_2(t))\phi(t)dt + O(n^{-3/2})$$

uniformly for $1 \leq j \leq j_n$ as $n \rightarrow \infty$. From L'Hospital's rule, we have that for any $r \geq 0$,

$$\int_x^\infty t^r \phi(t)dt \sim x^{r-1} \phi(x) \quad \text{as } x \rightarrow \infty. \tag{2.25}$$

Since $\min_{1 \leq j \leq j_n} u_{nj} \rightarrow \infty$ as $n \rightarrow \infty$ by (2.24), it follows from (2.25) that

$$\int_{u_{nj}}^\infty t^r \phi(t)dt \sim (u_{nj})^{r-1} \phi(u_{nj}) = \begin{cases} \frac{\phi(u_{nj})}{u_{nj}}, & \text{if } r = 0; \\ O((\max_{1 \leq j \leq j_n} u_{nj})^r) \frac{\phi(u_{nj})}{u_{nj}}, & \text{if } r > 0 \end{cases}$$

holds uniformly over $1 \leq j \leq j_n$. Furthermore, since the coefficients of $l_i(t)$ are uniformly bounded by $O(n^{-i/2})$ for $i = 1, 2$, we have

$$\begin{aligned} & \int_{u_{nj}}^{\infty} (1 + l_1(t) + l_2(t))\phi(t)dt \\ &= \left[1 + O\left(\frac{(\max_{1 \leq j \leq j_n} u_{nj})^3}{n^{1/2}}\right) + O\left(\frac{(\max_{1 \leq j \leq j_n} u_{nj})^6}{n}\right) \right] \frac{\phi(u_{nj})}{u_{nj}} \\ &= \left(1 + O\left(\frac{j_n^3}{n^2}\right)\right) \frac{\phi(u_{nj})}{u_{nj}} \end{aligned}$$

uniformly over $1 \leq j \leq j_n$ and thus obtain that

$$a_{nj} = \left(1 + O\left(\frac{j_n^3}{n^2}\right)\right) \frac{\phi(u_{nj})}{u_{nj}} + O(n^{-3/2}) \tag{2.26}$$

uniformly over $1 \leq j \leq j_n$. Therefore, we have

$$\sum_{j=1}^{j_n} a_{nj} = (1 + o(1)) \sum_{j=1}^{j_n} \frac{\phi(u_{nj})}{u_{nj}} + o(1).$$

In Lemma 2.2, by taking $x_n = nc_n^2/(1 - c_n^2)$ and $c_{nj} = x_n^{-1/2}(1 + o(1))$, where “ $o(1)$ ” is as indicated in (2.24), we then get

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{j_n} \frac{\phi(u_{nj})}{u_{nj}} = e^{-x}. \tag{2.27}$$

Step 4 Some terms in the maximum from (2.18) are negligible. From (2.24) again, we see

$$u_{nj_n}^2 \geq 2c_n^{-1}(1 - c_n^2)^{1/2}(a_n + b_n x) \frac{n^\delta - 2}{n^{1/2}}(1 + o(1)) \geq 6 \log n$$

for all large n . Hence,

$$\phi(u_{nj_n}) \leq \frac{1}{\sqrt{2\pi}} \exp(-3 \log n) = \frac{1}{\sqrt{2\pi} n^3}$$

for all large n . Then, it follows from (2.26) that $a_{nj_n} = O(n^{-3/2})$, and hence

$$\sum_{j=j_n+1}^p a_{nj} \leq (n - j_n)a_{nj_n} = O(n^{-1/2}).$$

This together with (2.27) yields (2.23). The proof is then completed. □

2.3 The Proof of Theorem 3

We begin with some preparation. The notation $k = k_n$ appears in the statement of Theorem 3. In the proof next we will write “ k_n ” if there is a danger of confusion, for example, a limit is taken for “ $n \rightarrow \infty$.” We write “ k ” in other occasions for clarity of formulas.

The following result characterizes the structure of the radius of the eigenvalues from the product ensemble.

Lemma 2.4 *Let k and z_1, \dots, z_n be as in (1.4). Let $\{s_{j,r}, 1 \leq r \leq k, j \geq 1\}$ be independent random variables and $s_{j,r}$ have the Gamma density $y^{j-1}e^{-y}I(y > 0)/(j-1)!$ for each j and r . Then, $\max_{1 \leq j \leq n} |z_j|^2$ and $\max_{1 \leq j \leq n} \prod_{r=1}^k s_{j,r}$ have the same distribution.*

Proof Let $\{s_{j,r}; 1 \leq r \leq k, j \geq 1\}$ be independent random variables and $s_{j,r}$ follow a Gamma(j) distribution with density function $y^{j-1}e^{-y}I(y \geq 0)/\Gamma(j)$ for all $1 \leq r \leq k$ and $j \geq 1$. Define $v_1(y) = \exp(-y), y > 0$, and set for $j \geq 2$,

$$v_j(y) = \int_0^\infty v_{j-1}(y/s) \frac{e^{-s}}{s} ds. \tag{2.28}$$

One can easily verify that for each $j \geq 1, v_j(y)$ is proportional to $w_j(y^{1/2})$, i.e., for some constants $d_j > 0$,

$$w_j(y^{1/2}) = d_j v_j(y), \quad y > 0. \tag{2.29}$$

Let z be any complex number with $Re(z) > 0$, and define for $j \geq 1$,

$$\gamma_j(z) = \int_0^\infty y^{z-1} v_j(y) dy.$$

Note that $\gamma_1(z) = \Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy$. For $j \geq 2$, by using (2.28),

$$\begin{aligned} \gamma_j(z) &= \int_0^\infty \frac{e^{-s}}{s} \left[\int_0^\infty y^{z-1} v_{j-1} \left(\frac{y}{s} \right) dy \right] ds \\ &= \int_0^\infty s^{z-1} e^{-s} \left[\int_0^\infty y^{z-1} v_{j-1}(y) dy \right] ds = \Gamma(z) \gamma_{j-1}(z). \end{aligned}$$

Thus, we have

$$\gamma_j(z) = \Gamma(z)^j, \quad j \geq 1. \tag{2.30}$$

Assume $Y_{nj}, 1 \leq j \leq n$, are independent random variables, and for each $1 \leq j \leq n$, the density of Y_{nj} is proportional to $y^{2j-1} w_k(y)$. By Lemma 1.1 and (1.1), $\max_{1 \leq j \leq n} |z_j|^2$ and $\max_{1 \leq j \leq n} Y_{nj}^2$ are identically distributed. Furthermore, since the density function of Y_{nj}^2 , denoted by $f_j(y)$, is proportional to $y^{j-1} w_k(y^{1/2})$, and thus proportional to $y^{j-1} v_k(y)$ from (2.29), we have from (2.30) that

$$f_j(y) = \frac{y^{j-1}v_k(y)}{\int_0^\infty y^{j-1}v_k(y)dy} = \frac{y^{j-1}v_k(y)}{\Gamma(j)^k}, \quad y > 0,$$

for $1 \leq j \leq n$. Let the characteristic function of $\log Y_{nj}^2$ be denoted by $g_j(t)$. Then, we have

$$\begin{aligned} g_j(t) &= \frac{1}{\Gamma(j)^k} \int_0^\infty e^{it \log y} y^{j-1} v_k(y) dy = \frac{1}{\Gamma(j)^k} \int_0^\infty y^{j-1+it} v_k(y) dy \\ &= \left(\frac{\Gamma(j+it)}{\Gamma(j)} \right)^k \end{aligned}$$

from (2.30). Since $\Gamma(j+it)/\Gamma(j)$ is the characteristic function of $\log s_{j,r}$, it follows that $\log Y_{nj}^2$ has the same distribution as that of $\sum_{r=1}^k \log s_{j,r}$, or equivalently, Y_{nj}^2 has the same distribution as that of $\prod_{r=1}^k s_{j,r}$ for $j \geq 1$. This implies the desired conclusion. \square

Lemma 2.5 *Let k be as in (1.4) and $\{s_{j,r}, 1 \leq r \leq k, j \geq 1\}$ be independent r.v.'s such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y > 0)/(j-1)!$ for all j, r . Set $\eta(x) = x - 1 - \log x$ and*

$$M_n(i) = \max_{n-i+1 \leq j \leq n} \left| \sum_{r=1}^k \left(\eta\left(\frac{s_{j,r}}{j}\right) - E\eta\left(\frac{s_{j,r}}{j}\right) \right) \right|, \quad 1 \leq i \leq n. \quad (2.31)$$

Set $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$. Then, for $1 \leq i \leq n$

$$\left| \max_{n-i+1 \leq j \leq n} \log \prod_{r=1}^k s_{j,r} - \max_{n-i+1 \leq j \leq n} \left(\frac{1}{j} \sum_{r=1}^k (s_{j,r} - j) + k \psi(j) \right) \right| \leq M_n(i).$$

Proof Set $Y_j = \prod_{r=1}^k s_{j,r}$ for $j \geq 1$. Then,

$$\log Y_j = \sum_{r=1}^k \log s_{j,r}$$

for $j \geq 1$. The moment-generating functions of $\log s_{j,r}$ are

$$m_j(t) = E(e^{t \log s_{j,r}}) = \frac{\Gamma(j+t)}{\Gamma(j)} \quad (2.32)$$

for $t > -j$. Therefore,

$$E(\log s_{j,r}) = \frac{d}{dt} m_j(t)|_{t=0} = \frac{\Gamma'(j)}{\Gamma(j)} = \psi(j)$$

by (1.5). Note that $\eta(x) = x - 1 - \log x$ for $x > 0$. Since $\eta(x) = \int_1^x \frac{s-1}{s} ds$, it is easy to verify that

$$0 \leq \eta(x) \leq \frac{(x - 1)^2}{2 \min(x, 1)}, \quad x > 0. \tag{2.33}$$

By using the expression $\log x = x - 1 - \eta(x)$, we can rewrite $\log Y_j$ as

$$\begin{aligned} \log Y_j &= \sum_{r=1}^k \log \frac{S_{j,r}}{j} + k \log j \\ &= \sum_{r=1}^k \frac{S_{j,r} - j}{j} - \sum_{r=1}^k \eta\left(\frac{S_{j,r}}{j}\right) + k \log j \\ &= \frac{1}{j} \sum_{r=1}^k (S_{j,r} - j) - \sum_{r=1}^k \eta\left(\frac{S_{j,r}}{j}\right) + k \log j \\ &= \frac{1}{j} \sum_{r=1}^k (S_{j,r} - j) + k\psi(j) - \sum_{r=1}^k \eta\left(\frac{S_{j,r}}{j}\right) + k(\log j - \psi(j)). \end{aligned}$$

Since $E(\log Y_j) = k\psi(j)$, we see that

$$\sum_{r=1}^k E\eta\left(\frac{S_{j,r}}{j}\right) = k(\log j - \psi(j))$$

and thus

$$\log Y_j = \frac{1}{j} \sum_{r=1}^k (S_{j,r} - j) + k\psi(j) - \sum_{r=1}^k \left(\eta\left(\frac{S_{j,r}}{j}\right) - E\eta\left(\frac{S_{j,r}}{j}\right) \right). \tag{2.34}$$

Note that for any two sequences of reals numbers $\{x_n\}$ and $\{y_n\}$,

$$\left| \max_{1 \leq j \leq n} x_j - \max_{1 \leq j \leq n} y_j \right| \leq \max_{1 \leq j \leq n} |x_j - y_j|.$$

Then, it follows from (2.34) that

$$\left| \max_{n-i+1 \leq j \leq n} \log Y_j - \max_{n-i+1 \leq j \leq n} \left(\frac{1}{j} \sum_{r=1}^k (S_{j,r} - j) + k\psi(j) \right) \right| \leq M_n(i).$$

□

We estimate $M_n(\cdot)$ next.

Lemma 2.6 *Let k be as in (1.4) and $M_n(i)$ be defined as in Lemma 2.5. Assume $\{j_n; n \geq 1\}$ is a sequence of numbers satisfying $1 \leq j_n \leq \frac{1}{2}n$ for all n . Then, for any*

sequence of positive integers $\{k_n\}$, $M_n(j_n) = O_P\left(\frac{j_n k_n^{1/2}}{n}\right)$. Further, if $\lim_{n \rightarrow \infty} k_n/n = 0$, then $M_n(j_n) = O_P\left(\frac{k_n \log n}{n}\right)$.

Proof By using the Minkowski inequality and (2.33), we get

$$\begin{aligned} E(M_n(j_n)) &\leq \sum_{n-j_n+1 \leq j \leq n} E \sum_{r=1}^{k_n} \left| \eta\left(\frac{s_{j,r}}{j}\right) - E\eta\left(\frac{s_{j,r}}{j}\right) \right| \\ &\leq \sum_{n-j_n+1 \leq j \leq n} \left[E \left(\sum_{r=1}^{k_n} \left| \eta\left(\frac{s_{j,r}}{j}\right) - E\eta\left(\frac{s_{j,r}}{j}\right) \right| \right)^2 \right]^{1/2} \\ &\leq \sum_{n-j_n+1 \leq j \leq n} \left[\sum_{r=1}^{k_n} E \left| \eta\left(\frac{s_{j,r}}{j}\right) - E\eta\left(\frac{s_{j,r}}{j}\right) \right|^2 \right]^{1/2} \\ &\leq k_n^{1/2} \sum_{n-j_n+1 \leq j \leq n} \left(E\eta\left(\frac{s_{j,1}}{j}\right)^2 \right)^{1/2} \\ &\leq \frac{1}{2} k_n^{1/2} \sum_{n-j_n+1 \leq j \leq n} \left\{ E \left[\left(\frac{s_{j,1} - j}{j} \right)^4 \left(\min\left(\frac{s_{j,1}}{j}, 1\right) \right)^{-2} \right] \right\}^{1/2}. \end{aligned}$$

Since $s_{j,1}$ has density $y^{j-1}e^{-y}I(y > 0)/(j - 1)!$, we see that $E(s_{j,1}^{-4}) = \frac{\Gamma(j-4)}{\Gamma(j)}$. By the Marcinkiewicz–Zygmund inequality (see, for example, Corollary 2 from [12]), we obtain $E(s_{j,1} - j)^8 \leq Kj^4$ for any $j \geq 1$, where K is a constant not depending on j . Then, it follows from Hölder’s inequality that

$$\begin{aligned} &E\left(\left(\frac{s_{j,1} - j}{j}\right)^4 \left(\min\left(\frac{s_{j,1}}{j}, 1\right)\right)^{-2}\right) \\ &\leq \left[E\left(\frac{s_{j,1} - j}{j}\right)^8 \cdot E\left(\min\left(\frac{s_{j,1}}{j}, 1\right)\right)^{-4} \right]^{1/2} \\ &\leq \left[E\left(\frac{s_{j,1} - j}{j}\right)^8 \cdot E\left(\left(\frac{j}{s_{j,1}}\right)^4 + 1\right) \right]^{1/2} \\ &\leq \left[\left(\frac{j^3}{(j-1)(j-2)(j-3)} + 1\right)^{1/2} \left(E\left(\frac{s_{j,1} - j}{j}\right)\right)^8 \right]^{1/2} \\ &\leq Cj^{-2} \end{aligned}$$

for any $j \geq 4$, where C is a constant. Combining the last two assertions, we get $E(M_n(j_n)) \leq O\left(\frac{j_n k_n^{1/2}}{n}\right)$. This implies the first conclusion.

Now, we prove the second one. Recall $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$ as in (1.5). By Formulas 6.3.18 and 6.4.12 from [1],

$$\psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \quad \text{and} \quad \psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + O\left(\frac{1}{x^3}\right) \quad (2.35)$$

as $x \rightarrow +\infty$. It is easy to check $E \log s_{j,1} = \frac{1}{\Gamma(j)} \int_0^\infty (\log y)y^{j-1}e^{-y}dy = \psi(j)$. Thus, from the first expression, we have

$$E\eta\left(\frac{s_{j,1}}{j}\right) = \log j - \psi(j) = O\left(\frac{1}{j}\right)$$

as $j \rightarrow \infty$. Hence, by (2.31),

$$M_n(j_n) \leq \max_{n-j_n+1 \leq j \leq n} \sum_{r=1}^{k_n} \eta\left(\frac{s_{j,r}}{j}\right) + O\left(\frac{k_n}{n}\right). \quad (2.36)$$

By Theorem 1 on page 217 from [40], we have that

$$P(s_{j,1} > j + j^{1/2}x) = (1 + o(1))(1 - \Phi(x)) \quad (2.37)$$

uniformly for $x \in (0, a_n)$ and $n/2 \leq j \leq n$ as $n \rightarrow \infty$, where $\{a_n; n \geq 1\}$ is an arbitrarily given sequence of positive numbers with $a_n = o(n^{1/6})$. By taking $r = 0$ in (2.25), we see that $1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$ as $x \rightarrow +\infty$. Now select $x = 2(\log n)^{1/2}$ in (2.37) to have

$$P(s_{j,1} > j + 2j^{1/2}(\log n)^{1/2}) = (1 + o(1))(1 - \Phi(2(\log n)^{1/2})) = O\left(\frac{1}{n^2}\right)$$

uniformly for $n/2 \leq j \leq n$ as $n \rightarrow \infty$. Similarly, we have

$$P(s_{j,1} < j - 2j^{1/2}(\log n)^{1/2}) = (1 + o(1))(1 - \Phi(2(\log n)^{1/2})) = O\left(\frac{1}{n^2}\right)$$

uniformly for $n/2 \leq j \leq n$ as $n \rightarrow \infty$. This implies

$$\sum_{r=1}^{k_n} \sum_{j=n-j_n+1}^n P(|s_{j,r} - j| > 2j^{1/2}(\log n)^{1/2}) = O\left(\frac{j_n k_n}{n^2}\right) = O\left(\frac{k_n}{n}\right) = o(1),$$

and thus, we get

$$\max_{1 \leq r \leq k_n} \max_{n-j_n+1 \leq j \leq n} \left| \frac{s_{j,r}}{j} - 1 \right| = O_P\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right).$$

Consequently,

$$\min_{1 \leq r \leq k_n} \min_{n-j_n+1 \leq j \leq n} \frac{s_{j,r}}{j} = 1 + O_P\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right).$$

By (2.36) and then (2.33), we obtain

$$\begin{aligned} M_n(j_n) &\leq k_n \cdot \max_{n-j_n+1 \leq j \leq n} \max_{1 \leq r \leq k_n} \eta \left(\frac{s_{j,r}}{j} \right) + O \left(\frac{k_n}{n} \right) \\ &\leq \frac{k_n}{2} \cdot \frac{\max_{n-j_n+1 \leq j \leq n} \max_{1 \leq r \leq k_n} \left| \frac{s_{j,r}}{j} - 1 \right|^2}{\min\{1, \min_{n-j_n+1 \leq j \leq n} \min_{1 \leq r \leq k_n} \frac{s_{j,r}}{j}\}} + O \left(\frac{k_n}{n} \right) \\ &= O_P \left(\frac{k_n \log n}{n} \right), \end{aligned}$$

proving the second conclusion. □

Review the notation we use before: $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$ as in (1.5) and

$$Y_j = \prod_{r=1}^k s_{j,r} \tag{2.38}$$

for $j \geq 1$, where $\{s_{j,r}, 1 \leq r \leq k, j \geq 1\}$ are independent random variables such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y > 0)/(j - 1)!$ for all j, r .

Lemma 2.7 *Let $\{j_n; n \geq 1\}$ and $\{k_n; n \geq 1\}$ be positive integers satisfying $\lim_{n \rightarrow \infty} \frac{j_n}{n} = 0$ and $\lim_{n \rightarrow \infty} \left(\frac{k_n}{n}\right)^{1/2} \frac{j_n}{(\log n)^{1/2}} = \infty$. Then, for any $x \in \mathbb{R}$,*

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{n-j_n} P \left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x \right) = 0. \tag{2.39}$$

Proof Fix $x \in \mathbb{R}$. It follows from (2.32) that for each $1 \leq j \leq n - j_n$ and any $t > 0$,

$$\begin{aligned} &P \left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x \right) \\ &\leq \frac{E(e^{t \log Y_j})}{\exp \left\{ t \left(k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x \right) \right\}} \\ &= \exp \left\{ k_n (\log(\Gamma(j+t)) - \log \Gamma(j)) - t \left(k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x \right) \right\} \\ &= \exp \left\{ k_n \int_0^t \psi(j+s) ds - t \left(k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x \right) \right\} \\ &= \exp \left\{ k_n \int_0^t [\psi(j+s) - \psi(j)] ds - t \left[k_n (\psi(n) - \psi(j)) + \left(\frac{k_n}{n}\right)^{1/2} x \right] \right\}. \end{aligned}$$

From (2.35), there exists an integer j_0 such that for all $j_0 \leq j \leq n - j_n$,

$$\log \frac{j+s}{j} \leq \psi(j+s) - \psi(j) = \int_0^s \psi'(j+v) dv \leq \frac{1.1s}{j}, \quad s \geq 0.$$

By the first inequality above, for all large n ,

$$\psi(n) - \psi(j) \geq \log \frac{n}{j} \geq \log \frac{n}{n - j_n} = -\log \left(1 - \frac{j_n}{n} \right) \geq \frac{0.999j_n}{n}, \quad j_0 \leq j \leq n - j_n.$$

Hence, by assumption $\left(\frac{k_n}{n}\right)^{1/2} = o\left(\frac{j_n k_n}{n}\right)$ we see that

$$k_n(\psi(n) - \psi(j)) + \left(\frac{k_n}{n}\right)^{1/2} x \geq 0.99k_n \log \frac{n}{j}, \quad j_0 \leq j \leq n - j_n$$

for all large n . Therefore, we have for $j_0 \leq j \leq n - j_n$

$$\begin{aligned} P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x\right) &\leq \exp \left\{ 1.1k_n \int_0^t \frac{s}{j} ds - 0.99tk_n(\log n - \log j) \right\} \\ &= \exp \left\{ k_n \left(\frac{1.1t^2}{2j} - 0.99t(\log n - \log j) \right) \right\} \end{aligned}$$

for all $t > 0$ and large n which does not depend on t . By selecting $t = 0.99j(\log n - \log j)$, we have

$$\begin{aligned} P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x\right) &\leq \exp \left\{ -0.44k_n j (\log n - \log j)^2 \right\}, \quad j_0 \leq j \leq n - j_n, \end{aligned} \tag{2.40}$$

for all large n . Note that

$$\begin{aligned} \min_{j_0 \leq j \leq n - j_n} j (\log n - \log j)^2 &\geq \min_{j_0 \leq s \leq n - j_n} s (\log n - \log s)^2 \\ &= \min_{j_0^{1/2} \leq s \leq (n - j_n)^{1/2}} s^2 (\log n - 2 \log s)^2 \\ &= \left(\min_{j_0^{1/2} \leq s \leq (n - j_n)^{1/2}} s (\log n - 2 \log s) \right)^2, \end{aligned}$$

where the last three minima are taken over all real numbers satisfying the corresponding constraints. It is easily seen that the minimum of $s(\log n - 2 \log s)$ for $j_0^{1/2} \leq s \leq (n - j_n)^{1/2}$ is achieved at the two end points of the interval, $t = j_0^{1/2}$ or $s = (n - j_n)^{1/2}$. Thus, for all large n ,

$$\begin{aligned} \min_{j_0 \leq j \leq n - j_n} j (\log n - \log j)^2 &\geq \min \left\{ j_0 (\log n - \log j_0)^2, (n - j_n) (\log n - \log(n - j_n))^2 \right\} \\ &\geq \min \left\{ \frac{1}{2} (\log n)^2, \frac{1}{2} \frac{j_n^2}{n} \right\}. \end{aligned}$$

From the given condition $\left(\frac{k_n}{n}\right)^{1/2} \frac{j_n}{(\log n)^{1/2}} = \infty$, we obtain

$$k_n \min_{j_0 \leq j \leq n-j_n} j(\log n - \log j)^2 \geq 10 \log n$$

for all large n . Therefore, combining all of the inequalities from (2.40) to the above, we have

$$\max_{j_0 \leq j \leq n-j_n} P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x\right) \leq \exp(-4.4 \log n) = n^{-4.4},$$

and hence,

$$\sum_{j=j_0}^{n-j_n} P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x\right) = O(n^{-3.4}) \rightarrow 0.$$

Finally, observe that, for each $1 \leq j < j_0$, $\log Y_j$ is a sum of k_n 's many i.i.d. random variables with $Ee^{t \log Y_j} < \infty$ for all $|t| < \frac{1}{2}$. Then, by the Chernoff bound (see, for instance, p. 27 from [14]),

$$P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} x\right) \rightarrow 0.$$

The last two assertions imply the desired result. □

Recall $\Lambda(x) = \exp(-e^{-x})$ for all $x \in \mathbb{R}$. We first prove the following proposition from which Theorem 3 will be obtained.

Proposition 2.1 *Let $\psi(x)$ be as in (1.5), $a(x)$ and $b(x)$ be as in Theorem 2.1, and z_j 's and k_n be as in Theorem 3. Define $\Phi_0(y) = \Lambda(y)$, $a_n = a(n/k_n)$, $b_n = b(n/k_n)$ if $\alpha = 0$, and $a_n = 0$, $b_n = 1$ if $\alpha \in (0, \infty]$. Then,*

$$\lim_{n \rightarrow \infty} P\left(\frac{\max_{1 \leq j \leq n} \log |z_j| - k_n \psi(n)/2}{(k_n/n)^{1/2}/2} \leq a_n + b_n y\right) = \Phi_\alpha(y), \quad y \in \mathbb{R}. \tag{2.41}$$

Proof For each of the three cases: $\alpha = 0$, $\alpha \in (0, \infty)$ and $\alpha = \infty$, we will show that there exists a sequence of positive integers $\{j_n\}$ with $1 \leq j_n \leq n/2$ such that

$$\lim_{n \rightarrow \infty} \sum_{j=1}^{n-j_n} P\left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} (a_n + b_n y)\right) = 0, \quad y \in \mathbb{R}; \tag{2.42}$$

$$\frac{M_n(j_n)}{(k_n/n)^{1/2} b_n} \text{ converges in probability to zero,} \tag{2.43}$$

where $M_n(\cdot)$ is defined as in Lemma 2.5, and

$$\lim_{n \rightarrow \infty} P \left(\frac{\max_{n-j_n+1 \leq j \leq n} \left(\frac{1}{j} \sum_{r=1}^{k_n} (s_{j,r} - j) + k_n \psi(j) \right) - k_n \psi(n)}{(k_n/n)^{1/2}} \leq a_n + b_n y \right) = \Phi_\alpha(y) \tag{2.44}$$

for $y \in \mathbb{R}$, where $\{s_{j,r}, 1 \leq r \leq k, j \geq 1\}$ are independent random variables such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y > 0)/(j - 1)!$ for all j and r . In fact, (2.44) implies

$$\frac{\max_{n-j_n+1 \leq j \leq n} \left(\frac{1}{j} \sum_{r=1}^{k_n} (s_{j,r} - j) + k_n \psi(j) \right) - k_n \psi(n)}{(k_n/n)^{1/2} b_n} - \frac{a_n}{b_n} \xrightarrow{d} \Phi_\alpha.$$

Review the definition of Y_j in (2.38). The above result together with (2.43), Lemmas 2.4 and 2.5 implies that

$$\frac{\max_{n-j_n+1 \leq j \leq n} \log Y_j - k_n \psi(n)}{(k_n/n)^{1/2} b_n} - \frac{a_n}{b_n} \xrightarrow{d} \Phi_\alpha.$$

Since (2.42) implies

$$\frac{\max_{1 \leq j \leq n-j_n} \log Y_j - k_n \psi(n)}{(k_n/n)^{1/2} b_n} - \frac{a_n}{b_n} \xrightarrow{d} 0,$$

the two limits above imply (2.41) due to the fact that $\max_{1 \leq j \leq n} \log |z_j|$ and $\frac{1}{2} \max_{1 \leq j \leq n} \log Y_j$ are identically distributed by Lemma 2.4.

Now, we start to verify equations (2.42)–(2.44) with a choice of j_n given by

$$j_n = \text{the integer part of } \left(\frac{n}{k_n} \right)^{1/2} n^{1/8} + 1 \tag{2.45}$$

for all large n . □

Proof of (2.42) It is easy to verify that the conditions in Lemma 2.7 are satisfied, and thus, (2.39) holds. In case $\alpha \in (0, \infty]$, $a_n = 0$ and $b_n = 1$, and (2.42) holds in this case. When $\alpha = 0$, $a_n + b_n y > 0$ for all large n , by applying (2.39) with $x = 0$ we have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{j=1}^{n-j_n} P \left(\log Y_j > k_n \psi(n) + \left(\frac{k_n}{n} \right)^{1/2} (a_n + b_n y) \right) \\ & \leq \lim_{n \rightarrow \infty} \sum_{j=1}^{n-j_n} P(\log Y_j > k_n \psi(n)) = 0, \end{aligned}$$

that is, (2.42) holds. This completes the proof of (2.42) for all three cases. \square

Proof of (2.43) To prove (2.43), it suffices to show $M_n(j_n) = O_P((\frac{k_n}{n})^{1/2}(\log n)^{-1})$ since $b_n \geq (\log n)^{-1/2}$ for all large n . We use Lemma 2.6 this time. When $\alpha \in (0, \infty]$, $j_n = O_P(n^{1/8})$ from (2.45), and then we have from the first conclusion in Lemma 2.6 that

$$M_n(j_n) = O_P\left(\frac{k_n^{1/2}}{n} j_n\right) = O_P\left(\left(\frac{k_n}{n}\right)^{1/2} n^{-3/8}\right) = O_P\left(\left(\frac{k_n}{n}\right)^{1/2} (\log n)^{-1}\right).$$

When $\alpha = 0$, we have from the two conclusions in Lemma 2.6 that

$$\begin{aligned} M_n(j_n) &= O_P\left(\min\left\{\frac{j_n k_n^{1/2}}{n}, \frac{k_n \log n}{n}\right\}\right) \\ &= \left(\frac{k_n}{n}\right)^{1/2} \cdot O_P\left(\min\left\{\frac{n^{1/8}}{k_n^{1/2}}, \left(\frac{k_n}{n}\right)^{1/2} \log n\right\}\right) \\ &= \left(\frac{k_n}{n}\right)^{1/2} \cdot O_P(n^{-1/8}) \\ &= O_P\left(\left(\frac{k_n}{n}\right)^{1/2} (\log n)^{-1}\right) \end{aligned}$$

since $\frac{n^{1/8}}{k_n^{1/2}} \leq n^{-1/8}$ if $k_n \geq n^{1/2}$ and $\frac{k_n^{1/2} \log n}{n^{1/2}} \leq n^{-1/8}$ if $k_n < n^{1/2}$. \square

Proof of (2.44) Set $T_n(j_n) = \max_{n-j_n+1 \leq j \leq n} \left(\frac{1}{j} \sum_{r=1}^{k_n} (s_{j,r} - j) + k_n \psi(j)\right)$. Then,

$$P\left(T_n(j_n) \leq k_n \psi(n) + \left(\frac{k_n}{n}\right)^{1/2} (a_n + b_n y)\right) \tag{2.46}$$

$$= \prod_{j=n-j_n+1}^n P\left(\sum_{r=1}^{k_n} (s_{j,r} - j) \leq j k_n (\psi(n) - \psi(j)) + j \left(\frac{k_n}{n}\right)^{1/2} (a_n + b_n y)\right). \tag{2.47}$$

Notice that $\sum_{r=1}^{k_n} s_{j,r}$ is a sum of $j k_n$ i.i.d. random variables with distribution $\text{Exp}(1)$, that is, it has density $e^{-x} I(x \geq 0)$. Since the mean and the variance of $\text{Exp}(1)$ are both equal to 1, we normalize the sum by

$$W_j := \frac{1}{\sqrt{j k_n}} \left(\left(\sum_{r=1}^{k_n} s_{j,r} \right) - j k_n \right).$$

By Theorem 1 on page 217 from [40], for any sequence of positive numbers δ_n such that $\delta_n = o((nk_n)^{1/6})$,

$$P(W_j > x) = (1 + o(1))(1 - \Phi(x)) \tag{2.48}$$

uniformly over $x \in [0, \delta_n]$ and $n/2 \leq j \leq n$ as $n \rightarrow \infty$. Now, reorganize the index in (2.47) to obtain

$$\begin{aligned} &P\left(T_n(j_n) \leq k_n\psi(n) + \left(\frac{k_n}{n}\right)^{1/2}(a_n + b_n y)\right) \\ &= \prod_{i=1}^{j_n} P\left(W_{n-i+1} \leq ((n-i+1)k_n)^{1/2}(\psi(n) - \psi(n-i+1)) \right. \\ &\quad \left. + \left(\frac{n-i+1}{n}\right)^{1/2}(a_n + b_n y)\right) \\ &= \prod_{i=1}^{j_n} (1 - a_{ni}), \end{aligned} \tag{2.49}$$

where $a_{ni} = P(W_{n-i+1} > x_{n,i})$ and

$$x_{n,i} = ((n-i+1)k_n)^{1/2}(\psi(n) - \psi(n-i+1)) + \left(1 - \frac{i-1}{n}\right)^{1/2}(a_n + b_n y). \tag{2.50}$$

Recalling (2.45), we know $j_n = o(n)$. From the second expression in (2.35), we have

$$\psi(n) - \psi(n-i+1) = \frac{i-1}{n} \left(1 + O\left(\frac{i}{n}\right)\right)$$

uniformly over $1 \leq i \leq j_n$ as $n \rightarrow \infty$. It follows that

$$\begin{aligned} &((n-i+1)k_n)^{1/2}(\psi(n) - \psi(n-i+1)) \\ &= \left(\frac{k_n}{n}\right)^{1/2}(i-1) \cdot \left(1 - \frac{i-1}{n}\right)^{1/2} \cdot \left(1 + O\left(\frac{i}{n}\right)\right) \\ &= \left(\frac{k_n}{n}\right)^{1/2}(i-1) \cdot \left(1 + O\left(\frac{i}{n}\right)\right) \\ &= \left(\frac{k_n}{n}\right)^{1/2}(i-1) \cdot \left(1 + O(n^{-3/8})\right) \end{aligned}$$

uniformly over $1 \leq i \leq j_n$ as $n \rightarrow \infty$. Since $a_n + b_n y = O((\log n)^{1/2})$,

$$\begin{aligned} \left(\left(1 - \frac{i-1}{n}\right)^{1/2} - 1\right)(a_n + b_n y) &= (i-1) \cdot O\left(\frac{(\log n)^{1/2}}{n}\right) \\ &= \left(\frac{k_n}{n}\right)^{1/2}(i-1) \cdot O\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right) \end{aligned}$$

uniformly over $1 \leq i \leq j_n$. Therefore, by combining the above two expansions we get

$$x_{n,i} = \left(1 + O(n^{-3/8})\right) \left(\frac{k_n}{n}\right)^{1/2} (i - 1) + a_n + b_n y \tag{2.51}$$

uniformly over $1 \leq i \leq j_n$. We emphasize that the above is true when $i = j_n = 1$, which can be seen directly from (2.50). This fact will be used later. \square

Finally, we prove (2.44) by considering the three cases: $\alpha = 0$, $\alpha \in (0, \infty)$ and $\alpha = \infty$.

Case 1 $\alpha = 0$. Since

$$a_n = a(n/k_n) \sim (\log(n/k_n))^{1/2} \text{ and } b_n = b(n/k_n) = (\log(n/k_n))^{-1/2} \rightarrow 0,$$

we have

$$\min_{1 \leq i \leq j_n} x_{n,i} \rightarrow \infty \text{ and } \max_{1 \leq i \leq j_n} x_{n,i} = O\left(\frac{k_n^{1/2} j_n}{n^{1/2}} + (\log n)^{1/2}\right) = O(n^{1/8}).$$

It follows from (2.48) that

$$a_{ni} = (1 + o(1))(1 - \Phi(x_{n,i}))$$

uniformly over $1 \leq i \leq j_n$. In Lemma 2.2, choose $x_n = n/k_n$, j_n as in (2.45) and $c_{nj} = (1 + O(n^{-3/8}))\left(\frac{k_n}{n}\right)^{1/2}$ as in (2.51) to obtain

$$\sum_{i=1}^{j_n} a_{ni} = (1 + o(1)) \sum_{i=1}^{j_n} (1 - \Phi(x_{n,i})) \rightarrow e^{-y}.$$

Further, it is easily seen that $\max_{1 \leq i \leq j_n} a_{ni} \rightarrow 0$. Applying (2.22) to (2.49), we arrive at

$$P\left(\frac{T_n(j_n) - k_n \psi(n)}{\left(\frac{k_n}{n}\right)^{1/2}} \leq a_n + b_n y\right) \rightarrow e^{-e^{-y}} = \Phi_0(y), \quad y \in \mathbb{R},$$

that is, we get (2.44) for $\alpha = 0$.

Case 2 We see that $j_n \sim \alpha^{-1/2} n^{1/8}$ from (2.45). By definition, $a_n = 0$ and $b_n = 1$. Then, it follows from (2.51) that

$$x_{n,i} = (1 + o(1))\alpha^{1/2}(i - 1) + y$$

holds uniformly over $1 \leq i \leq j_n$ as $n \rightarrow \infty$. We claim that

$$a_{ni} = (1 + o(1))(1 - \Phi(x_{n,i})) \tag{2.52}$$

uniformly over $1 \leq i \leq j_n$. In fact, review that (2.48) holds if $0 < x = o(n^{1/3})$. Evidently, $\max_{1 \leq i \leq j_n} |x_{n,i}| = O(n^{1/8})$. But, there is a possibility that $x_{n,i} < 0$ for small values of i . Let $j_0 > 1$ be an integer such that $\min_{j_0 \leq i \leq j_n} x_{n,i} > 0$. Then, we have from (2.48) that (2.52) holds uniformly over $j_0 \leq i \leq j_n$. By using the standard central limit theorem, we know (2.48) holds as well for each $i = 1, \dots, j_0 - 1$. Therefore, for each $i \geq 1$,

$$\lim_{n \rightarrow \infty} a_{ni} = 1 - \Phi(\alpha^{1/2}(i - 1) + y) \quad \text{and} \quad \sum_{i \geq 1} (1 - \Phi(\alpha^{1/2}(i - 1) + y)) < \infty \tag{2.53}$$

by the fact $1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$ as $x \rightarrow +\infty$. We now apply Lemma 2.1 to show (2.44). By defining $a_{ni} = 0$ for all $i > j_n$, with (2.53), we only need to verify the following two conditions: $\sup_{n \geq n_0, 1 \leq i \leq j_n} a_{ni} < 1$ for some integer n_0 and $\lim_{n \rightarrow \infty} \sum_{i=1}^{j_n} a_{ni} = \sum_{i=1}^{\infty} (1 - \Phi(\alpha^{-1/2}(i - 1) + y))$. The first one follows from (2.52) and the fact that $x_{n,i} \geq \frac{1}{2}\alpha^{1/2}(i - 1) + y \geq y$ for $1 \leq i \leq j_n$ for all large n . The second condition can be easily verified by the dominated convergence theorem since $a_{ni} \leq 2(1 - \Phi(\frac{1}{2}\alpha^{1/2}(i - 1) + y))$ for all $1 \leq i \leq j_n$ as n is sufficiently large and $\sum_{i=1}^{\infty} 2(1 - \Phi(\frac{1}{2}\alpha^{1/2}(i - 1) + y)) < \infty$.

Case 3 $\alpha = \infty$. From (2.45), $0 \leq (\frac{k_n}{n})(j_n - 1) \leq n^{1/8}$ and thus $x_{n,i} = O(n^{1/8})$ by (2.51). In particular, we have $x_{n,1} = y$, and for all large n , $x_{n,i} > 0$ if $2 \leq i \leq j_n$ and $j_n \geq 2$. Therefore,

$$a_{ni} = (1 + o(1))(1 - \Phi(x_{n,i}))$$

uniformly over $1 \leq i \leq j_n$ from (2.48). From (2.51), $a_{n1} \rightarrow 1 - \Phi(y)$ as $n \rightarrow \infty$. Obviously, $x_{n,i} \geq \frac{i}{3}(\frac{k_n}{n})^{1/2}$ if $2 \leq i \leq j_n$ and $j_n \geq 2$. Thus, use the fact $1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$ as $x \rightarrow +\infty$ to see that, for large n ,

$$\begin{aligned} I(j_n \geq 2) \sum_{i=2}^{j_n} a_{ni} &\leq 2 \sum_{i=2}^{\infty} \left(1 - \Phi\left(\frac{i}{3}\left(\frac{k_n}{n}\right)^{1/2}\right)\right) \\ &\leq \sum_{i=2}^{\infty} \exp\left\{-\frac{k_n}{18n}i^2\right\} \\ &\leq \int_{-\infty}^{\infty} \exp\left\{-\frac{k_n}{18n}x^2\right\} dx = 3\sqrt{2\pi}\left(\frac{n}{k_n}\right)^{1/2} \end{aligned}$$

since $\exp\left\{-\frac{k_n}{18n}i^2\right\} \leq \int_{i-1}^i \exp\left\{-\frac{k_n}{18n}x^2\right\} dx$ for all $i \geq 2$. Thus, $I(j_n \geq 2) \sum_{i=2}^{j_n} a_{ni} \rightarrow 0$. This and the fact $I(j_n \geq 2) \cdot \max_{2 \leq i \leq j_n} a_{ni} \rightarrow 0$ imply that $I(j_n \geq 2)(1 - \prod_{i=2}^{j_n} (1 - a_{ni})) \rightarrow 0$ as $n \rightarrow \infty$. So, we have from (2.49) that

$$\begin{aligned}
 &P\left(T_n(j_n) \leq k_n\psi(n) + \left(\frac{k_n}{n}\right)^{1/2}(a_n + b_n y)\right) \\
 &= \prod_{i=1}^{j_n} (1 - a_{ni}) \\
 &= (1 - a_{n1}) \cdot \left[1 + I(j_n \geq 2) \left(-1 + \prod_{i=2}^{j_n} (1 - a_{ni})\right)\right] \rightarrow \Phi(y) = \Phi_\infty(y)
 \end{aligned}$$

as $n \rightarrow \infty$. Reviewing the notation of $T_n(j_n)$ defined above (2.46), we get (2.44) for the case $\alpha = \infty$. The proof of the proposition is then completed. \square

Proof of Theorem 3 We use the same notation as in Proposition 2.1. We first show the following:

(i) If $\lim_{n \rightarrow \infty} k_n/n = 0$, particularly for $k_n \equiv k$, then

$$\frac{2(n/k_n)^{1/2}}{b_n} \left(\frac{\max_{1 \leq j \leq n} |z_j|}{n^{k_n/2}} - 1\right) - \frac{a_n}{b_n} \text{ converges weakly to cdf } \exp(-e^{-x}). \tag{2.54}$$

(ii) If $\lim_{n \rightarrow \infty} k_n/n = \alpha \in (0, \infty)$, then

$$\frac{\max_{1 \leq j \leq n} |z_j|}{n^{k_n/2}} \text{ converges weakly to cdf } \Phi_\alpha\left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\right), \quad x > 0. \tag{2.55}$$

To do so, for $\alpha \in [0, \infty)$, define

$$V_n = \frac{\max_{1 \leq j \leq n} \log |z_j| - k_n\psi(n)/2}{(k_n/n)^{1/2}b_n/2} - \frac{a_n}{b_n}.$$

Then, V_n converges in distribution to Θ_α by Proposition 2.1, where Θ_α is a random variable with cdf $\Phi_\alpha(y)$. Trivially,

$$\begin{aligned}
 \max_{1 \leq j \leq n} |z_j| &= \exp\left\{\frac{1}{2}k_n\psi(n) + \frac{1}{2}\left(\frac{k_n}{n}\right)^{1/2}(a_n + b_n V_n)\right\} \\
 &= \exp\left\{\frac{1}{2}\left(k_n\psi(n) + \left(\frac{k_n}{n}\right)^{1/2}a_n\right)\right\} \cdot \exp\left\{\frac{1}{2}\left(\frac{k_n}{n}\right)^{1/2}b_n V_n\right\}.
 \end{aligned} \tag{2.56}$$

If $\alpha = 0$, then $\frac{k_n}{n} \rightarrow 0$, $a_n = a\left(\frac{n}{k_n}\right) \sim (\log \frac{n}{k_n})^{1/2} \rightarrow \infty$, $b_n = b\left(\frac{n}{k_n}\right) = (\log \frac{n}{k_n})^{-1/2} \rightarrow 0$, and $\left(\frac{k_n}{n}\right)^{1/2}a_n \sim \left(\frac{k_n}{n}\right)^{1/2}b_n^{-1} \rightarrow 0$ as $n \rightarrow \infty$. Using (2.35) and

expanding (2.56), we get

$$\begin{aligned} \max_{1 \leq j \leq n} |z_j| &= \exp \left\{ \frac{1}{2} k_n \log n + O\left(\frac{k_n}{n}\right) + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} a_n \right\} \\ &\quad \left(1 + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} b_n V_n + O_P\left(\frac{k_n b_n^2}{n}\right) \right) \\ &= n^{k_n/2} \left(1 + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} a_n + O\left(\frac{k_n}{n}\right) \right) \left(1 + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} b_n V_n + O_P\left(\frac{k_n}{n}\right) \right) \\ &= n^{k_n/2} \left(1 + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} a_n + \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} b_n V_n + O_P\left(\frac{k_n a_n^2}{n}\right) \right), \end{aligned}$$

which yields that

$$\frac{2(n/k_n)^{1/2}}{b_n} \left(\frac{\max_{1 \leq j \leq n} |z_j|}{n^{k_n/2}} - 1 \right) - \frac{a_n}{b_n} = V_n + O_P\left(\left(\frac{k_n}{n}\right)^{1/2} \left(\log \frac{n}{k_n}\right)^{3/2}\right)$$

converges in distribution to Λ by the Slutsky lemma. We obtain (2.54).

Now, assume $\alpha \in (0, \infty)$. In this case, $a_n = 0$ and $b_n = 1$. Then, from (2.56),

$$\max_{1 \leq j \leq n} |z_j| = \exp \left\{ \frac{1}{2} k_n \psi(n) \right\} \exp \left\{ \frac{1}{2} \left(\frac{k_n}{n}\right)^{1/2} V_n \right\}.$$

Using expansion $\psi(n) = \log n - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$ from (2.35), we have

$$\frac{\max_{1 \leq j \leq n} |z_j|}{n^{k_n/2}} = \exp \left\{ -\frac{1}{4} \alpha + o(1) \right\} \cdot \exp \left\{ \left(\frac{1}{2} \alpha^{1/2} + o(1)\right) V_n \right\},$$

which converges weakly to the distribution of $e^{-\alpha/4} \exp\left(\frac{1}{2} \alpha^{1/2} \Theta_\alpha\right)$, given by $\Phi_\alpha\left(\frac{1}{2} \alpha^{1/2} + 2\alpha^{-1/2} \log y\right)$, $y > 0$. We get (2.55).

From (2.54), it is easy to see

$$\begin{aligned} \frac{2(n/k_n)^{1/2}}{b_n} &= \left(\frac{n}{k_n} \log \frac{n}{k_n}\right)^{1/2} = \alpha_n \quad \text{and} \quad \frac{a_n}{b_n} = \log \frac{n}{k_n} - \log \log \frac{n}{k_n} \\ &\quad - \frac{1}{2} \log(2\pi) = \beta_n. \end{aligned}$$

Thus, we obtain (a) of Theorem 3. The part (b) follows from (2.55), and the part (c) is yielded from Proposition 2.1 with $\Phi_\infty(x) = \Phi(x)$. This completes the proof of the theorem. □

2.4 The Verifications of (1.2) and (1.6)

Verification of (1.2) First, by the Taylor expansion,

$$\begin{aligned}
 H_k(y) &= e^{-y} \sum_{j=0}^{k-1} \frac{y^j}{j!} = 1 - e^{-y} \sum_{j=k}^{\infty} \frac{y^j}{j!} \\
 &= 1 - \frac{y^k e^{-y}}{k!} \left(1 + y \sum_{j=k+1}^{\infty} \frac{y^{j-k-1}}{(k+1) \cdots j} \right)
 \end{aligned}$$

for all $y \in \mathbb{R}$ and $k \geq 1$. Notice that the absolute value of the above sum is bounded by $\sum_{j=k+1}^{\infty} \frac{1}{(k+1) \cdots j} \leq 1 + \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$ uniformly for all $k \geq 1$ and $|y| \leq 1$. This says

$$H_k(y) = 1 - \frac{y^k}{k!} e^{-y} (1 + O(y)) = 1 - \frac{y^k}{k!} (1 + O(y))$$

as $y \rightarrow 0$ uniformly for all $k \geq 1$. Hence,

$$\begin{aligned}
 \log \prod_{k=1}^{\infty} H_k(y) &= \sum_{k=1}^{\infty} \log H_k(y) \\
 &= -(1 + O(y)) \sum_{k=1}^{\infty} \frac{y^k}{k!} = -y(1 + O(y))
 \end{aligned}$$

since $\sum_{k=1}^{\infty} \frac{y^k}{k!} = e^y - 1 \sim y$ as $y \rightarrow 0$. Therefore,

$$1 - \prod_{k=1}^{\infty} H_k(y) = 1 - e^{-y(1+O(y))} \sim y$$

as $y \rightarrow 0$. Taking $y = x^{-2}$ and letting $x \rightarrow \infty$, we get (1.2). □

Verification of (1.6) Given parameter $\beta > 0$, set

$$F_{\beta}(x) = \prod_{j=0}^{\infty} \Phi(x + \beta j)$$

for $x \in \mathbb{R}$. From integration by parts, we know $1 - \Phi(x) = \frac{1}{\sqrt{2\pi} x} e^{-x^2/2} (1 + O(x^{-2}))$ as $x \rightarrow +\infty$. Use $\log(1 - t) = -t(1 + O(t))$ as $t \rightarrow 0$ to have

$$\begin{aligned}
 \log \Phi(x) &= -\frac{1}{\sqrt{2\pi} x} e^{-x^2/2} (1 + O(x^{-2})) \left(1 + O\left(\frac{1}{x} e^{-x^2/2}\right) \right) \\
 &= -\frac{1}{\sqrt{2\pi} x} e^{-x^2/2} (1 + a(x))
 \end{aligned}$$

as $x \rightarrow +\infty$, where $a(x)$ is defined over $[1, \infty)$ and $|a(x)| \leq Cx^{-2}$ for all $x \geq 1$ and C is a constant not depending on x . Thus,

$$\begin{aligned} \log F_\beta(x) &= \sum_{j=0}^\infty \log \Phi(x + \beta j) \\ &= -\frac{1}{\sqrt{2\pi}} \sum_{j=0}^\infty \frac{1}{x + \beta j} e^{-(x+\beta j)^2/2} (1 + a(x + \beta j)) \\ &= -(1 + o(1)) \frac{1}{\sqrt{2\pi}} \sum_{j=0}^\infty \frac{1}{x + \beta j} e^{-(x+\beta j)^2/2} \end{aligned} \tag{2.57}$$

as $x \rightarrow +\infty$. Observe that

$$\frac{1}{x + \beta(j + 1)} e^{-(x+\beta(j+1))^2/2} \leq \int_{x+\beta j}^{x+\beta(j+1)} \frac{1}{t} e^{-t^2/2} dt \leq \frac{1}{x + \beta j} e^{-(x+\beta j)^2/2}$$

for all $x > 0$ and $j \geq 0$. Sum the above over all $j \geq 1$ to obtain

$$\int_{x+\beta}^\infty \frac{1}{t} e^{-t^2/2} dt \leq \sum_{j=1}^\infty \frac{1}{x + \beta j} e^{-(x+\beta j)^2/2} \leq \frac{1}{x + \beta} e^{-(x+\beta)^2/2} + \int_{x+\beta}^\infty \frac{1}{t} e^{-t^2/2} dt$$

for all $x > 0$. Write $\int_{x+\beta}^\infty \frac{1}{t} e^{-t^2/2} dt = -\int_{x+\beta}^\infty \frac{1}{t^2} (e^{-t^2/2})' dt$. From the integration by parts, $\int_{x+\beta}^\infty \frac{1}{t} e^{-t^2/2} dt \sim \frac{1}{(x+\beta)^2} e^{-(x+\beta)^2/2}$ as $x \rightarrow +\infty$. Since $\beta > 0$, we have $\frac{1}{(x+\beta)^2} e^{-(x+\beta)^2/2} = o(\frac{1}{x} e^{-x^2/2})$ and $\frac{1}{x+\beta} e^{-(x+\beta)^2/2} = o(\frac{1}{x} e^{-x^2/2})$ as $x \rightarrow +\infty$. It follows from (2.57) that

$$\log F_\beta(x) \sim -\frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

as $x \rightarrow +\infty$. In other words, the first term in the sum appeared in (2.57) dominates the sum. Thus,

$$1 - F_\beta(x) = 1 - e^{\log F_\beta(x)} \sim -\log F_\beta(x) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

as $x \rightarrow +\infty$. Observe that the above approximation is free of the choice of β . Since $F_{\sqrt{\alpha}}(x) = \Phi_{\alpha}(x)$ for $x > 0$. Replacing “ x ” by “ $\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x$,” we arrive at

$$\begin{aligned} & 1 - \Phi_{\alpha} \left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x \right) \\ &= 1 - F_{\sqrt{\alpha}} \left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x \right) \\ &\sim \frac{1}{\sqrt{2\pi}} \left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x \right)^{-1} \exp \left\{ - \left(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x \right)^2 / 2 \right\} \\ &\sim \frac{\sqrt{\alpha} e^{-\alpha/8}}{2\sqrt{2\pi}} \frac{1}{x \log x} e^{-2(\log x)^2/\alpha} \end{aligned}$$

as $x \rightarrow +\infty$. At last,

$$P(e^{N(0,1)} \geq x) = P(N(0, 1) \geq \log x) \sim \frac{1}{\sqrt{2\pi} \log x} e^{-(\log x)^2/2}$$

as $x \rightarrow +\infty$. This verifies (1.6) and the statement below. \square

Acknowledgements We thank Drs. Ming Gao, Wenqing Hu, Jing Wang, Ke Wang and Gongjun Xu for helping us check the proofs. We also thank a referee for her or his suggestion on drawing Fig. 1.

References

1. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions. Dover, New York (1972)
2. Akemann, G., Baik, J., Francesco, P.D.: The Oxford Handbook of Random Matrix Theory (Oxford Handbooks in Mathematics). Oxford University Press, Oxford (2001)
3. Akemann, G., Burda, Z.: Universal microscopic correlation functions for products of independent Ginibre matrices. *J. Phys. A: Math. Theor.* **45**(46), 465201 (2012)
4. Bai, Z.D.: Methodologies in spectral analysis of large dimensional random matrices, a review. *Stat. Sin.* **9**, 9611–9677 (1999)
5. Bai, Z.D., Yin, Y.Q., Krishnaiah, P.R.: On the limiting empirical distribution function of the eigenvalues of a multivariate F matrix. *Theory Probab. Appl.* **32**, 490–500 (1987)
6. Baik, J., Deift, P., Johansson, K.: On the distribution of the length of the longest increasing subsequence of random permutations. *J. Am. Math. Soc.* **12**(4), 1119–1178 (1999)
7. Balakrishnan, N., Cohen, A.C.: Order Statistics and Inference: Estimation Methods. Academic Press, San Diego (1991)
8. Bordenave, C.: On the spectrum of sum and product of non-Hermitian random matrices. *Electron. Commun. Probab.* **16**, 104–113 (2011)
9. Burda, Z.: Free products of large random matrices—a short review of recent developments. *J. Phys. Conf. Ser.* **473**, 012002 (2013). Also available at <http://arxiv.org/pdf/1309.2568v2>
10. Burda, Z., Janik, R.A., Waclaw, B.: Spectrum of the product of independent random Gaussian matrices. *Phys. Rev. E* **81**, 041132 (2010)
11. Chafaï, D., Péché, S.: A note on the second order universality at the edge of Coulomb gases on the plane. *J. Stat. Phys.* **156**(2), 368–383 (2014)
12. Chow, Y.S., Teicher, H.: Probability Theory: Independence, Interchangeability, Martingales, 3rd edn. Springer, Berlin (2003)
13. Collins, B.: Product of random projections, Jacobi ensembles and universality problems arising from free probability. *Probab. Theory Relat. Fields* **133**, 315–344 (2005)

14. Dembo, A., Zeitouni, O.: *Large Deviations Techniques and Applications*, 2nd edn. Springer, New York (1998)
15. Di Francesco, P., Gaudin, F.M., Itzykson, C., Lesage, F.: Laughlin's wave functions, Coulomb gases and expansions of the discriminant. *Int. J. Mod. Phys. A* **9**, 4257–4351 (1994)
16. Diaconis, P., Evans, S.: Linear functionals of eigenvalues of random matrices. *Trans. Am. Math. Soc.* **353**, 2615–2633 (2001)
17. Dong, Z., Jiang, T., Li, D.: Circular law and arc law for truncation of random unitary matrix. *J. Math. Phys.* **53**, 013301–013314 (2012)
18. Eaton, M.: *Multivariate Statistics: A Vector Space Approach*. IMS Lecture Notes Monograph, vol. 53, Institute of Mathematical Statistics, Beachwood, OH (2007)
19. Edelman, A.: The probability that a random real Gaussian matrix has k real eigenvalues, related distributions, and the circular law. *J. Multivar. Anal.* **60**, 203–232 (1997)
20. Erdős, L., Knowles, A., Yau, H., Yin, J.: Spectral statistics of Erdős–Rényi graphs II: eigenvalue spacing and the extreme eigenvalues. *Commun. Math. Phys.* **314**(3), 587–640 (2012)
21. Forrester, P.J., Mays, A.: Pfaffian point process for the Gaussian real generalised eigenvalue problem. *Probab. Theory Relat. Fields* **154**(1–2), 1–47 (2011)
22. Forrester, P.J., Nagao, T.: Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble. *J. Phys. A: Math. Theor.* **41**, 375003 (2008)
23. Ginibre, J.: Statistical ensembles of complex, quaternion, and real matrices. *J. Math. Phys.* **6**, 440–449 (1965)
24. Götze, F., Tikhomirov, T.: On the Asymptotic Spectrum of Products of Independent Random Matrices. (2010). <http://arxiv.org/pdf/1309.2568v2>
25. Haake, F.: *Quantum Signatures of Chaos*. Springer Series in Synergetics, vol. 54. Springer, New York (2010)
26. Hough, J.B., Krishnapur, M., Peres, Y., Virág, B.: *Zeros of Gaussian Analytic Functions and Determinantal Point Processes*. American Mathematical Society, Providence, RI (2009)
27. Jiang, T.: Approximation of Haar distributed matrices and limiting distributions of eigenvalues of Jacobi ensembles. *Probab. Theory Relat. Fields* **144**(1), 221–246 (2009)
28. Jiang, T.: The entries of Haar-invariant matrices from the classical compact groups. *J. Theor. Probab.* **23**(4), 1227–1243 (2010)
29. Johansson, K.: From Gumbel to Tracy–Widom. *Probab. Theory Relat. Fields* **138**, 75–112 (2007)
30. Johansson, K.: Shape fluctuations and random matrices. *Commun. Math. Phys.* **209**, 437–476 (2000)
31. Johnstone, I.: Multivariate analysis and Jacobi ensembles: largest eigenvalue, Tracy–Widom limits and rates of convergence. *Ann. Stat.* **36**(6), 2638–2716 (2008)
32. Johnstone, I.: On the distribution of the largest eigenvalue in principal components analysis. *Ann. Stat.* **29**, 295–327 (2001)
33. Khoruzhenko, B.A., Sommers, H.J.: *The Oxford Handbook of Random Matrix Theory* (Oxford Handbooks in Mathematics). Oxford University Press, New York (2001)
34. Kostlan, E.: On the spectra of Gaussian matrices. *Linear Algebra Appl* **162–164**, 385–388 (1992)
35. Krishnapur, M.: From random matrices to random analytic functions. *Ann. Probab.* **37**(1), 314–346 (2009)
36. Kuijlaars, A.B.J., López-García, A.: The normal matrix model with a monomial potential, a vector equilibrium problem, and multiple orthogonal polynomials on a star. *Nonlinearity* **28**, 347–406 (2015)
37. Lehmann, N., Sommers, H.J.: Eigenvalue statistics of random real matrices. *Phys. Rev. Lett.* **67**, 941–944 (1991)
38. O'Rourke, S., Soshnikov, A.: Products of independent non-Hermitian random matrices. *Electr. J. Probab.* **16**(81), 2219–2245 (2011)
39. O'Rourke, S., Renfrew, D., Soshnikov, A., Vu, V.: Products of Independent Elliptic Random Matrices. <http://arxiv.org/pdf/1403.6080v2> (2014)
40. Petrov, V.V.: *Sums of Independent Random Variables*. Springer, Berlin (1975)
41. Petz, D., Hiai, F.: Logarithmic energy as entropy functional. In: Carlen, E., Harrell, E.M., Loss, M. (eds.) *Advances in Differential Equations and Mathematical Physics*. Contemporary Math, vol. 217, pp. 205–221 (1998)
42. Ramírez, J., Rider, B., Virág, B.: Beta ensembles, stochastic Airy spectrum and a diffusion. *J. Am. Math. Soc.* **24**, 919–944 (2011)
43. Reiss, R.D.: Uniform approximation to distributions of extreme order statistics. *Adv. Appl. Probab.* **13**, 533–547 (1981)

44. Resnick, S.I.: *Extreme Values, Regular Variation and Point Processes*. Springer, New York (2007)
45. Rider, B.C.: A limit theorem at the edge of a non-Hermitian random matrix ensemble. *J. Phys. A* **36**(12), 3401–3409 (2003)
46. Rider, B.C.: Order statistics and Ginibre's ensembles. *J. Stat. Phys.* **114**, 1139–1148 (2004)
47. Rider, B.C., Sinclair, C.D.: Extremal laws for the real Ginibre ensemble. *Ann. Appl. Probab.* **24**(4), 1621–1651 (2014)
48. Tao, T., Vu, V.: Random matrices: universality of local eigenvalue statistics. *Acta Math.* **206**(1), 127–204 (2011)
49. Tracy, C.A., Widom, H.: Level-spacing distributions and Airy kernel. *Commun. Math. Phys.* **159**, 151–174 (1994)
50. Tracy, C.A., Widom, H.: On the orthogonal and symplectic matrix ensembles. *Commun. Math. Phys.* **177**, 727–754 (1996)
51. Tracy, C.A., Widom, H.: Distribution functions for largest eigenvalues and their applications. *Proc. ICM Beijing* **1**, 587–596 (2002)
52. Wachter, K.W.: The limiting empirical measure of multiple discriminant ratios. *Ann. Stat.* **8**, 937–957 (1980)
53. Życzkowski, K., Sommers, H.: Truncation of random unitary matrices. *J. Phys. A: Math. Gen.* **33**, 2045–2057 (2000)