# Spectral Radii of Large Non-Hermitian Random Matrices

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**Abstract**

By using the independence structure of points following a determinantal point process, we study the radii of the spherical ensemble, the truncation of the circular unitary ensemble and the product ensemble with parameter $n$ and $k$. The limiting distributions of the three radii are obtained. They are not the Tracy-Widom distribution. In particular, for the product ensemble, we show that the limiting distribution has a transition phenomenon: when $k/n \to 0$, $k/n \to \alpha \in (0, \infty)$ and $k/n \to \infty$, the liming distribution is the Gumbel distribution, a new distribution $\mu$ and the logarithmic normal distribution, respectively. The cumulative distribution function (cdf) of $\mu$ is the infinite product of some normal distribution functions. Another new distribution $\nu$ is also obtained for the spherical ensemble such that the cdf of $\nu$ is the infinite product of the cdfs of some Poisson-distributed random variables.

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# 1 Introduction

The largest eigenvalues of the three Hermitian matrices (Gaussian orthogonal ensemble, Gaussian unitary ensemble and Gaussian symplectic ensemble) are proved to converge to the Tracy-Widom laws by Tracy and Widom (1994, 1996). Since then there have been very active research in this direction. For example, Baik et al. (1999) establish a connection between the longest increasing subsequence problem and the Tracy-Widom law. The relationships among the largest eigenvalues, combinatorics, growth processes, random tilings and the determinantal point processes are found [see, e.g., Tracy-Widom (2002) and Johansson (2007) and the literature therein]. In the studies of the high-dimensional statistics, Johnstone (2001, 2008) and Jiang (2009) prove that the largest eigenvalues of Wishart and Jacobi matrices converge to the Tracy-Widom law. Ramírez et al. (2011) obtain the asymptotic distribution of the largest eigenvalues of beta-Hermite ensemble. Recently, a research interest is the universality of the largest eigenvalues of matrices with non-Gaussian entries; see, for example, Tao and Vu (2011), Erdős et al. (2012) and the references therein.

In this paper we will study the largest absolute values of the eigenvalues of some non-Hermitian matrices. Initiated by Ginibre (1965) for the study of Gaussian random matrices (real, complex and symplectic), the interest has continued and theoretical results are found to have many applications in quantum chromodynamics, chaotic quantum systems and growth processes; see more descriptions from the paper by Akemann, Baik and Francesco (2001). The applications also include dissipative quantum maps [Haake (2010)] and fractional quantum-Hall effect [Di Francesco et al. (1994)]. We refer the readers to Khoruzhenko and Sommers (2001) for more details.

For a matrix $\mathbf{M}$ with eigenvalues $z_1, \cdots, z_n$, the quantity $\max_{1\le j\le n}|z_j|$ is refereed to as the spectral radius of $\mathbf{M}$. In the pioneer work by Rider (2003, 2004) and Rider and Sinclair (2014), the spectral radii of the real, complex and symplectic Ginibre ensembles are studied. For the complex Ginibre ensemble, it is shown that the spectral radius converges to the Gumbel distribution. This phenomenon is very different from the Tracy-Widom distribution. The key observation is that the absolute values of the eigenvalues of the complex Ginibre ensemble are independent random variables with the Gamma distributions; see Kostlan (1992). Later it is found that the independence phenomenon is true not only for the complex Ginibre ensemble, but also true for other complex-valued determinantal point processes; see, for example, Hough et al. (2009) for further details.

In this paper, we will study the largest radii of three rotation-invariant and non-Hermitian random matrices: the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are independent complex Ginibre ensembles, the truncation of circular unitary ensemble, and the product ensemble $\prod_{j=1}^{k}\mathbf{X}_j$ where $\mathbf{X}_1, \cdots, \mathbf{X}_k$ are independent $n\times n$ complex Ginibre ensembles. The spectral radius of the first one converges to a new distribution $\nu$, that of the second one converges to the Gumbel distribution, and that of the third one, depending on the ratio

$\alpha := \lim_{n\to\infty} k_n/n$, converges to the Gumbel distribution when $\alpha = 0$, a new distribution $\mu$ when $\alpha \in (0, \infty)$ and the logarithmic normal distribution when $\alpha = \infty$.

Our analysis of the spectral radius is based on the following result. It is a special case of Theorem 1.2 from Chafaï and Péché (2014) which is another version of Theorem 4.7.1 from Hough et al. (2009).

**Lemma 1.1** *(Independence of radius) Assume the density function of* $(Z_1, \cdots, Z_n) \in \mathbb{C}^n$ *is proportional to* $\prod_{1\le j<k\le n} |z_j - z_k|^2 \cdot \prod_{j=1}^n \varphi(|z_j|)$*, where* $\varphi(x) \ge 0$ *for all* $x \ge 0$*. Let* $Y_1, \cdots, Y_n$ *be independent r.v.'s such that the density of* $Y_j$ *is proportional to* $y^{2j-1}\varphi(y)I(y \ge 0)$ *for each* $1 \le j \le n$*. Then,* $g(|Z_1|, \cdots, |Z_n|)$ *and* $g(Y_1, \cdots, Y_n)$ *have the same distribution for any symmetric function* $g(y_1, \cdots, y_n)$*.*

In this paper, we will study the limiting distributions of $\max_{1\le j\le n} |z_j|$ for three specific $\varphi$'s. In addition to Lemma 1.1, Chafaï and Péché (2014) also prove that the limiting distribution of $\max_{1\le j\le n} |z_j|$ is the Gumbel distribution under certain assumptions on $\varphi(x)$. However, their results do not apply to our three ensembles because their assumptions are not satisfied. See further elaborations in the section "strategy of the proofs".

Now we present our results on the three ensembles in Subsections 1.1, 1.2 and 1.3, respectively. After this the strategy of the proofs and some comments are given. We compare the limiting curves appearing in this paper in Figure 1.

### 1.1 Spherical Ensemble

Let $\mathbf{A}$ and $\mathbf{B}$ be two $n \times n$ matrices and all of the $2n^2$ entries of the matrices are i.i.d. $\mathbb{C}N(0, 1)$-distributed random variables. Then $\mathbf{A}^{-1}\mathbf{B}$ is called a spherical ensemble [Hough et al. (2009)]. It has a connection to the matrix $F$ distribution in statistics literature; see, for instance, p. 331 from Eaton (2007). Let $z_1, \cdots, z_n$ be the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$. Then their joint probability density function is given by

$$C \cdot \prod_{j<k} |z_j - z_k|^2 \cdot \prod_{k=1}^{n} \frac{1}{(1 + |z_k|^2)^{n+1}} \tag{1.1}$$

where $C$ is a normalizing constant; see, for example, Krishnapur (2009). The joint density of $z_1, \cdots, z_n$ of the real analogue of the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are i.i.d. real Ginibre ensembles, is given by Forrester and Nagao (2008) and Forrester and Mays (2011).

The empirical distribution of the eigenvalues has an asymptotic distribution $\mu$ with density $\frac{1}{\pi(1+|z|^2)^2}$ (Bordenave, 2011). When mapping the eigenvalues on the complex plane to the Riemann sphere through the stereographic projection, the induced (pushforward) measure of $\mu$ is the uniform distribution on the sphere.The spectral distribution of the singular values of $\mathbf{A}^{-1}\mathbf{B}$, which is the same as the eigenvalues of the $F$-matrix $(\mathbf{A}\mathbf{A}^*)^{-1}(\mathbf{B}\mathbf{B}^*)$, converges weakly to a non-random distribution; see, for instance, Wachter (1980) and Bai et al. (1987).

In this paper, we say

$$X_n \text{ converges weakly to the cdf } F(x) \text{ or a random variable } X$$

if the probability distribution of $X_n$ converges weakly to that generated by the cumulative distribution function (cdf) $F(x)$ or $X$. Now we study the spectral radius.

**THEOREM 1** *Let $z_1, \cdots, z_n$ have the density as in (1.1). Define $H_k(x) = e^{-x} \sum_{j=0}^{k-1} \frac{x^j}{j!}$ for $k \geq 1$. Then $\frac{1}{\sqrt{n}} \max_{1 \leq j \leq n} |z_j|$ converges weakly to probability distribution function $H(x) = \prod_{k=1}^{\infty} H_k(x^{-2})$ for $x > 0$ and $H(x) = 0$ for $x \leq 0$.*

Observe that $H_k(x)$ is the cdf $P(\text{Poi}(x) \leq k-1)$ for each $k \geq 1$, where $\text{Poi}(x)$ is a Poisson random variable with parameter $x > 0$. So $H(x)$ is the product of those cdfs evaluated at $x^{-2}$. Moreover, we have

$$1 - H(x) \sim \frac{1}{x^2} \tag{1.2}$$

as $x \to +\infty$. So $H(x)$ is heavy-tailed. This property will be verified in Section 2.4. It will be very helpful to provide some other descriptions of $H(x)$ in terms of, say, its Taylor expansion or a differential equation.

Johnstone (2008) proves that, under a trivial transformation, the largest singular value of the $F$-matrix $(\mathbf{AA}^*)^{-1}(\mathbf{BB}^*)$ asymptotically follows a Tracy-Widom distribution. Here, the spectral radius converges weakly to the new distribution $H(x)$.

## 1.2 Truncation of Circular Unitary Ensemble

Now we consider the truncation of the circular unitary ensemble. Let $\mathbf{U}$ be an $n \times n$ Haar-invariant unitary matrix [see, e.g., Diaconis and Evans (2001) and Jiang (2009, 2010)]. For $n > p \geq 1$, write

$$\mathbf{U} = \begin{pmatrix} \mathbf{A} & \mathbf{C}^* \\ \mathbf{B} & \mathbf{D} \end{pmatrix}$$

where $\mathbf{A}$, as a truncation of $\mathbf{U}$, is a $p \times p$ submatrix. Let $z_1, \cdots, z_p$ be the eigenvalues of $\mathbf{A}$. It is known from Zyczkowski and Sommers (2000) that their density function is

$$C \cdot \prod_{1 \leq j < k \leq p} |z_j - z_k|^2 \prod_{j=1}^{p} (1 - |z_j|^2)^{n-p-1} \tag{1.3}$$

where $C$ is a normalizing constant. Assuming $c = \lim \frac{p}{n}$, Życzkowski and Sommers (2000) show that the empirical distribution of $z_i$'s converges to the distribution with density proportional to $\frac{1}{(1-|z|^2)^2}$ for $|z| \leq c$ if $c \in (0, 1)$. Dong et al. (2012) prove that the empirical distribution goes to the circular law and the arc law as $c = 0$ and $c = 1$, respectively.

Collins (2005) proves that $\mathbf{A}^*\mathbf{A}$ forms a Jacobi ensemble. Johansson (2000) and Jiang (2009) show that a transform of the largest eigenvalue of $\mathbf{A}^*\mathbf{A}$ converges weakly to the Tracy-Widom distribution. For the spectral radius $\max_{1\le j\le p}|z_j|$ of $\mathbf{A}$ itself, we obtain the following result.

**THEOREM 2** *Assume that $z_1, \cdots, z_p$ have density as in (1.3) and there exist constants $h_1, h_2 \in (0,1)$ such that $h_1 < \frac{p}{n} < h_2$ for all $n \ge 2$. Then $(\max_{1\le j\le p}|z_j| - A_n)/B_n$ converges weakly to the cdf $\Lambda(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, where $A_n = c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$, $B_n = \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,*

$$c_n = \Big(\frac{p-1}{n-1}\Big)^{1/2}, \quad b_n = b\Big(\frac{nc_n^2}{1-c_n^2}\Big), \quad a_n = a\Big(\frac{nc_n^2}{1-c_n^2}\Big)$$

*with*

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2}\log(\sqrt{2\pi}\log y) \quad \text{and} \quad b(y) = (\log y)^{-1/2}$$

*for $y > 3$.*

Trivially, in the above theorem, $\{A_n; n \ge 3\}$ is bounded and $B_n$ has the scale of $(n\log n)^{-1/2}$.

### 1.3 Product Ensemble

Given integer $k \ge 1$. Assume $\mathbf{X}_1, \cdots, \mathbf{X}_k$ are i.i.d. $n \times n$ random matrices and the $n^2$ entries of $\mathbf{X}_1$ are i.i.d. with distribution $\mathbb{C}N(0,1)$. Let $z_1, \cdots, z_n$ be the eigenvalues of the product $\prod_{j=1}^k \mathbf{X}_j$. It is known that their joint density function is

$$C \prod_{1\le j<l\le n} |z_j - z_l|^2 \prod_{j=1}^n w_k(|z_j|) \tag{1.4}$$

where $C$ is a normalizing constant and $w_k(z)$ has a recursive formula given by $w_1(z) = \exp(-|z|^2)$ and

$$w_k(z) = 2\pi \int_0^\infty w_{k-1}\Big(\frac{z}{r}\Big)\exp(-r^2)\frac{dr}{r}$$

for all integer $k \ge 2$; see, e.g., Akemann and Burda (2012). The function $w_k(z)$ also has a representation in terms of the so-called Meijer G-function; see the previous reference.

Götze and Tikhomirov (2010) prove that the mean of the empirical distribution of $\{z_j/n^{k/2},\ 1 \le j \le n\}$ converges to a distribution with density $\frac{1}{k\pi}|z|^{\frac{2}{k}-2}$ for $|z| \le 1$. Later, Bordenave (2011), O'Rourke and Soshnikov (2011) and O'Rourke et al. (2014) further generalize this result to the almost sure convergence. The Gaussian case was first considered by Burda et al. (2010) and Burda (2013) through investigating the limit of the kernel of a determinantal point process. Second, the empirical distribution of the singular values of

$\mathbf{X}_1\mathbf{X}_2/n$ converges weakly to a non-random distribution, see, for instance, Theorem 2.10 from Bai (1999).

Now we consider the largest radius and the result is given below. We allow $k$ changes with $n$ in this paper. First, we need some notation. Let $\Phi$ denote the cumulative distribution function of $N(0,1)$. For $\alpha \in (0,\infty)$, define

$$\Phi_\alpha(x) = \prod_{j=0}^{\infty} \Phi\Big(x + j\alpha^{1/2}\Big),$$

and $\Phi_\infty(x) = \Phi(x)$. The digamma function $\psi$ is defined by

$$\psi(z) = \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.5}$$

where $\Gamma(z)$ is the Gamma function.

**THEOREM 3** *Let $k = k_n$ be a sequence of positive integers. The following holds.*
*(a). If $\lim_{n\to\infty} k_n/n = 0$, particularly for $k_n \equiv k$, then $\alpha_n\big(n^{-k_n/2}\max_{1\le j\le n}|z_j| - 1\big) - \beta_n$ converges weakly to the cdf $\exp(-e^{-x})$, where*

$$\alpha_n = \Big(\frac{n}{k_n}\log\frac{n}{k_n}\Big)^{1/2} \quad \textit{and} \quad \beta_n = \log\frac{n}{k_n} - \log\log\frac{n}{k_n} - \frac{1}{2}\log(2\pi).$$

*(b). If $\lim_{n\to\infty} k_n/n = \alpha \in (0,\infty)$, then*

$$\frac{\max_{1\le j\le n}|z_j|}{n^{k_n/2}} \ \textit{converges weakly to the cdf}\ \ \Phi_\alpha\Big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2}\log x\Big), \qquad x > 0.$$

*(c). If $\lim_{n\to\infty} k_n/n = \infty$, then*

$$\frac{\max_{1\le j\le n}\log|z_j| - k_n\psi(n)/2}{\sqrt{k_n/n}/2} \ \textit{converges weakly to}\ N(0,1).$$

Taking $k = 1$ in (a) of Theorem 3, the corresponding limiting result is obtained by Rider (2003). Here we not only get the result for finite $k$, but for all possible range of $k_n$, which leads to the three transition zones: $k_n/n \to \alpha$ with $\alpha = 0$, $\alpha \in (0,\infty)$ and $\alpha = \infty$.

It is not known to the authors if Theorem 3 still holds if the entries of matrices $\mathbf{X}_i$'s are not Gaussian. In fact, it is not clear even for the largest radius of $\mathbf{X}_1$.

Theorem 3 gives the asymptotic distribution of the largest radius of the product of independent (non-Hermitian) Ginibre ensembles. A natural question is what if the Ginibre ensembles $X_j$'s are replaced by (Hermitian) Wigner matrices $Y_j := (X_j + X_j^*)/2$? As stated in the Introduction, the largest eigenvalue of each $Y_j$ asymptotically follows the Tracy-Widom law.

Second, let $\eta_{\max}(\mathbf{X}_i)$ be the largest singular value of $\mathbf{X}_i$ for each $i$. It is proved that $\eta_{\max}(\mathbf{X}_1)/\sqrt{n} \to 2$ in probability (see, e.g., Bai, 1999). Let $\lambda_1(\mathbf{X}_i), \cdots, \lambda_n(\mathbf{X}_i)$ be the eigenvalues of $\mathbf{X}_i$ for each $i$. Obviously, $\max_{1\le j\le n}|\lambda_j(\mathbf{X}_i)| \le \eta_{\max}(\mathbf{X}_i)$ for each $i$. From Rider (2003) or (a) of Theorem 3, we know $\max_{1\le j\le n}|\lambda_j(\mathbf{X}_i)|/\sqrt{n} \to 1$ in probability for each $i$. It is interesting to see the first limit is 2 and the second is 1. Now, the assertion (b) of Theorem 3 says that, though $\max_{1\le j\le n}|\lambda_j(\mathbf{X}_i)|/\sqrt{n} \to 1$ for each $i$, the radius of $\prod_{i=1}^k(\mathbf{X}_i/\sqrt{n}\,)$ goes to a distribution with support $[0,\infty)$.

Finally, let us look at the tail behavior of the distribution in (b) of Theorem 3. In fact we have

$$1-\Phi_\alpha\Big(\frac{1}{2}\alpha^{1/2}+2\alpha^{-1/2}\log y\Big) \quad\sim\quad \frac{C}{x\log x}e^{-2(\log x)^2/\alpha} \tag{1.6}$$

as $x\to+\infty$, where $C=\frac{\sqrt{\alpha}e^{-\alpha/8}}{2\sqrt{2\pi}}$. It is different from that of $e^{N(0,1)}$, the standard logarithmic normal distribution: $P(e^{N(0,1)}\ge x)\sim\frac{1}{\sqrt{2\pi}\,\log x}e^{-(\log x)^2/2}$ as $x\to+\infty$. This will be verified in Section 2.4.

**Plots of the density curves of the limiting spectral radii.** To visually compare the limiting distributions of the spectral radii, we plot the density curves in Fig. 1 for the following distributions: distribution $H(x)$ in Theorem 1, Gumbel distribution $\Lambda(x)=\exp(-e^{-x})$ in Theorems 2 and 3, and distributions $\widetilde{\Phi}_\alpha(x):=\Phi_\alpha\Big(\frac{1}{2}\alpha^{1/2}+2\alpha^{-1/2}\log x\Big)$ in Theorem 3.

**Strategy of the proofs**. By using Lemma 1.1, the absolute values of eigenvalues $|z_i|$'s are "independent". So we are dealing with the maxima of independent random variables with different distributions. The first step is to identify these distributions. In fact, we show that the largest radii in Theorems 1, 2 and 3 are essentially the maximum of independent random variables of density $y^{2j-1}(1+y^2)^{-(n+1)}$ for $1\le j\le n$ in (2.2), the maximum of independent random variables with beta distributions in (2.18), and the maximum of products of i.i.d. Gamma-distributed random variables in Lemma 2.4, respectively. The next step is the analysis of their tail probabilities through moderate deviations. This procedure costs the major efforts.

Assuming the function $\varphi(t)$ in Lemma 1.1 has the form of $e^{-nV(t)}$ ($V(t)$ not depending on $n$) with $V(t)=t^\alpha$ for some $\alpha\ge 1$ or $V(t)$ being a special type of convex functions, Chafaï and Péché (2014) shows that the limiting distribution of $\max_{1\le j\le n}|z_j|$ is always the Gumbel distribution. In our Theorems 1, 2 and 3, $V(t)$ either depends on $n$ or not the type of special convex functions required in their theorems. In particular, we see different limiting distributions in Theorems 1 and 3.

**Comments**:

1. It is noteworthy to mention that, though the main idea is analyzing the maxima of independent random variables, the proofs are not trivial. In the classical study of the maxima of i.i.d. random variables, the limiting distributions are only of three types: Fréchet dis-

tribution, Gumbel distribution and Weibull distribution; see, for example, Resnick (2007). However, the limiting distributions appeared in Theorems 1 and (b) of Theorem 3 are new.

2. The eigenvalues of the three random matrices investigated in this paper are rotation-invariant. This special property gives us the advantage of independence by Lemma 1.1. When the eigenvalues are not of the invariant property, it seems there have no good understanding on the largest radii. For example, there is no invariance property for $z_1, \cdots, z_n$ if they have the joint density

$$f(z_1, \cdots, z_n) = C \cdot \prod_{1 \le j < k \le n} |z_j - z_k|^2 \cdot \exp\Big\{ -n \sum_{j=1}^{n} \Big( \frac{(\mathrm{Re} z_j)^2}{1+\tau} + \frac{(\mathrm{Im} z_j)^2}{1-\tau} \Big) \Big\}$$

where $\tau \in (-1, 1)$ is a parameter and $C$ is a normalizing constant [Lemma 4 from Petz and Hiai (1998)]. See also similar examples on page 3403 from Rider (2003) and (1.1) from Kuijlaars and López (2015).

3. In this paper, we work on matrices with complex Gaussian entries. A similar study may be done for matrices with real and symplectic Gaussian random variables. For example, we know from Ginibre (1965), Lehmann and Sommers (1991) and Edelman (1997) that the densities of the eigenvalues of real and symplectic Ginibre ensembles are also explicit. Rider (2003) and Rider and Sinclair (2014) obtain the limiting distributions of the largest radii for the real and symplectic cases. It is possible that our current work can be carried out to the three real and symplectic analogues: the spherical ensemble $\mathbf{A}^{-1}\mathbf{B}$ where $\mathbf{A}$ and $\mathbf{B}$ are real or symplectic Ginibre ensembles [further information can be seen from Forrester and Nagao (2008) and Forrester and Mays (2011)], the truncation of Haar-invariant orthogonal or symplectic matrices [see, e.g., Jiang (2010)] and $\prod_{j=1}^{k} \mathbf{X}_j$ where $\mathbf{X}_1, \cdots, \mathbf{X}_k$ are i.i.d. real or symplectic Ginibre ensembles.

4. Tracy and Widom (1994, 1996) prove that the largest eigenvalues of the Gaussian orthogonal, unitary and symplectic ensembles converge to the Tracy-Widom laws. Recently there have been an active research on the universality of the eigenvalues of non-Gaussian matrices; see, for example, Tao and Vu (2011), Erdős et al. (2012) and the references therein. In particular, Erdős et al. generalize the results by Tracy-Widom to the matrices with non-Gaussian entries. Our Theorems 1, 2 and 3 consider the eigenvalues of matrices with Gaussian entries. It will be interesting to study the universality of the three results for the matrices with non-Gaussian entries.

Finally, the organization of the rest of paper is as follows. We will prove Theorems 1, 2 and 3 in Sections 2.1, 2.2 and 2.3, respectively. The verifications of (1.2) and (1.6) are given in Section 2.4.

Plots of density curves for limiting spectral radii

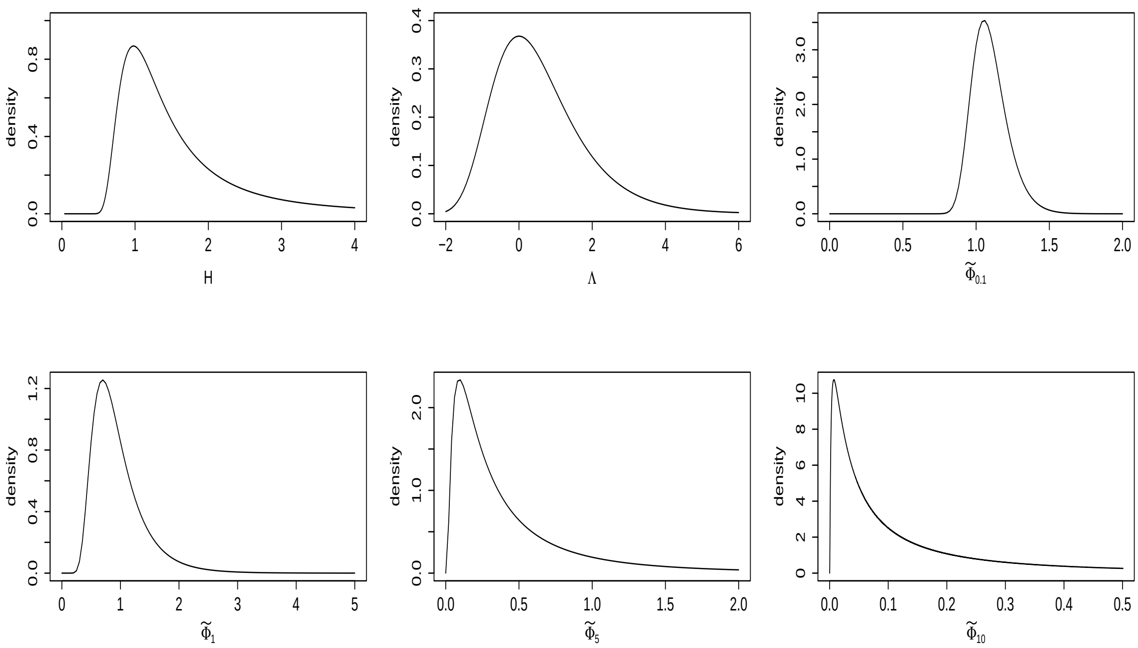


Figure 1: *Distribution* $H(x)$ *is given in Theorem 1, Gumbel distribution* $\Lambda(x) = \exp(-e^{-x})$ *is given in Theorems 2 and 3, and distributions* $\widetilde{\Phi}_\alpha(x) := \Phi_\alpha\big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2}\log x\big)$ *for* $\alpha = 0.1, 1, 5, 10$ *are given in Theorem 3.*

# 2 Proofs

In this section, we will prove Theorems 1, 2 and 3 in each subsection.

## 2.1 The Proof of Theorem 1

We start with a lemma.

**Lemma 2.1** *Let* $a_{ni} \in [0, 1)$ *be constants for* $i \geq 1, n \geq 1$ *and* $\sup_{n\geq 1, i\geq 1} a_{ni} < 1$. *For each* $i \geq 1$, $a_i := \lim_{n\to\infty} a_{ni}$. *Assume* $c_n := \sum_{i=1}^{\infty} a_{ni} < \infty$ *for each* $n \geq 1$ *and* $c := \sum_{i=1}^{\infty} a_i < \infty$, *and* $\lim_{n\to\infty} c_n = c$. *Then*

$$\lim_{n\to\infty} \prod_{i=1}^{\infty}(1 - a_{ni}) = \prod_{i=1}^{\infty}(1 - a_i).$$

**Proof.** Note that $\prod_{i=1}^{\infty}(1 - a_{ni})$ and $\prod_{i=1}^{\infty}(1 - a_i)$ are well defined, and $\prod_{i=1}^{\infty}(1 - a_{ni}) > 0$ for each $n \geq 1$ and $\prod_{i=1}^{\infty}(1 - a_i) > 0$. It suffices to show that

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} \log(1 - a_{ni}) = \sum_{i=1}^{\infty} \log(1 - a_i). \tag{2.1}$$

Note that for each fixed $k > 1$

$$\begin{aligned}
\limsup_{n\to\infty} \sum_{i=1}^{\infty} |a_{ni} - a_i| &\leq \limsup_{n\to\infty} \sum_{i=1}^{k} |a_{ni} - a_i| + \limsup_{n\to\infty} \sum_{i=k+1}^{\infty} |a_{ni} - a_i| \\
&= \limsup_{n\to\infty} \sum_{i=k+1}^{\infty} |a_{ni} - a_i| \\
&\leq \limsup_{n\to\infty} \sum_{i=k+1}^{\infty} a_{ni} + \sum_{i=k+1}^{\infty} a_i \\
&= \limsup_{n\to\infty} (c_n - \sum_{i=1}^{k} a_{ni}) + c - \sum_{i=1}^{k} a_i \\
&= 2(c - \sum_{i=1}^{k} a_i),
\end{aligned}$$

which goes to zero as $k \to \infty$. Therefore, we have

$$\lim_{n\to\infty} \sum_{i=1}^{\infty} |a_{ni} - a_i| = 0.$$

Set $a = \sup_{n\geq 1, i\geq 1} a_{ni}$. Then $0 \leq a < 1$. It follows that

$$\begin{aligned} |\sum_{i=1}^{\infty}\log(1-a_{ni}) - \sum_{i=1}^{\infty}\log(1-a_i)| &\leq \sum_{i=1}^{\infty}|\log(1-a_{ni}) - \log(1-a_i)| \\ &= \sum_{i=1}^{\infty}|\int_{a_i}^{a_{ni}}\frac{1}{1-t}dt| \\ &\leq \frac{1}{1-a}\sum_{i=1}^{\infty}|a_{ni}-a_i| \to 0 \end{aligned}$$

as $n\to\infty$, proving (2.1). □

**Proof of Theorem 1**. By Lemma 1.1 and (1.1), $\max_{1\leq j\leq n}|z_j|$ and $\max_{1\leq j\leq n} Y_{nj}$ have the same distribution, where $Y_{n1}, \cdots, Y_{nn}$ are independent such that $Y_{nj}$ has the probability density function (pdf) proportional to $y^{2j-1}(1+y^2)^{-(n+1)}I(y\geq 0)$ for $1\leq j\leq n$. Thus, to prove the theorem, it suffices to show

$$\lim_{n\to\infty} P\Big(\frac{1}{\sqrt{n}}\max_{1\leq j\leq n} Y_{nj}\leq x\Big) = H(x) \tag{2.2}$$

for each $x>0$.

Let $X_i$, $i\geq 1$ be a sequence of i.i.d. random variables with cumulative distribution function (cdf) $F$. Let $X_{1:n}\leq X_{2:n}\leq\cdots\leq X_{n:n}$ be the order statistics of $X_1, X_2, \cdots, X_n$ for each $n\geq 1$. Then from page 14 on the book by Balakrishnan and Cohen (1991), we know that the cdf of $X_{i:n}$ is given by

$$F_{i:n}(x) = \sum_{r=i}^{n}\binom{n}{r}F(x)^r(1-F(x))^{n-r} = \frac{n!}{(i-1)!(n-i)!}\int_0^{F(x)} t^{i-1}(1-t)^{n-i}dt \tag{2.3}$$

for each $1\leq i\leq n$. If $F$ has a probability density function $f$, then the pdf of $X_{i:n}$ is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}F(x)^{i-1}(1-F(x))^{n-i}f(x). \tag{2.4}$$

The monotonicity of the order statistics implies that $F_{i:n}(x)$ is non-increasing in $i$ for each $x$, that is,

$$F_{1:n}(x)\geq F_{2:n}(x)\geq\cdots\geq F_{n:n}(x). \tag{2.5}$$

Let $\{u_n,\ n\geq 1\}$ be a sequence of constants such that $\lim_{n\to\infty} n(1-F(u_n)) =: \tau\in(0,\infty)$. Write $\tau_n = n(1-F(u_n))$. Then it follows from the first equality in equation (2.3) that

$$
\begin{aligned}
F_{n-i+1:n}(u_n) &= \sum_{r=n-i+1}^{n} \binom{n}{r} F(u_n)^r (1-F(u_n))^{n-r} \\
&= \sum_{j=0}^{i-1} \binom{n}{j} \Big(1-\frac{\tau_n}{n}\Big)^{n-j} \Big(\frac{\tau_n}{n}\Big)^{j} \\
&= \sum_{j=0}^{i-1} \frac{1}{j!} \cdot \prod_{l=1}^{j-1} \Big(1-\frac{l}{n}\Big) \cdot \Big(1-\frac{\tau_n}{n}\Big)^{n-j} (\tau_n)^j \\
&\to e^{-\tau} \sum_{j=0}^{i-1} \frac{\tau^j}{j!} = H_i(\tau) \qquad (2.6)
\end{aligned}
$$

as $n \to \infty$ for each fixed integer $i \geq 1$.

Now, we take $F(y) = \frac{y^2}{1+y^2}$ for $y > 0$. Fix $x > 0$, set $u_n = u_n(x) = \sqrt{n}x$. Then $\lim_{n\to\infty} n(1-F(u_n)) = x^{-2}$. Then from (2.6)

$$
\lim_{n\to\infty} F_{n-i+1:n}(u_n) = H_i(x^{-2}) \qquad (2.7)
$$

for each fixed integer $i \geq 1$. For each $n \geq 1$, define

$$
a_{ni} = \begin{cases} 1 - F_{n-i+1:n}(u_n), & \text{if } 1 \leq i \leq n; \\ 0, & \text{if } i > n. \end{cases}
$$

Then it follows from (2.5) that $\sup_{n\geq 1, i \geq 1} a_{ni} = \sup_{n \geq 1} a_{n1}$. By the first identity in (2.3),

$$
a_{n1} = 1 - \Big(\frac{nx^2}{1+nx^2}\Big)^n = 1 - \Big(1 + \frac{x^{-2}}{n}\Big)^{-n}
$$

is increasing in $n$. Hence,

$$
\sup_{n\geq 1, i\geq 1} a_{ni} = \lim_{n\to\infty} a_{n1} = 1 - \exp(-x^{-2}) \in (0,1).
$$

From (2.7) we have $\lim_{n\to\infty} a_{ni} = 1 - H_i(x^{-2}) =: a_i$ for each $i \geq 1$. Moreover, we have

$$
\begin{aligned}
\sum_{i=1}^{\infty} a_{ni} &= E\Big[\sum_{i=1}^{n} I(X_{n-i+1:n} > u_n)\Big] \\
&= E\Big[\sum_{i=1}^{n} I(X_i > u_n)\Big] \\
&= n(1-F(u_n)) \to x^{-2}
\end{aligned}
$$

and

$$
\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} (1 - H_i(x^{-2})) = \exp(-x^{-2}) \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \frac{(x^{-2})^k}{k!}.
$$

By exchanging the ordering of the sums, we know $\sum_{i=1}^{\infty}\sum_{k=i}^{\infty}\frac{(x^{-2})^k}{k!}=\sum_{k=1}^{\infty}\sum_{i=1}^{k}\frac{(x^{-2})^k}{k!}=\sum_{k=1}^{\infty}\frac{(x^{-2})^k}{(k-1)!}=x^{-2}\exp\{x^{-2}\}$. It follows that $\sum_{i=1}^{\infty}a_i=x^{-2}$. By Lemma 2.1, we have

$$\lim_{n\to\infty}\prod_{i=1}^{n}F_{i:n}(\sqrt{n}x)=\lim_{n\to\infty}\prod_{i=1}^{\infty}(1-a_{ni})=\prod_{i=1}^{\infty}H_i(x^{-2})=H(x)\quad \text{for } x>0.$$

From (2.4) we obtain the pdf of $X_{j:n}$ given by

$$f_{j:n}(y)=\frac{2n!}{(j-1)!(n-j)!}y^{2j-1}(1+y^2)^{-(n+1)},\quad y>0,$$

which is also the pdf of $Y_{nj}$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty}P\Big(\frac{1}{\sqrt{n}}\max_{1\le j\le n}Y_{nj}\le x\Big) &= \lim_{n\to\infty}\prod_{j=1}^{n}P(Y_{nj}\le\sqrt{n}x)\\ &= \lim_{n\to\infty}\prod_{j=1}^{n}F_{j:n}(\sqrt{n}x)=H(x)\end{aligned}$$

for $x>0$. This completes the proof of Theorem 1. ■

## 2.2 The Proof of Theorem 2

In the rest of the paper, we will need the following notation. The symbol $C_n\sim D_n$ as $n\to\infty$ means that $\lim_{n\to\infty}\frac{C_n}{D_n}=1$. Similarly, $C_n(t)\sim D_n(t)$ uniformly over $t\in T_n$ if $\lim_{n\to\infty}\sup_{t\in T_n}|\frac{C_n(t)}{D_n(t)}-1|=0$. Also, $C_n(t)=O(D_n(t))$ uniformly over $t\in T_n$ if $\sup_{t\in T_n}|\frac{C_n(t)}{D_n(t)}|$ is bounded. We write $C_n(t)=o(D_n(t))$ uniformly over $t\in T_n$ if $\sup_{t\in T_n}|\frac{C_n(t)}{D_n(t)}|$ converges to zero as $n\to\infty$.

For random variables $\{X_n;\,n\ge1\}$ and constants $\{a_n;\,n\ge1\}$, we write $X_n=O_P(a_n)$ if $\lim_{x\to+\infty}\lim_{n\to\infty}P(|\frac{X_n}{a_n}|\ge x)=0$. In particular, if $X_n=O_P(a_n)$ and $\{b_n;\,n\ge1\}$ is a sequence of constants with $\lim_{n\to\infty}b_n=\infty$, then $\frac{X_n}{a_nb_n}\to0$ in probability as $n\to\infty$.

Recall $\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}\,dt$ for $x\in\mathbb{R}$. Let also $a(x)$ and $b(x)$ be as in Theorem 2.

**Lemma 2.2** *Let $\{j_n,n\ge1\}$ and $\{x_n,n\ge1\}$ be positive numbers with $\lim_{n\to\infty}x_n=\infty$ and $\lim_{n\to\infty}j_nx_n^{-1/2}(\log x_n)^{1/2}=\infty$. For fixed $y\in\mathbb{R}$, if $\{c_{n,j},1\le j\le j_n,n\ge1\}$ are real numbers such that $\lim_{n\to\infty}\max_{1\le j\le j_n}|c_{n,j}x_n^{1/2}-1|=0$, then*

$$\lim_{n\to\infty}\sum_{j=1}^{j_n}\big(1-\Phi((j-1)c_{n,j}+a(x_n)+b(x_n)y\big)=e^{-y};\tag{2.8}$$

$$\lim_{n\to\infty}\sum_{j=1}^{j_n}\frac{1}{(j-1)c_{n,j}+a(x_n)+b(x_n)y}\phi((j-1)c_{n,j}+a(x_n)+b(x_n)y\big)=e^{-y}.\tag{2.9}$$

**Proof.** From the definition, it is easy to see that $\lim_{n\to\infty} a(x_n) = +\infty$ and $\lim_{n\to\infty} b(x_n) = 0$ and $\min_{1\le j\le j_n} c_{n,j} > 0$ as $n$ is large enough. Thus, $\min_{1\le j\le j_n}[(j-1)c_{n,j} + a(x_n) + b(x_n)y] \to +\infty$ as $n \to \infty$. By using the integration by parts we see that $1 - \Phi(x) \sim \frac{\phi(x)}{x}\big(1 + O(x^{-2})\big)$ as $x \to \infty$. Therefore, (2.8) follows from (2.9). Now let us prove (2.9).

It is easy to verify that for $y \in \mathbb{R}$

$$\exp\big(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\big) \sim \frac{\sqrt{2\pi}\log x_n}{\sqrt{x_n}} e^{-y} \tag{2.10}$$

as $n \to \infty$. For large $n$, define

$$l_n = \text{ the integer part of } \min\Big\{\frac{j_n^{1/2} x_n^{1/4}}{(\log x_n)^{1/4}}, \frac{x_n^{1/2}}{(\log x_n)^{1/4}}\Big\}.$$

Then, as $n \to \infty$,

$$j_n > l_n \to \infty, \quad l_n x_n^{-1/2}(\log x_n)^{1/2} \to \infty \ \text{ and } \ l_n x_n^{-1/2} \to 0. \tag{2.11}$$

Fix $y \in \mathbb{R}$ and set

$$u_{nj} = (j-1)c_{n,j} + a(x_n) + b(x_n)y, \qquad 1 \le j \le j_n.$$

Then we conclude the following facts:
**Fact 1:** Uniformly over $1 \le j \le l_n$,

$$u_{nj} = (\log x_n)^{1/2}(1 + o(1)) \tag{2.12}$$

by using the third assertion in (2.11) and

$$u_{nj}^2 = (a(x_n) + b(x_n)y)^2 + 2(j-1)\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1)) + o(1); \tag{2.13}$$

**Fact 2:** Uniformly over $l_n < j \le j_n$, which is different from the assumption on (2.12) and (2.13),

$$u_{nj} \ge (\log x_n)^{1/2}(1 + o(1)) \tag{2.14}$$

and

$$u_{nj}^2 \ge (a(x_n) + b(x_n)y)^2 + 2(j-1)\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1)) + o(1). \tag{2.15}$$

It then follows from (2.12), (2.13) and (2.10) that

$$\begin{aligned}
\sum_{j=1}^{l_n} \frac{1}{u_{nj}}\phi(u_{nj}) \ &\sim \ \frac{\exp\big(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\big)}{(2\pi\log x_n)^{1/2}} \sum_{j=1}^{l_n} \exp\Big(-(j-1)\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\Big) \\
&\sim \ \frac{\exp\big(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\big)}{(2\pi\log x_n)^{1/2}} \frac{1}{1 - \exp\big(-\frac{(\log x_n)^{1/2}}{x_n^{1/2}}(1 + o(1))\big)} \qquad (2.16) \\
&\sim \ \frac{\exp\big(-\frac{1}{2}(a(x_n) + b(x_n)y)^2\big)}{(2\pi\log x_n)^{1/2}} \frac{x_n^{1/2}}{(\log x_n)^{1/2}} \sim e^{-y}, \qquad (2.17)
\end{aligned}$$

where the middle limit in (2.11) is used in the second step. Similarly, it follows from (2.14), (2.15) and (2.10) that

$$\begin{aligned}
&\sum_{j=l_n+1}^{j_n} \frac{1}{u_{nj}}\phi(u_{nj})\\
\leq\ & \frac{\exp\big(-\frac{1}{2}(a(x_n)+b(x_n)y)^2\big)}{(2\pi\log x_n)^{1/2}} \sum_{j=l_n+1}^{\infty} \exp\Big(-(j-1)\frac{(\log x_n)^{1/2}}{{x_n}^{1/2}}(1+o(1))\Big)\\
\leq\ & \frac{\exp\big(-\frac{1}{2}(a(x_n)+b(x_n)y)^2\big)}{(2\pi\log x_n)^{1/2}} \frac{1}{1-\exp\big(-\frac{(\log x_n)^{1/2}}{{x_n}^{1/2}}(1+o(1))\big)}\\
& \qquad\qquad\qquad \times\exp\Big(-l_n\frac{(\log x_n)^{1/2}}{{x_n}^{1/2}}(1+o(1))\Big)\\
=\ & O\Big(\exp\Big(-l_n\frac{(\log x_n)^{1/2}}{{x_n}^{1/2}}(1+o(1))\Big)\Big)\to 0
\end{aligned}$$

by using (2.16) and (2.17) in the equality and the middle assertion in (2.11) in the last step. By adding up the above equation and (2.17), we obtain (2.9). ■

Let $\{U_i;\ i\geq 1\}$ be a sequence of i.i.d. random variables uniformly distributed over $(0,1)$, and let $U_{1:n}\leq U_{2:n}\leq\cdots\leq U_{n:n}$ be the order statistics of $U_1, U_2,\cdots, U_n$ for each $n\geq 1$. Recall $\mathcal{B}$ is the collection of all Borel sets on $\mathbb{R}$. The following lemma is a special case of Proposition 2.10 from Reiss (1981).

**LEMMA 2.3** *There exists a constant $C>0$ such that for all $r>k\geq 1$,*

$$\begin{aligned}
&\sup_{B\in\mathcal{B}}\Big|P\Big(\frac{r^{3/2}}{\sqrt{(r-k)k}}\big(U_{r-k+1:r}-\frac{r-k}{r}\big)\in B\Big)-\int_B(1+l_1(t)+l_2(t))\phi(t)dt\Big|\\
\leq\ & C\cdot\Big(\frac{r}{(r-k)k}\Big)^{3/2}
\end{aligned}$$

*where for $i=1,2$, $l_i(t)$ is a polynomial in $t$ of degree $\leq 3i$, depending on $r$ and $k$, and all of its coefficients are of order $O((\frac{r}{(r-k)k})^{i/2})$.*

**Proof of Theorem 2**. Review the density formula in (1.3). Set $m_n=n-p$. For ease of notation, we sometimes write $m$ for $m_n$. By assumption, $h_2'<\frac{m_n}{n}<h_1'$ for all $n\geq 2$ where $h_i'=1-h_i\in(0,1)$ for $i=1,2$. Then we need to prove $(\max_{1\leq j\leq p}|z_j|-A_n)/B_n$ converges weakly to the cdf $\exp(-e^{-x})$, where $A_n=c_n+\frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n$, $B_n=\frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n$,

$$c_n=\Big(\frac{p-1}{n-1}\Big)^{1/2},\quad b_n=b\Big(\frac{nc_n^2}{1-c_n^2}\Big),\quad a_n=a\Big(\frac{nc_n^2}{1-c_n^2}\Big)$$

with

$$a(x)=(\log x)^{1/2}-(\log x)^{-1/2}\log(\sqrt{2\pi}\log x)\quad\text{and}\quad b(x)=(\log x)^{-1/2}$$

for $x > 3$. We proceed this through several steps.

*Step 1: Reducing the problem to the maximum of independent random variables with beta distributions.* Let $U_i$, $i \geq 1$ be a sequence of i.i.d. random variables uniformly distributed over $(0,1)$, and $U_{1:n} \leq U_{2:n} \leq \cdots \leq U_{n:n}$ be the order statistics of $U_1, U_2, \cdots, U_n$ for each $n \geq 1$. From (2.4), the density function of $U_{j:m_n+j-1}$ is

$$f_{j:m_n+j-1}(x) = \frac{(m_n+j-1)!}{(j-1)!(m_n-1)!} x^{j-1}(1-x)^{m_n-1}, \qquad x \in (0,1).$$

Denote the corresponding cdf as $F_{j:m_n+j-1}(x)$. Notice the pdf of $(U_{j:m_n+j-1})^{1/2}$ is proportional to $x^{2j-1}(1-x^2)^{m_n-1}$. For each $n \geq 2$, let $\{Y_{nj};\ 1 \leq j \leq p\}$ be independent random variables such that $Y_{nj}$ and $(U_{j:m_n+j-1})^{1/2}$ have the same distribution. By Lemma 1.1 and (1.1), $\max_{1\leq j\leq p}|z_j|$ and $\max_{1\leq j\leq p} Y_{nj}$ have the same distribution. We claim that, to prove the theorem, it suffices to show

$$\lim_{n\to\infty} P\Big(\max_{1\leq j\leq p} Y_{nj}^2 \leq \beta_n(x)\Big) = \exp(-e^{-x}) \tag{2.18}$$

for every $x \in \mathbb{R}$, where $\beta_n(x) = c_n^2 + c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}(a_n + b_n x)$. In fact, (2.18) implies that

$$W_n := \frac{1}{b_n}\left(c_n^{-1}(1-c_n^2)^{-1/2}(n-1)^{1/2}\big(\max_{1\leq j\leq p} Y_{nj}^2 - c_n^2\big) - a_n\right) \xrightarrow{d} \Lambda, \tag{2.19}$$

where $\Lambda$ is a probability distribution with cdf $\exp(-e^{-x})$, $x \in \mathbb{R}$. From Taylor's expansion

$$\begin{aligned}
\max_{1\leq j\leq p} Y_{nj} &= \Big(c_n^2 + c_n(1-c_n^2)^{1/2}(n-1)^{-1/2}(a_n + b_n W_n)\Big)^{1/2} \\
&= c_n\left(1 + c_n^{-1}(1-c_n^2)^{1/2}\frac{a_n + b_n W_n}{(n-1)^{1/2}}\right)^{1/2} \\
&= c_n\left(1 + \frac{1}{2}c_n^{-1}(1-c_n^2)^{1/2}\frac{a_n + b_n W_n}{(n-1)^{1/2}} + O_P\Big(\frac{a_n^2}{n-1}\Big)\right) \\
&= c_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}a_n + \frac{1}{2}(1-c_n^2)^{1/2}(n-1)^{-1/2}b_n W_n \\
&\qquad + O_P\Big(\frac{\log n}{n}\Big) \\
&= A_n + B_n W_n + O_p\Big(\frac{\log n}{n}\Big),
\end{aligned}$$

where we use the facts $a_n \to \infty$, $b_n \to 0$ and $c_n \in (0,1)$ in the above. Since $B_n$ has the scale of $(n\log n)^{-1/2}$, by (2.19),

$$\frac{\max_{1\leq j\leq p} Y_{nj} - A_n}{B_n} = W_n + O_p\Big(\frac{(\log n)^{3/2}}{n^{1/2}}\Big) \to \Lambda$$

weakly, which leads to the desired conclusion. Now we proceed to show (2.18).

*Step 2: A preparation.* We claim that

$$1 - F_{1:m_n}(x) \leq 1 - F_{2:m_n+1}(x) \leq \cdots \leq 1 - F_{p:m_n+p-1}(x) \tag{2.20}$$

for $x \in (0, 1)$. In fact, since for each $1 < j \leq p$,

$$U_{j:m_n+j-1} = \begin{cases} U_{j-1:m_n+j-2}, & \text{if } U_{m_n+j-1} \leq U_{j-1:m_n+j-2}; \\ \min(U_{j:m_n+j-2}, U_{m_n+j-1}), & \text{if } U_{m_n+j-1} > U_{j-1:m_n+j-2}, \end{cases}$$

which implies that $U_{j-1:m_n+j-2} \leq U_{j:m_n+j-1}$ for $1 < j \leq p$. This yields (2.20).

For each $n \geq 2$, set $a_{nj} = 1 - F_{p+1-j:m_n+p-j}(\beta_n(x)) = 1 - F_{p+1-j:n-j}(\beta_n(x))$ for $1 \leq j \leq p$. From (2.20), for each $n$, $a_{ni}$ is non-increasing in $i$. Since $Y_{nj}^2$ and $U_{j:m_n+j-1}$ are identically distributed, we have

$$P\Big(\max_{1\leq j\leq p} Y_{nj}^2 \leq \beta_n(x)\Big) = \prod_{j=1}^{p} P(Y_{nj}^2 \leq \beta_n(x)) = \prod_{j=1}^{p}(1 - a_{nj}). \tag{2.21}$$

It is easy to check the following holds: suppose $\{l_n;\, n \geq 1\}$ is sequence of positive integers. Let $z_{ni} \in [0, 1)$ be constants for all $1 \leq i \leq l_n$ with $\max_{1\leq i\leq l_n} z_{ni} \to 0$ and $\sum_{i=1}^{l_n} z_{ni} \to z \in [0, \infty)$. Then

$$\lim_{n\to\infty} \prod_{i=1}^{l_n}(1 - z_{ni}) = e^{-z}. \tag{2.22}$$

Next we will use (2.21) and (2.22) to prove (2.18). In fact, we only need to verify that

$$\sum_{j=1}^{p} a_{nj} \to e^{-x} \tag{2.23}$$

and

$$\max_{1\leq j\leq p} a_{nj} = a_{n1} \to 0.$$

*Step 3: The analysis of dominated terms in the maximum from (2.18).* Fix $\delta \in (\frac{1}{2}, \frac{2}{3})$. Let $j_n = [n^\delta]$, the integer part of $n^\delta$. For $1 \leq j \leq j_n$, define

$$u_{nj} = \frac{(n-j)^{3/2}}{((p-j)m_n)^{1/2}} \left(\beta_n(x) - \frac{p-j}{n-j}\right).$$

Meanwhile, we rewrite

$$\beta_n(x) = \frac{p-1}{n-1} + \frac{((p-1)m_n)^{1/2}}{(n-1)^{3/2}}(a_n + b_n x).$$

Then we see that uniformly over $1 \le j \le j_n$,

$$\begin{aligned} u_{nj} &= \Big(\frac{p-1}{n-1} - \frac{p-j}{n-j}\Big) \cdot \frac{(n-j)^{3/2}}{((p-j)m_n)^{1/2}} \\ &\qquad\qquad + \Big(\frac{n-j}{n-1}\Big)^{3/2} \cdot \Big(\frac{p-1}{p-j}\Big)^{1/2}(a_n + b_n x) \\ &= \Big(\frac{p-j}{p-1}\Big)^{-1/2} \cdot \Big(\frac{n-j}{n-1}\Big)^{1/2} \cdot \Big(\frac{n-p}{p-1}\Big)^{1/2} \cdot \Big(\frac{n-1}{n}\Big)^{-1/2} \cdot \frac{j-1}{n^{1/2}} \\ &\qquad\qquad + \Big(\frac{n-j}{n-1}\Big)^{3/2} \cdot \Big(\frac{p-j}{p-1}\Big)^{-1/2}(a_n + b_n x). \end{aligned}$$

Now, $\frac{n-p}{p-1} = \frac{1-c_n^2}{c_n^2}$. Also, given $\tau \in \mathbb{R}$, trivially $\Big(\frac{p-j}{p-1}\Big)^{\tau} = 1 + O(\frac{j}{n})$ and $\Big(\frac{n-j}{n-1}\Big)^{\tau} = 1 + O(\frac{j}{n})$ uniformly for all $1 \le j \le j_n$. Since $a_n \sim (\log n)^{1/2}$ and $b_n = o(1)$, we have

$$\begin{aligned} u_{nj} &= \frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n}(j-1)(1+o(1)) + a_n + b_n x + O\Big(\frac{j \log n}{n}\Big) \\ &= \frac{(1-c_n^2)^{1/2}}{n^{1/2}c_n}(j-1)(1+o(1)) + a_n + b_n x \end{aligned} \tag{2.24}$$

uniformly for all $1 \le j \le j_n$.

In Lemma 2.3, take $r = n - j$ and $k = n - p$ to have

$$\sup_{B \in \mathcal{B}} \Big| P(V_{p-j+1:n-j} \in B) - \int_B (1 + l_1(t) + l_2(t))\phi(t)dt \Big| = O(n^{-3/2})$$

uniformly over $1 \le j \le j_n$ as $n \to \infty$, where

$$V_{p-j+1:n-j} = \frac{(n-j)^{3/2}}{((n-j)(n-p))^{1/2}}\Big(U_{p-j+1:n-j} - \frac{p-j}{n-j}\Big)$$

and where, for $i = 1, 2$, $l_i(t)$ is a polynomial in $t$ of degree $\le 3i$, depending on $n$, and all of its coefficients are of order $O(n^{-i/2})$ by the assumption $h_1 < \frac{p}{n} < h_2$ for all $n \ge 2$. Now, by taking $B = (u_{nj}, \infty)$ we obtain

$$a_{nj} = P(V_{p-j+1:n-j} > u_{nj}) = \int_{u_{nj}}^{\infty} (1 + l_1(t) + l_2(t))\phi(t)dt + O(n^{-3/2})$$

uniformly for $1 \le j \le j_n$ as $n \to \infty$. From L'Hospital's rule, we have that for any $r \ge 0$

$$\int_x^{\infty} t^r \phi(t)dt \sim x^{r-1}\phi(x) \qquad \text{as } x \to \infty. \tag{2.25}$$

Since $\min_{1 \le j \le j_n} u_{nj} \to \infty$ as $n \to \infty$ by (2.24), it follows from (2.25) that

$$\int_{u_{nj}}^{\infty} t^r \phi(t)dt \sim (u_{nj})^{r-1}\phi(u_{nj}) = \begin{cases} \frac{\phi(u_{nj})}{u_{nj}}, & \text{if } r = 0; \\ O((\max_{1 \le j \le j_n} u_{nj})^r)\frac{\phi(u_{nj})}{u_{nj}}, & \text{if } r > 0 \end{cases}$$

holds uniformly over $1 \le j \le j_n$. Furthermore, since the coefficients of $l_i(t)$ are uniformly bounded by $O(n^{-i/2})$ for $i = 1, 2$, we have

$$\begin{aligned}
& \int_{u_{nj}}^{\infty} (1 + l_1(t) + l_2(t))\phi(t)dt \\
= & \Big[1 + O\Big(\frac{(\max_{1\le j\le j_n} u_{nj})^3}{n^{1/2}}\Big) + O\Big(\frac{(\max_{1\le j\le j_n} u_{nj})^6}{n}\Big)\Big] \frac{\phi(u_{nj})}{u_{nj}} \\
= & (1 + O(\frac{j_n^3}{n^2}))\frac{\phi(u_{nj})}{u_{nj}}
\end{aligned}$$

uniformly over $1 \le j \le j_n$, and thus obtain that

$$a_{nj} = \left(1 + O(\frac{j_n^3}{n^2})\right)\frac{\phi(u_{nj})}{u_{nj}} + O(n^{-3/2}) \tag{2.26}$$

uniformly over $1 \le j \le j_n$. Therefore, we have

$$\sum_{j=1}^{j_n} a_{nj} = (1 + o(1))\sum_{j=1}^{j_n}\frac{\phi(u_{nj})}{u_{nj}} + o(1).$$

In Lemma 2.2, by taking $x_n = nc_n^2/(1 - c_n^2)$ and $c_{nj} = x_n^{-1/2}(1 + o(1))$ where "$o(1)$" is as indicated in (2.24), we then get

$$\lim_{n\to\infty}\sum_{j=1}^{j_n}\frac{\phi(u_{nj})}{u_{nj}} = e^{-x}. \tag{2.27}$$

*Step 4: Some terms in the maximum from (2.18) are negligible.* From (2.24) again, we see

$$u_{nj_n}^2 \ge 2c_n^{-1}(1 - c_n^2)^{1/2}(a_n + b_n x)\frac{n^{\delta} - 2}{n^{1/2}}(1 + o(1)) \ge 6\log n$$

for all large $n$. Hence,

$$\phi(u_{nj_n}) \le \frac{1}{\sqrt{2\pi}}\exp(-3\log n) = \frac{1}{\sqrt{2\pi}n^3}$$

for all large $n$. Then it follows from (2.26) that $a_{nj_n} = O(n^{-3/2})$, and hence

$$\sum_{j=j_n+1}^{p} a_{nj} \le (n - j_n)a_{nj_n} = O(n^{-1/2}).$$

This together with (2.27) yields (2.23). The proof is then completed. ■

## 2.3 The Proof of Theorem 3

We begin with some preparation. The notation $k = k_n$ appears in the statement of Theorem 3. In the proof next we will write "$k_n$" if there is a danger of confusion, for example, a limit is taken for "$n \to \infty$". We write "$k$" in other occasions for clarity of formulas.

The following result characterizes the structure of the radius of the eigenvalues from the product ensemble.

**LEMMA 2.4** *Let $k$ and $z_1, \cdots, z_n$ be as in (1.4). Let $\{s_{j,r},\ 1 \le r \le k, j \ge 1\}$ be independent random variables and $s_{j,r}$ have the Gamma density $y^{j-1}e^{-y}I(y > 0)/(j-1)!$ for each $j$ and $r$. Then $\max_{1\le j\le n} |z_j|^2$ and $\max_{1\le j\le n} \prod_{r=1}^k s_{j,r}$ have the same distribution.*

**Proof**. Let $\{s_{j,r};\ 1 \le r \le k, j \ge 1\}$ be independent random variables and $s_{j,r}$ follow a Gamma($j$) distribution with density function $y^{j-1}e^{-y}I(y \ge 0)/\Gamma(j)$ for all $1 \le r \le k$ and $j \ge 1$. Define $v_1(y) = \exp(-y)$, $y > 0$, and set for $j \ge 2$

$$v_j(y) = \int_0^\infty v_{j-1}(y/s)\frac{e^{-s}}{s}ds. \tag{2.28}$$

One can easily verify that for each $j \ge 1$, $v_j(y)$ is proportional to $w_j(y^{1/2})$, i.e., for some constants $d_j > 0$,

$$w_j(y^{1/2}) = d_j v_j(y), \qquad y > 0. \tag{2.29}$$

Let $z$ be any complex number with $Re(z) > 0$, and define for $j \ge 1$

$$\gamma_j(z) = \int_0^\infty y^{z-1}v_j(y)dy.$$

Note that $\gamma_1(z) = \Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$. For $j \ge 2$, by using (2.28),

$$\begin{aligned}\gamma_j(z) &= \int_0^\infty \frac{e^{-s}}{s}\Big[\int_0^\infty y^{z-1}v_{j-1}\big(\frac{y}{s}\big)\,dy\Big]\,ds \\ &= \int_0^\infty s^{z-1}e^{-s}\Big[\int_0^\infty y^{z-1}v_{j-1}(y)\,dy\Big]\,ds = \Gamma(z)\gamma_{j-1}(z).\end{aligned}$$

Thus, we have

$$\gamma_j(z) = \Gamma(z)^j, \qquad j \ge 1. \tag{2.30}$$

Assume $Y_{nj}$, $1 \le j \le n$ are independent random variables, and for each $1 \le j \le n$, the density of $Y_{nj}$ is proportional to $y^{2j-1}w_k(y)$. By Lemma 1.1 and (1.1), $\max_{1\le j\le n} |z_j|^2$ and $\max_{1\le j\le n} Y_{nj}^2$ are identically distributed. Furthermore, since the density function of $Y_{nj}^2$, denoted by $f_j(y)$, is proportional to $y^{j-1}w_k(y^{1/2})$, and thus proportional to $y^{j-1}v_k(y)$ from (2.29), we have from (2.30) that

$$f_j(y) = \frac{y^{j-1}v_k(y)}{\int_0^\infty y^{j-1}v_k(y)dy} = \frac{y^{j-1}v_k(y)}{\Gamma(j)^k}, \quad y > 0,$$

for $1 \le j \le n$. Let the characteristic function of $\log Y_{nj}^2$ be denoted by $g_j(t)$. Then we have

$$g_j(t) = \frac{1}{\Gamma(j)^k} \int_0^{\infty} e^{it\log y} y^{j-1} v_k(y) dy = \frac{1}{\Gamma(j)^k} \int_0^{\infty} y^{j-1+it} v_k(y) dy = \left(\frac{\Gamma(j+it)}{\Gamma(j)}\right)^k$$

from (2.30). Since $\Gamma(j+it)/\Gamma(j)$ is the characteristic function of $\log s_{j,r}$, it follows that $\log Y_{nj}^2$ has the same distribution as that of $\sum_{r=1}^k \log s_{j,r}$, or equivalently, $Y_{nj}^2$ has the same distribution as that of $\prod_{r=1}^k s_{j,r}$ for $j \ge 1$. This implies the desired conclusion. ■

**Lemma 2.5** *Let $k$ be as in (1.4) and $\{s_{j,r},\ 1 \le r \le k, j \ge 1\}$ be independent r.v.'s such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y>0)/(j-1)!$ for all $j, r$. Set $\eta(x) = x - 1 - \log x$ and*

$$M_n(i) = \max_{n-i+1 \le j \le n} \Big| \sum_{r=1}^k \Big(\eta\Big(\frac{s_{j,r}}{j}\Big) - E\eta\Big(\frac{s_{j,r}}{j}\Big)\Big)\Big|, \quad 1 \le i \le n. \tag{2.31}$$

*Set $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$. Then for $1 \le i \le n$*

$$\Big|\max_{n-i+1 \le j \le n} \log \prod_{r=1}^k s_{j,r} - \max_{n-i+1 \le j \le n} \Big(\frac{1}{j}\sum_{r=1}^k (s_{j,r} - j) + k\,\psi(j)\Big)\Big| \le M_n(i).$$

**Proof**. Set $Y_j = \prod_{r=1}^k s_{j,r}$ for $j \ge 1$. Then,

$$\log Y_j = \sum_{r=1}^k \log s_{j,r}$$

for $j \ge 1$. The moment generating functions of $\log s_{j,r}$ is

$$m_j(t) = E(e^{t\log s_{j,r}}) = \frac{\Gamma(j+t)}{\Gamma(j)} \tag{2.32}$$

for $t > -j$. Therefore,

$$E(\log s_{j,r}) = \frac{d}{dt} m_j(t)|_{t=0} = \frac{\Gamma'(j)}{\Gamma(j)} = \psi(j)$$

by (1.5). Note that $\eta(x) = x - 1 - \log x$ for $x > 0$. Since $\eta(x) = \int_1^x \frac{s-1}{s} ds$, it is easy to verify that

$$0 \le \eta(x) \le \frac{(x-1)^2}{2\min(x,1)}, \qquad x > 0. \tag{2.33}$$

By using the expression $\log x = x - 1 - \eta(x)$ we can rewrite $\log Y_j$ as

$$\begin{aligned}
\log Y_j &= \sum_{r=1}^{k} \log \frac{s_{j,r}}{j} + k \log j \\
&= \sum_{r=1}^{k} \frac{s_{j,r} - j}{j} - \sum_{r=1}^{k} \eta\Big(\frac{s_{j,r}}{j}\Big) + k \log j \\
&= \frac{1}{j} \sum_{r=1}^{k} (s_{j,r} - j) - \sum_{r=1}^{k} \eta\Big(\frac{s_{j,r}}{j}\Big) + k \log j \\
&= \frac{1}{j} \sum_{r=1}^{k} (s_{j,r} - j) + k\psi(j) - \sum_{r=1}^{k} \eta\Big(\frac{s_{j,r}}{j}\Big) + k(\log j - \psi(j)).
\end{aligned}$$

Since $E(\log Y_j) = k\psi(j)$, we see that

$$\sum_{r=1}^{k} E\eta\Big(\frac{s_{j,r}}{j}\Big) = k(\log j - \psi(j))$$

and thus

$$\log Y_j = \frac{1}{j} \sum_{r=1}^{k} (s_{j,r} - j) + k\psi(j) - \sum_{r=1}^{k} \Big(\eta\Big(\frac{s_{j,r}}{j}\Big) - E\eta\Big(\frac{s_{j,r}}{j}\Big)\Big). \tag{2.34}$$

Note that for any two sequences of reals numbers $\{x_n\}$ and $\{y_n\}$,

$$|\max_{1 \le j \le n} x_j - \max_{1 \le j \le n} y_j| \le \max_{1 \le j \le n} |x_j - y_j|.$$

Then it follows from (2.34) that

$$\Big| \max_{n-i+1 \le j \le n} \log Y_j - \max_{n-i+1 \le j \le n} \Big( \frac{1}{j} \sum_{r=1}^{k} (s_{j,r} - j) + k\psi(j) \Big) \Big| \le M_n(i). \qquad \blacksquare$$

We estimate $M_n(\cdot)$ next.

**Lemma 2.6** *Let $k$ be as in (1.4) and $M_n(i)$ be defined as in Lemma 2.5. Assume $\{j_n;\ n \ge 1\}$ is a sequence of numbers satisfying $1 \le j_n \le \frac{1}{2}n$ for all $n$. Then, for any sequence of positive integers $\{k_n\}$, $M_n(j_n) = O_P\Big(\frac{j_n k_n^{1/2}}{n}\Big)$. Further, if $\lim_{n\to\infty} k_n/n = 0$, then $M_n(j_n) = O_P\Big(\frac{k_n \log n}{n}\Big)$.*

**Proof.** By using the Minkowski inequality and (2.33) we get

$$\begin{aligned}
E(M_n(j_n)) &\leq \sum_{n-j_n+1\leq j\leq n} E\sum_{r=1}^{k_n}\Big|\eta\Big(\frac{s_{j,r}}{j}\Big)-E\eta\Big(\frac{s_{j,r}}{j}\Big)\Big| \\
&\leq \sum_{n-j_n+1\leq j\leq n} \Big[E\Big(\sum_{r=1}^{k_n}\Big|\eta\Big(\frac{s_{j,r}}{j}\Big)-E\eta\Big(\frac{s_{j,r}}{j}\Big)\Big|\Big)^2\Big]^{1/2} \\
&\leq \sum_{n-j_n+1\leq j\leq n} \Big[\sum_{r=1}^{k_n}E\Big|\eta\Big(\frac{s_{j,r}}{j}\Big)-E\eta\Big(\frac{s_{j,r}}{j}\Big)\Big|^2\Big]^{1/2} \\
&\leq k_n^{1/2}\sum_{n-j_n+1\leq j\leq n}\Big(E\eta\Big(\frac{s_{j,1}}{j}\Big)^2\Big)^{1/2} \\
&\leq \frac{1}{2}k_n^{1/2}\sum_{n-j_n+1\leq j\leq n}\Big\{E\Big[\Big(\frac{s_{j,1}-j}{j}\Big)^4\Big(\min\Big(\frac{s_{j,1}}{j},1\Big)\Big)^{-2}\Big]\Big\}^{1/2}.
\end{aligned}$$

Since $s_{j,1}$ has density $y^{j-1}e^{-y}I(y>0)/(j-1)!$, we see that $E\big(s_{j,1}^{-4}\big)=\frac{\Gamma(j-4)}{\Gamma(j)}$. By the Marcinkiewicz-Zygmund inequality (see, for example, Corollary 2 from Chow and Teicher, 2003), we obtain $E(s_{j,1}-j)^8\leq Kj^4$ for any $j\geq 1$ where $K$ is a constant not depending on $j$. Then, it follows from Hölder's inequality that

$$\begin{aligned}
& E\Big(\Big(\frac{s_{j,1}-j}{j}\Big)^4\Big(\min\Big(\frac{s_{j,1}}{j},1\Big)\Big)^{-2}\Big) \\
\leq\ & \Big[E\Big(\frac{s_{j,1}-j}{j}\Big)^8\cdot E\Big(\min\Big(\frac{s_{j,1}}{j},1\Big)\Big)^{-4}\Big]^{1/2} \\
\leq\ & \Big[E\Big(\frac{s_{j,1}-j}{j}\Big)^8\cdot E\Big(\Big(\frac{j}{s_{j,1}}\Big)^4+1\Big)\Big]^{1/2} \\
\leq\ & \Big[\Big(\frac{j^3}{(j-1)(j-2)(j-3)}+1\Big)^{1/2}\Big(E\Big(\frac{s_{j,1}-j}{j}\Big)^8\Big]^{1/2} \\
\leq\ & Cj^{-2}
\end{aligned}$$

for any $j\geq 4$ where $C$ is a constant. Combining the last two assertions, we get $E(M_n(j_n))\leq O(\frac{j_nk_n^{1/2}}{n})$. This implies the first conclusion.

Now we prove the second one. Recall $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ for $x>0$ as in (1.5). By Formulas 6.3.18 and 6.4.12 from Abramowitz and Stegun (1972),

$$\psi(x)=\log x-\frac{1}{2x}+O\Big(\frac{1}{x^2}\Big)\quad\text{and}\quad\psi'(x)=\frac{1}{x}+\frac{1}{2x^2}+O\Big(\frac{1}{x^3}\Big)\tag{2.35}$$

as $x\to+\infty$. It is easy to check $E\log s_{j,1}=\frac{1}{\Gamma(j)}\int_0^\infty(\log y)y^{j-1}e^{-y}\,dy=\psi(j)$. Thus, from the first expression, we have

$$E\eta\Big(\frac{s_{j,1}}{j}\Big)=\log j-\psi(j)=O\Big(\frac{1}{j}\Big)$$

as $j \to \infty$. Hence, by (2.31),

$$M_n(j_n) \le \max_{n-j_n+1\le j\le n} \sum_{r=1}^{k_n} \eta\Big(\frac{s_{j,r}}{j}\Big) + O\Big(\frac{k_n}{n}\Big). \tag{2.36}$$

By Theorem 1 on page 217 from Petrov (1975), we have that

$$P(s_{j,1} > j + j^{1/2}x) = (1+o(1))(1-\Phi(x)) \tag{2.37}$$

uniformly for $x \in (0, a_n)$ and $n/2 \le j \le n$ as $n \to \infty$, where $\{a_n;\, n \ge 1\}$ is an arbitrarily given sequence of positive numbers with $a_n = o(n^{1/6})$. By taking $r = 0$ in (2.25), we see that $1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}\,x} e^{-x^2/2}$ as $x \to +\infty$. Now select $x = 2(\log n)^{1/2}$ in (2.37) to have

$$P\big(s_{j,1} > j + 2j^{1/2}(\log n)^{1/2}\big) = (1+o(1))\big(1 - \Phi(2(\log n)^{1/2}\big) = O(\frac{1}{n^2})$$

uniformly for $n/2 \le j \le n$ as $n \to \infty$. Similarly we have

$$P\big(s_{j,1} < j - 2j^{1/2}(\log n)^{1/2}\big) = (1+o(1))\big(1 - \Phi(2(\log n)^{1/2}\big) = O\Big(\frac{1}{n^2}\Big)$$

uniformly for $n/2 \le j \le n$ as $n \to \infty$. This implies

$$\sum_{r=1}^{k_n} \sum_{j=n-j_n+1}^{n} P\big(|s_{j,r} - j| > 2j^{1/2}(\log n)^{1/2}\big) = O\Big(\frac{j_n k_n}{n^2}\Big) = O\Big(\frac{k_n}{n}\Big) = o(1),$$

and thus we get

$$\max_{1\le r\le k_n} \max_{n-j_n+1\le j\le n} \Big|\frac{s_{j,r}}{j} - 1\Big| = O_P\Big(\frac{(\log n)^{1/2}}{n^{1/2}}\Big).$$

Consequently,

$$\min_{1\le r\le k_n} \min_{n-j_n+1\le j\le n} \frac{s_{j,r}}{j} = 1 + O_P\Big(\frac{(\log n)^{1/2}}{n^{1/2}}\Big).$$

By (2.36) and then (2.33), we obtain

$$\begin{aligned} M_n(j_n) &\le k_n \cdot \max_{n-j_n+1\le j\le n} \max_{1\le r\le k_n} \eta(\frac{s_{j,r}}{j}) + O(\frac{k_n}{n}) \\ &\le \frac{k_n}{2} \cdot \frac{\max_{n-j_n+1\le j\le n} \max_{1\le r\le k_n} |\frac{s_{j,r}}{j} - 1|^2}{\min\{1, \min_{n-j_n+1\le j\le n} \min_{1\le r\le k_n} \frac{s_{j,r}}{j}\}} + O\Big(\frac{k_n}{n}\Big) \\ &= O_P\Big(\frac{k_n \log n}{n}\Big), \end{aligned}$$

proving the second conclusion. ■

Review the notation we use before: $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for $x > 0$ as in (1.5) and

$$Y_j = \prod_{r=1}^{k} s_{j,r} \tag{2.38}$$

for $j \ge 1$, where $\{s_{j,r},\, 1 \le r \le k, j \ge 1\}$ are independent random variables such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y > 0)/(j-1)!$ for all $j, r$.

**LEMMA 2.7** *Let* $\{j_n;\, n\geq 1\}$ *and* $\{k_n;\, n\geq 1\}$ *be positive integers satisfying* $\lim_{n\to\infty}\frac{j_n}{n}=0$ *and* $\lim_{n\to\infty}(\frac{k_n}{n})^{1/2}\frac{j_n}{(\log n)^{1/2}}=\infty$. *Then for any* $x\in\mathbb{R}$,

$$\lim_{n\to\infty}\sum_{j=1}^{n-j_n}P\Big(\log Y_j>k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)=0. \tag{2.39}$$

**Proof.** Fix $x\in\mathbb{R}$. It follows from (2.32) that for each $1\leq j\leq n-j_n$ and any $t>0$,

$$\begin{aligned}
& P\Big(\log Y_j>k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)\\
\leq\ & \frac{E(e^{t\log Y_j})}{\exp\big\{t(k_n\psi(n)+(\frac{k_n}{n})^{1/2}x)\big\}}\\
=\ & \exp\Big\{k_n(\log(\Gamma(j+t)-\log\Gamma(j)))-t\Big(k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)\Big\}\\
=\ & \exp\Big\{k_n\int_0^t\psi(j+s)ds-t\Big(k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)\Big\}\\
=\ & \exp\Big\{k_n\int_0^t[\psi(j+s)-\psi(j)]ds-t\Big[k_n(\psi(n)-\psi(j))+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big]\Big\}.
\end{aligned}$$

From (2.35), there exist an integer $j_0$ such that for all $j_0\leq j\leq n-j_n$

$$\log\frac{j+s}{j}\leq\psi(j+s)-\psi(j)=\int_0^s\psi'(j+v)dv\leq\frac{1.1s}{j},\quad s\geq 0.$$

By the first inequality above, for all large $n$,

$$\psi(n)-\psi(j)\geq\log\frac{n}{j}\geq\log\frac{n}{n-j_n}=-\log(1-\frac{j_n}{n})\geq\frac{0.999j_n}{n},\quad j_0\leq j\leq n-j_n.$$

Hence, by assumption $(\frac{k_n}{n})^{1/2}=o(\frac{j_nk_n}{n})$ we see that

$$k_n(\psi(n)-\psi(j))+\Big(\frac{k_n}{n}\Big)^{1/2}x\geq 0.99k_n\log\frac{n}{j},\quad j_0\leq j\leq n-j_n$$

for all large $n$. Therefore we have for $j_0\leq j\leq n-j_n$

$$\begin{aligned}
& P\Big(\log Y_j>k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)\\
\leq\ & \exp\Big\{1.1k_n\int_0^t\frac{s}{j}ds-0.99tk_n(\log n-\log j)\Big\}\\
=\ & \exp\Big\{k_n\Big(\frac{1.1t^2}{2j}-0.99t(\log n-\log j)\Big)\Big\}
\end{aligned}$$

for all $t>0$ and large $n$ which does not depend on $t$. By selecting $t=0.99j(\log n-\log j)$ we have

$$\begin{aligned}
& P\Big(\log Y_j>k_n\psi(n)+\Big(\frac{k_n}{n}\Big)^{1/2}x\Big)\\
\leq\ & \exp\big\{-0.44k_nj(\log n-\log j)^2\big\},\quad j_0\leq j\leq n-j_n,
\end{aligned} \tag{2.40}$$

for all large $n$. Note that

$$\begin{aligned}
\min_{j_0 \le j \le n-j_n} j(\log n - \log j)^2 &\ge \min_{j_0 \le s \le n-j_n} s(\log n - \log s)^2 \\
&= \min_{j_0^{1/2} \le s \le (n-j_n)^{1/2}} s^2(\log n - 2\log s)^2 \\
&= \Big( \min_{j_0^{1/2} \le s \le (n-j_n)^{1/2}} s(\log n - 2\log s) \Big)^2,
\end{aligned}$$

where the last three minima are taken over all real numbers satisfying the corresponding constraints. It is easily seen that the minimum of $s(\log n - 2\log s)$ for $j_0^{1/2} \le s \le (n-j_n)^{1/2}$ is achieved at the two end points of the interval, $t = j_0^{1/2}$ or $s = (n-j_n)^{1/2}$. Thus, for all large $n$,

$$\begin{aligned}
\min_{j_0 \le j \le n-j_n} j(\log n - \log j)^2 &\ge \min\left\{ j_0(\log n - \log j_0)^2, (n-j_n)(\log n - \log(n-j_n))^2 \right\} \\
&\ge \min\Big\{ \frac{1}{2}(\log n)^2, \ \frac{1}{2}\frac{j_n^2}{n} \Big\}.
\end{aligned}$$

From the given condition $(\frac{k_n}{n})^{1/2}\frac{j_n}{(\log n)^{1/2}} = \infty$, we obtain

$$k_n \min_{j_0 \le j \le n-j_n} j(\log n - \log j)^2 \ge 10 \log n$$

for all large $n$. Therefore, combining all of the inequalities from (2.40) to the above, we have

$$\max_{j_0 \le j \le n-j_n} P\Big( \log Y_j > k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2} x \Big) \le \exp(-4.4 \log n) = n^{-4.4},$$

and hence

$$\sum_{j=j_0}^{n-j_n} P\Big( \log Y_j > k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2} x \Big) = O(n^{-3.4}) \to 0.$$

Finally, observe that, for each $1 \le j < j_0$, $\log Y_j$ is a sum of $k_n$'s many i.i.d. random variables with $Ee^{t\log Y_j} < \infty$ for all $|t| < \frac{1}{2}$. Then, by the Chernoff bound (see, for instance, p. 27 from Dembo and Zeitouni, 1998),

$$P\Big( \log Y_j > k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2} x \Big) \to 0.$$

The last two assertions imply the desired result. ■

Recall $\Lambda(x) = \exp(-e^{-x})$ for all $x \in \mathbb{R}$. We first prove the following proposition from which Theorem 3 will be obtained.

**PROPOSITION 2.1** *Let $\psi(x)$ be as in (1.5), $a(x)$ and $b(x)$ be as in Theorem 2.1, and $z_j$'s and $k_n$ be as in Theorem 3. Define $\Phi_0(y) = \Lambda(y)$, $a_n = a(n/k_n)$, $b_n = b(n/k_n)$ if $\alpha = 0$, and $a_n = 0$, $b_n = 1$ if $\alpha \in (0, \infty]$. Then*

$$\lim_{n\to\infty} P\Big( \frac{\max_{1\le j\le n} \log|z_j| - k_n\psi(n)/2}{(k_n/n)^{1/2}/2} \le a_n + b_n y \Big) = \Phi_\alpha(y), \quad y \in \mathbb{R}. \tag{2.41}$$

**Proof.** For each of the three cases: $\alpha = 0$, $\alpha \in (0, \infty)$, and $\alpha = \infty$ we will show that there exists a sequence of positive integers $\{j_n\}$ with $1 \le j_n \le n/2$ such that

$$\lim_{n\to\infty} \sum_{j=1}^{n-j_n} P\Big(\log Y_j > k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big) = 0, \quad y \in \mathbb{R}; \tag{2.42}$$

$$\frac{M_n(j_n)}{(k_n/n)^{1/2}b_n} \text{ converges in probability to zero} \tag{2.43}$$

where $M_n(\cdot)$ is defined as in Lemma 2.5, and

$$\begin{aligned}&\lim_{n\to\infty} P\Big(\frac{\max_{n-j_n+1\le j\le n}\big(\frac{1}{j}\sum_{r=1}^{k_n}(s_{j,r}-j) + k_n\psi(j)\big) - k_n\psi(n)}{(k_n/n)^{1/2}} \le a_n + b_n y\Big)\\ =\ & \Phi_\alpha(y)\end{aligned} \tag{2.44}$$

for $y \in \mathbb{R}$, where $\{s_{j,r},\ 1 \le r \le k, j \ge 1\}$ are independent random variables such that $s_{j,r}$ has density $y^{j-1}e^{-y}I(y > 0)/(j-1)!$ for all $j$ and $r$. In fact, (2.44) implies

$$\frac{\max_{n-j_n+1\le j\le n}\big(\frac{1}{j}\sum_{r=1}^{k_n}(s_{j,r}-j) + k_n\psi(j)\big) - k_n\psi(n)}{(k_n/n)^{1/2}b_n} - \frac{a_n}{b_n} \xrightarrow{d} \Phi_\alpha.$$

Review the definition of $Y_j$ in (2.38). The above result together with (2.43), Lemmas 2.4 and 2.5 implies that

$$\frac{\max_{n-j_n+1\le j\le n} \log Y_j - k_n\psi(n)}{(k_n/n)^{1/2}b_n} - \frac{a_n}{b_n} \xrightarrow{d} \Phi_\alpha.$$

Since (2.42) implies

$$\frac{\max_{1\le j\le n-j_n} \log Y_j - k_n\psi(n)}{(k_n/n)^{1/2}b_n} - \frac{a_n}{b_n} \xrightarrow{d} 0,$$

the two limits above imply (2.41) due to the fact that $\max_{1\le j\le n}\log|z_j|$ and $\frac{1}{2}\max_{1\le j\le n}\log Y_j$ are identically distributed by Lemma 2.4.

Now we start to verify equations (2.42)-(2.44) with a choice of $j_n$ given by

$$j_n = \text{ the integer part of } \Big(\frac{n}{k_n}\Big)^{1/2} n^{1/8} + 1 \tag{2.45}$$

for all large $n$.

*Proof of* (2.42). It is easy to verify that the conditions in Lemma 2.7 are satisfied, and thus (2.39) holds. In case $\alpha \in (0, \infty]$, $a_n = 0$ and $b_n = 1$, and (2.42) holds in this case. When $\alpha = 0$, $a_n + b_n y > 0$ for all large $n$, by applying (2.39) with $x = 0$ we have

$$\begin{aligned}&\lim_{n\to\infty} \sum_{j=1}^{n-j_n} P\Big(\log Y_j > k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big)\\ \le\ & \lim_{n\to\infty} \sum_{j=1}^{n-j_n} P(\log Y_j > k_n\psi(n)) = 0,\end{aligned}$$

that is, (2.42) holds. This completes the proof of (2.42) for all three cases.

*Proof of* (2.43). To prove (2.43), it suffices to show $M_n(j_n) = O_P((\frac{k_n}{n})^{1/2}(\log n)^{-1})$ since $b_n \geq (\log n)^{-1/2}$ for all large $n$. We use Lemma 2.6 this time. When $\alpha \in (0, \infty]$, $j_n = O_P(n^{1/8})$ from (2.45), and then we have from the first conclusion in Lemma 2.6 that

$$M_n(j_n) = O_P\Big(\frac{k_n^{1/2}}{n} j_n\Big) = O_P\Big(\Big(\frac{k_n}{n}\Big)^{1/2} n^{-3/8}\Big) = O_P\Big(\Big(\frac{k_n}{n}\Big)^{1/2} (\log n)^{-1}\Big).$$

When $\alpha = 0$, we have from the two conclusions in Lemma 2.6 that

$$\begin{aligned} M_n(j_n) &= O_P\Big(\min\Big\{\frac{j_n k_n^{1/2}}{n}, \frac{k_n \log n}{n}\Big\}\Big) \\ &= \Big(\frac{k_n}{n}\Big)^{1/2} \cdot O_P\Big(\min\Big\{\frac{n^{1/8}}{k_n^{1/2}}, \Big(\frac{k_n}{n}\Big)^{1/2} \log n\Big\}\Big) \\ &= \Big(\frac{k_n}{n}\Big)^{1/2} \cdot O_P\big(n^{-1/8}\big) \\ &= O_P\Big(\Big(\frac{k_n}{n}\Big)^{1/2} (\log n)^{-1}\Big) \end{aligned}$$

since $\frac{n^{1/8}}{k_n^{1/2}} \leq n^{-1/8}$ if $k_n \geq n^{1/2}$ and $\frac{k_n^{1/2} \log n}{n^{1/2}} \leq n^{-1/8}$ if $k_n < n^{1/2}$.

*Proof of* (2.44). Set $T_n(j_n) = \max_{n-j_n+1 \leq j \leq n} \big(\frac{1}{j}\sum_{r=1}^{k_n}(s_{j,r} - j) + k_n\psi(j)\big)$. Then

$$P\Big(T_n(j_n) \leq k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big) \tag{2.46}$$

$$= \prod_{j=n-j_n+1}^{n} P\Big(\sum_{r=1}^{k_n}(s_{j,r} - j) \leq jk_n(\psi(n) - \psi(j)) + j\Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big). \tag{2.47}$$

Notice $\sum_{r=1}^{k_n} s_{j,r}$ is a sum of $jk_n$ i.i.d. random variables with distribution Exp(1), that is, it has density $e^{-x}I(x \geq 0)$. Since the mean and the variance of Exp(1) are both equal to 1, we normalize the sum by

$$W_j := \frac{1}{\sqrt{jk_n}}\Big(\big(\sum_{r=1}^{k_n} s_{j,r}\big) - jk_n\Big).$$

By Theorem 1 on page 217 from Petrov (1975), for any sequence of positive numbers $\delta_n$ such that $\delta_n = o((nk_n)^{1/6})$,

$$P(W_j > x) = (1 + o(1))(1 - \Phi(x)) \tag{2.48}$$

uniformly over $x \in [0, \delta_n]$ and $n/2 \leq j \leq n$ as $n \to \infty$. Now reorganize the index in (2.47) to obtain

$$\begin{aligned}
& P\Big(T_n(j_n) \le k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big) \\
= & \prod_{i=1}^{j_n} P\Big(W_{n-i+1} \le \big((n-i+1)k_n\big)^{1/2}(\psi(n) - \psi(n-i+1)) \\
& \qquad\qquad\qquad + \Big(\frac{n-i+1}{n}\Big)^{1/2}(a_n + b_n y)\Big) \\
= & \prod_{i=1}^{j_n}(1 - a_{ni}), \qquad (2.49)
\end{aligned}$$

where $a_{ni} = P\big(W_{n-i+1} > x_{n,i}\big)$ and

$$x_{n,i} = ((n-i+1)k_n)^{1/2}(\psi(n) - \psi(n-i+1)) + \Big(1 - \frac{i-1}{n}\Big)^{1/2}(a_n + b_n y). \qquad (2.50)$$

Recalling (2.45), we know $j_n = o(n)$. From the second expression in (2.35) we have

$$\psi(n) - \psi(n-i+1) = \frac{i-1}{n}\Big(1 + O\Big(\frac{i}{n}\Big)\Big)$$

uniformly over $1 \le i \le j_n$ as $n \to \infty$. It follows that

$$\begin{aligned}
& ((n-i+1)k_n)^{1/2}(\psi(n) - \psi(n-i+1)) \\
= & \Big(\frac{k_n}{n}\Big)^{1/2}(i-1) \cdot \Big(1 - \frac{i-1}{n}\Big)^{1/2} \cdot \Big(1 + O\Big(\frac{i}{n}\Big)\Big) \\
= & \Big(\frac{k_n}{n}\Big)^{1/2}(i-1) \cdot \Big(1 + O\Big(\frac{i}{n}\Big)\Big) \\
= & \Big(\frac{k_n}{n}\Big)^{1/2}(i-1) \cdot \Big(1 + O\big(n^{-3/8}\big)\Big)
\end{aligned}$$

uniformly over $1 \le i \le j_n$ as $n \to \infty$. Since $a_n + b_n y = O((\log n)^{1/2})$,

$$\begin{aligned}
\Big(\Big(1 - \frac{i-1}{n}\Big)^{1/2} - 1\Big)(a_n + b_n y) &= (i-1) \cdot O\Big(\frac{(\log n)^{1/2}}{n}\Big) \\
&= \Big(\frac{k_n}{n}\Big)^{1/2}(i-1) \cdot O\Big(\frac{(\log n)^{1/2}}{n^{1/2}}\Big)
\end{aligned}$$

uniformly over $1 \le i \le j_n$. Therefore, by combining the above two expansions we get

$$x_{n,i} = \Big(1 + O\big(n^{-3/8}\big)\Big)\Big(\frac{k_n}{n}\Big)^{1/2}(i-1) + a_n + b_n y \qquad (2.51)$$

uniformly over $1 \le i \le j_n$. We emphasize the above is true when $i = j_n = 1$, which can be seen directly from (2.50). This fact will be used later.

Finally, we prove (2.44) by considering the three cases: $\alpha = 0$, $\alpha \in (0, \infty)$ and $\alpha = \infty$.

*Case 1:* $\alpha = 0$. Since

$$a_n = a(n/k_n) \sim (\log(n/k_n))^{1/2} \ \text{ and } \ b_n = b(n/k_n) = (\log(n/k_n))^{-1/2} \to 0,$$

we have

$$\min_{1\le i\le j_n} x_{n,i} \to \infty \text{ and } \max_{1\le i\le j_n} x_{n,i} = O\Big(\frac{k_n^{1/2} j_n}{n^{1/2}} + (\log n)^{1/2}\Big) = O\big(n^{1/8}\big).$$

It follows from (2.48) that

$$a_{ni} = (1+o(1))(1-\Phi(x_{n,i}))$$

uniformly over $1 \le i \le j_n$. In Lemma 2.2, choose $x_n = n/k_n$, $j_n$ as in (2.45) and $c_{nj} = (1+O(n^{-3/8}))\big(\frac{k_n}{n}\big)^{1/2}$ as in (2.51) to obtain

$$\sum_{i=1}^{j_n} a_{ni} = (1+o(1))\sum_{i=1}^{j_n}(1-\Phi(x_{n,i})) \to e^{-y}.$$

Further, it is easily seen that $\max_{1\le i\le j_n} a_{ni} \to 0$. Applying (2.22) to (2.49), we arrive at

$$P\Big(\frac{T_n(j_n) - k_n\psi(n)}{(\frac{k_n}{n})^{1/2}} \le a_n + b_n y\Big) \to e^{-e^{-y}} = \Phi_0(y), \quad y \in \mathbb{R},$$

that is, we get (2.44) for $\alpha = 0$.

*Case 2*: We see that $j_n \sim \alpha^{-1/2} n^{1/8}$ from (2.45). By definition, $a_n = 0$ and $b_n = 1$. Then it follows from (2.51) that

$$x_{n,i} = (1+o(1))\alpha^{1/2}(i-1) + y$$

holds uniformly over $1 \le i \le j_n$ as $n \to \infty$. We claim that

$$a_{ni} = (1+o(1))(1-\Phi(x_{n,i})) \tag{2.52}$$

uniformly over $1 \le i \le j_n$. In fact, review that (2.48) holds if $0 < x = o(n^{1/3})$. Evidently, $\max_{1\le i\le j_n}|x_{n,i}| = O(n^{1/8})$. But there is a possibility that $x_{n,i} < 0$ for small values of $i$. Let $j_0 > 1$ be an integer such that $\min_{j_0\le i\le j_n} x_{n,i} > 0$. Then we have from (2.48) that (2.52) holds uniformly over $j_0 \le i \le j_n$. By using the standard central limit theorem, we know (2.48) holds as well for each $i = 1, \cdots, j_0 - 1$. Therefore, for each $i \ge 1$,

$$\lim_{n\to\infty} a_{ni} = 1 - \Phi\big(\alpha^{1/2}(i-1)+y\big) \quad \text{and} \quad \sum_{i\ge 1}\big(1-\Phi\big(\alpha^{1/2}(i-1)+y\big)\big) < \infty \tag{2.53}$$

by the fact $1-\Phi(x) \sim \frac{1}{\sqrt{2\pi}x}e^{-x^2/2}$ as $x \to +\infty$. We now apply Lemma 2.1 to show (2.44). By defining $a_{ni} = 0$ for all $i > j_n$, with (2.53), we only need to verify the following two conditions: $\sup_{n\ge n_0, 1\le i\le j_n} a_{ni} < 1$ for some integer $n_0$ and $\lim_{n\to\infty}\sum_{i=1}^{j_n} a_{ni} = \sum_{i=1}^{\infty}(1 - \Phi(\alpha^{-1/2}(i-1)+y))$. The first one follows from (2.52) and the fact that $x_{n,i} \ge \frac{1}{2}\alpha^{1/2}(i-1) + y \ge y$ for $1 \le i \le j_n$ for all large $n$. The second condition can be easily verified by the dominated convergence theorem since $a_{ni} \le 2(1-\Phi(\frac{1}{2}\alpha^{1/2}(i-1)+y))$ for all $1 \le i \le j_n$ as $n$ is sufficiently large and $\sum_{i=1}^{\infty} 2(1-\Phi(\frac{1}{2}\alpha^{1/2}(i-1)+y) < \infty$.

*Case 3:* $\alpha = \infty$. From (2.45), $0 \le (\frac{k_n}{n})(j_n - 1) \le n^{1/8}$ and thus $x_{n,i} = O(n^{1/8})$ by (2.51). In particular, we have $x_{n,1} = y$, and for all large $n$, $x_{n,i} > 0$ if $2 \le i \le j_n$ and $j_n \ge 2$. Therefore,

$$a_{ni} = (1 + o(1))(1 - \Phi(x_{n,i}))$$

uniformly over $1 \le i \le j_n$ from (2.48). From (2.51), $a_{n1} \to 1 - \Phi(y)$ as $n \to \infty$. Obviously, $x_{n,i} \ge \frac{i}{3}(\frac{k_n}{n})^{1/2}$ if $2 \le i \le j_n$ and $j_n \ge 2$. Thus, use the fact $1 - \Phi(x) \sim \frac{1}{\sqrt{2\pi}\,x} e^{-x^2/2}$ as $x \to +\infty$ to see that, for large $n$,

$$\begin{aligned} I(j_n \ge 2) \sum_{i=2}^{j_n} a_{ni} &\le 2 \sum_{i=2}^{\infty} \Big(1 - \Phi\Big(\frac{i}{3}\Big(\frac{k_n}{n}\Big)^{1/2}\Big) \\ &\le \sum_{i=2}^{\infty} \exp\Big\{ - \frac{k_n}{18n} i^2 \Big\} \\ &\le \int_{-\infty}^{\infty} \exp\Big\{ - \frac{k_n}{18n} x^2 \Big\}\, dx = 3\sqrt{2\pi}\Big(\frac{n}{k_n}\Big)^{1/2} \end{aligned}$$

since $\exp\big\{ - \frac{k_n}{18n} i^2 \big\} \le \int_{i-1}^{i} \exp\big\{ - \frac{k_n}{18n} x^2 \big\}\, dx$ for all $i \ge 2$. Thus, $I(j_n \ge 2) \sum_{i=2}^{j_n} a_{ni} \to 0$. This and the fact $I(j_n \ge 2) \cdot \max_{2 \le i \le j_n} a_{ni} \to 0$ imply that $I(j_n \ge 2)\big(1 - \prod_{i=2}^{j_n}(1 - a_{ni})\big) \to 0$ as $n \to \infty$. So we have from (2.49) that

$$\begin{aligned} & P\Big(T_n(j_n) \le k_n \psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n y)\Big) \\ = & \prod_{i=1}^{j_n}(1 - a_{ni}) \\ = & (1 - a_{n1}) \cdot \Big[1 + I(j_n \ge 2)\big( - 1 + \prod_{i=2}^{j_n}(1 - a_{ni})\big)\Big] \to \Phi(y) = \Phi_\infty(y) \end{aligned}$$

as $n \to \infty$. Reviewing the notation of $T_n(j_n)$ defined above (2.46), we get (2.44) for the case $\alpha = \infty$. The proof of the proposition is then completed. ■

**Proof of Theorem 3**. We use the same notation as in Proposition 2.1. We first show the following:

(i) If $\lim_{n\to\infty} k_n/n = 0$, particularly for $k_n \equiv k$, then

$$\frac{2(n/k_n)^{1/2}}{b_n}\Big(\frac{\max_{1\le j\le n}|z_j|}{n^{k_n/2}} - 1\Big) - \frac{a_n}{b_n} \;\; \text{converges weakly to cdf } \;\exp(-e^{-x}). \qquad (2.54)$$

(ii) If $\lim_{n\to\infty} k_n/n = \alpha \in (0, \infty)$, then

$$\frac{\max_{1\le j\le n}|z_j|}{n^{k_n/2}} \;\; \text{converges weakly to cdf } \; \Phi_\alpha\Big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2}\log x\Big), \quad x > 0. \qquad (2.55)$$

To do so, for $\alpha \in [0, \infty)$, define

$$V_n = \frac{\max_{1\le j\le n} \log|z_j| - k_n\psi(n)/2}{(k_n/n)^{1/2} b_n/2} - \frac{a_n}{b_n}.$$

Then $V_n$ converges in distribution to $\Theta_\alpha$ by Proposition 2.1, where $\Theta_\alpha$ is a random variable with cdf $\Phi_\alpha(y)$. Trivially,

$$\begin{aligned}\max_{1\le j\le n} |z_j| &= \exp\Big\{\frac{1}{2}k_n\psi(n) + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2}(a_n + b_n V_n)\Big\} \\ &= \exp\Big\{\frac{1}{2}\Big(k_n\psi(n) + \Big(\frac{k_n}{n}\Big)^{1/2} a_n\Big)\Big\}\cdot\exp\Big\{\frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} b_n V_n\Big\}. \qquad (2.56)\end{aligned}$$

If $\alpha = 0$, then $\frac{k_n}{n} \to 0$, $a_n = a(\frac{n}{k_n}) \sim (\log\frac{n}{k_n})^{1/2} \to \infty$, $b_n = b(\frac{n}{k_n}) = (\log\frac{n}{k_n})^{-1/2} \to 0$, and $(\frac{k_n}{n})^{1/2} a_n \sim (\frac{k_n}{n})^{1/2} b_n^{-1} \to 0$ as $n\to\infty$. Using (2.35) and expanding (2.56) we get

$$\begin{aligned}\max_{1\le j\le n} |z_j| &= \exp\Big\{\frac{1}{2}k_n\log n + O\Big(\frac{k_n}{n}\Big) + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} a_n\Big\}\Big(1 + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} b_n V_n + O_P\Big(\frac{k_n b_n^2}{n}\Big)\Big) \\ &= n^{k_n/2}\Big(1 + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} a_n + O\Big(\frac{k_n}{n}\Big)\Big)\Big(1 + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} b_n V_n + O_P\Big(\frac{k_n}{n}\Big)\Big) \\ &= n^{k_n/2}\Big(1 + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} a_n + \frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} b_n V_n + O_P\Big(\frac{k_n a_n^2}{n}\Big)\Big),\end{aligned}$$

which yields that

$$\frac{2(n/k_n)^{1/2}}{b_n}\Big(\frac{\max_{1\le j\le n}|z_j|}{n^{k_n/2}} - 1\Big) - \frac{a_n}{b_n} = V_n + O_P\Big(\Big(\frac{k_n}{n}\Big)^{1/2}\Big(\log\frac{n}{k_n}\Big)^{3/2}\Big)$$

converges in distribution to $\Lambda$ by the Slutsky lemma. We obtain (2.54).

Now assume $\alpha \in (0, \infty)$. In this case, $a_n = 0$ and $b_n = 1$. Then from (2.56),

$$\max_{1\le j\le n} |z_j| = \exp\Big\{\frac{1}{2}k_n\psi(n)\Big\}\exp\Big\{\frac{1}{2}\Big(\frac{k_n}{n}\Big)^{1/2} V_n\Big\}.$$

Using expansion $\psi(n) = \log n - \frac{1}{2n} + O(\frac{1}{n^2})$ from (2.35) we have

$$\frac{\max_{1\le j\le n}|z_j|}{n^{k_n/2}} = \exp\Big\{-\frac{1}{4}\alpha + o(1)\Big\}\cdot\exp\Big\{\Big(\frac{1}{2}\alpha^{1/2} + o(1)\Big)V_n\Big\},$$

which converges weakly to the distribution of $e^{-\alpha/4}\exp\big(\frac{1}{2}\alpha^{1/2}\Theta_\alpha\big)$, given by $\Phi_\alpha(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2}\log y)$, $y > 0$. We get (2.55).

From (2.54) it is easy to see

$$\frac{2(n/k_n)^{1/2}}{b_n} = \Big(\frac{n}{k_n}\log\frac{n}{k_n}\Big)^{1/2} = \alpha_n \quad \text{and} \quad \frac{a_n}{b_n} = \log\frac{n}{k_n} - \log\log\frac{n}{k_n} - \frac{1}{2}\log(2\pi) = \beta_n.$$

Thus we obtain (a) of Theorem 3. The part (b) follows from (2.55) and the part (c) is yielded from Proposition 2.1 with $\Phi_\infty(x) = \Phi(x)$. This completes the proof of the theorem. ■

## 2.4 The Verifications of (1.2) and (1.6)

**Verification of (1.2)**. First, by the Taylor expansion,

$$
\begin{aligned}
H_k(y) = e^{-y}\sum_{j=0}^{k-1}\frac{y^j}{j!} &= 1 - e^{-y}\sum_{j=k}^{\infty}\frac{y^j}{j!} \\
&= 1 - \frac{y^k e^{-y}}{k!}\Big(1 + y\sum_{j=k+1}^{\infty}\frac{y^{j-k-1}}{(k+1)\cdots j}\Big)
\end{aligned}
$$

for all $y \in \mathbb{R}$ and $k \geq 1$. Notice that the absolute value of the above sum is bounded by $\sum_{j=k+1}^{\infty}\frac{1}{(k+1)\cdots j} \leq 1 + \sum_{j=1}^{\infty}\frac{1}{j^2} < \infty$ uniformly for all $k \geq 1$ and $|y| \leq 1$. This says

$$
H_k(y) = 1 - \frac{y^k}{k!}e^{-y}(1 + O(y)) = 1 - \frac{y^k}{k!}(1 + O(y))
$$

as $y \to 0$ uniformly for all $k \geq 1$. Hence

$$
\begin{aligned}
\log\prod_{k=1}^{\infty}H_k(y) &= \sum_{k=1}^{\infty}\log H_k(y) \\
&= -(1 + O(y))\sum_{k=1}^{\infty}\frac{y^k}{k!} = -y(1 + O(y))
\end{aligned}
$$

since $\sum_{k=1}^{\infty}\frac{y^k}{k!} = e^y - 1 \sim y$ as $y \to 0$. Therefore,

$$
1 - \prod_{k=1}^{\infty}H_k(y) = 1 - e^{-y(1+O(y))} \sim y
$$

as $y \to 0$. Taking $y = x^{-2}$ and letting $x \to \infty$, we get (1.2). ■

**Verification of (1.6)**. Given parameter $\beta > 0$, set

$$
F_\beta(x) = \prod_{j=0}^{\infty}\Phi\Big(x + \beta j\Big)
$$

for $x \in \mathbb{R}$. From integration by parts, we know $1 - \Phi(x) = \frac{1}{\sqrt{2\pi}\,x}e^{-x^2/2}(1 + O(x^{-2}))$ as $x \to +\infty$. Use $\log(1-t) = -t(1 + O(t))$ as $t \to 0$ to have

$$
\begin{aligned}
\log\Phi(x) &= -\frac{1}{\sqrt{2\pi}\,x}e^{-x^2/2}(1 + O(x^{-2}))\Big(1 + O\Big(\frac{1}{x}e^{-x^2/2}\Big)\Big) \\
&= -\frac{1}{\sqrt{2\pi}\,x}e^{-x^2/2}(1 + a(x))
\end{aligned}
$$

as $x \to +\infty$, where $a(x)$ is defined over $[1, \infty)$ and $|a(x)| \le Cx^{-2}$ for all $x \ge 1$ and $C$ is a constant not depending on $x$. Thus,

$$\begin{aligned}
\log F_\beta(x) &= \sum_{j=0}^{\infty} \log \Phi\Big(x + \beta j\Big) \\
&= -\frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{1}{x+\beta j} e^{-(x+\beta j)^2/2}(1 + a(x+\beta j)) \\
&= -(1+o(1)) \frac{1}{\sqrt{2\pi}} \sum_{j=0}^{\infty} \frac{1}{x+\beta j} e^{-(x+\beta j)^2/2} \qquad (2.57)
\end{aligned}$$

as $x \to +\infty$. Observe

$$\frac{1}{x+\beta(j+1)} e^{-(x+\beta(j+1))^2/2} \le \int_{x+\beta j}^{x+\beta(j+1)} \frac{1}{t} e^{-t^2/2}\, dt \le \frac{1}{x+\beta j} e^{-(x+\beta j)^2/2}$$

for all $x > 0$ and $j \ge 0$. Sum the above over all $j \ge 1$ to obtain

$$\int_{x+\beta}^{\infty} \frac{1}{t} e^{-t^2/2}\, dt \le \sum_{j=1}^{\infty} \frac{1}{x+\beta j} e^{-(x+\beta j)^2/2} \le \frac{1}{x+\beta} e^{-(x+\beta)^2/2} + \int_{x+\beta}^{\infty} \frac{1}{t} e^{-t^2/2}\, dt$$

for all $x > 0$. Write $\int_{x+\beta}^{\infty} \frac{1}{t} e^{-t^2/2}\, dt = -\int_{x+\beta}^{\infty} \frac{1}{t^2} (e^{-t^2/2})'\, dt$. From the integration by parts, $\int_{x+\beta}^{\infty} \frac{1}{t} e^{-t^2/2}\, dt \sim \frac{1}{(x+\beta)^2} e^{-(x+\beta)^2/2}$ as $x \to +\infty$. Since $\beta > 0$, we have $\frac{1}{(x+\beta)^2} e^{-(x+\beta)^2/2} = o\big(\frac{1}{x} e^{-x^2/2}\big)$ and $\frac{1}{x+\beta} e^{-(x+\beta)^2/2} = o\big(\frac{1}{x} e^{-x^2/2}\big)$ as $x \to +\infty$. It follows from (2.57) that

$$\log F_\beta(x) \sim -\frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

as $x \to +\infty$. In other words, the first term in the sum appeared in (2.57) dominates the sum. Thus,

$$1 - F_\beta(x) = 1 - e^{\log F_\beta(x)} \sim -\log F_\beta(x) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

as $x \to +\infty$. Observe that the above approximation is free of the choice of $\beta$. Since $F_{\sqrt{\alpha}}(x) = \Phi_\alpha(x)$ for $x > 0$. Replacing "$x$" by "$\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x$", we arrive at

$$\begin{aligned}
& 1 - \Phi_\alpha\big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\big) \\
= \;& 1 - F_{\sqrt{\alpha}}\big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\big) \\
\sim \;& \frac{1}{\sqrt{2\pi}} \big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\big)^{-1} \exp\Big\{ -\big(\frac{1}{2}\alpha^{1/2} + 2\alpha^{-1/2} \log x\big)^2/2 \Big\} \\
\sim \;& \frac{\sqrt{\alpha} e^{-\alpha/8}}{2\sqrt{2\pi}} \frac{1}{x \log x} e^{-2(\log x)^2/\alpha}
\end{aligned}$$

as $x \to +\infty$. At last

$$P\big(e^{N(0,1)} \geq x\big) = P(N(0,1) \geq \log x) \sim \frac{1}{\sqrt{2\pi}\,\log x} e^{-(\log x)^2/2}$$

as $x \to +\infty$. This verifies (1.6) and the statement below. ■

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