Moments of Traces for Circular Beta-ensembles

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Abstract Let $\theta_1, \cdots, \theta_n$ be random variables from Dyson's circular β-ensemble with probability density function $Const \cdot \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^{\beta}$. For each $n \ge 2$ and $\beta > 0$, we obtain some inequalities on $\mathbb{E}\big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\big]$, where $Z_n = (e^{i\theta_1}, \cdots, e^{i\theta_n})$ and p_μ is the power-sum symmetric function for partition μ. When $\beta = 2$, our inequalities recover an identity by Diaconis and Evans for Haar-invariant unitary matrices. Further, we have

$$(a) \quad \lim_{n\to\infty} \mathbb{E}\left[p_\mu(Z_n)\overline{p_\nu(Z_n)}\right] = \delta_{\mu\nu}\left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu \quad \text{for any } \beta > 0 \text{ and partitions } \mu, \nu;$$

$$(b) \quad \lim_{m\to\infty} \mathbb{E}\left[|p_m(Z_n)|^2\right] = n \quad \text{for any } \beta > 0 \text{ and } n \ge 2,$$

where $l(\mu)$ is the length of μ and z_μ is explicit on μ. These results apply to the three important ensembles: COE ($\beta = 1$), CUE ($\beta = 2$) and CSE ($\beta = 4$). The main tool is the Jack function.

1 Introduction

Let M_n be an $n \times n$ Haar-invariant unitary matrix, that is, the entries of unitary matrix M_n are random variables satisfying that the probability distribution of the entries of M_n is the same as that of UM_n and that of M_nU for any $n \times n$ unitary matrix U. Diaconis and Evans (Theorem 2.1 from [4]) proved that

(a) Consider $a = (a_1, \cdots, a_k)$ and $b = (b_1, \cdots, b_k)$ with $a_j, b_j \in \{0, 1, 2, \cdots\}$. Then for $n \ge \sum_{j=1}^k ja_j \vee \sum_{j=1}^k jb_j$,

$$\mathbb{E}\Big[\prod_{j=1}^{k} (\mathrm{Tr}(M_n^j))^{a_j}\overline{(\mathrm{Tr}(M_n^j))^{b_j}}\Big] = \delta_{ab}\prod_{j=1}^{k} j^{a_j} a_j! \tag{1.1}$$

where δ_{ab} is Kronecker's delta.

(b) For any positive integers j and k,

$$\mathbb{E}\big[\mathrm{Tr}(M_n^j)\overline{\mathrm{Tr}(M_n^k)}\big] = \delta_{jk} \cdot j \wedge n. \tag{1.2}$$

The idea of the proof is based on the group representation theory of unitary group $U(n)$. Some other derivations for (1.1) and (1.2) are given in [5, 20, 21, 22].

Notice an $n \times n$ Haar-invariant unitary matrix is also called a CUE, which belongs to the Circular Ensembles of three members: the Circular Orthogonal Ensemble (COE), the Circular Unitary

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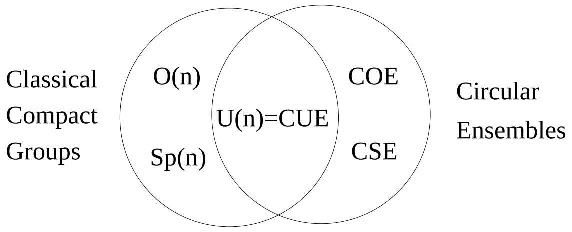

Figure 1: *Circular Ensembles and Haar-invariant Matrices from Classical Compact Groups*

Ensemble (CUE) and the Circular Symplectic Ensemble (CSE), see Figure 1 for the relationship, where the left circle consists of matrices which generate the Haar measures on the orthogonal group $O(n)$, Haar measure on the unitary group $U(n)$ and Haar measure on the real symplectic group $Sp(n)$, respectively.

Let $e^{i\theta_1}, \cdots, e^{i\theta_n}$ be the the eigenvalues of an $n \times n$ Haar-invariant unitary matrix, or equivalently, an $n \times n$ CUE, it is known (see, e.g., [10, 19]) that the density function of $\theta_1, \cdots \theta_n$ is $f(\theta_1, \cdots, \theta_n|\beta)$ with $\beta = 2$, where

$$f(\theta_1, \cdots, \theta_n|\beta) = (2\pi)^{-n} \cdot \frac{\Gamma(1+\beta/2)^n}{\Gamma(1+\beta n/2)} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta \tag{1.3}$$

with $\beta > 0$ and $\theta_i \in [0, 2\pi)$ for $1 \le i \le n$. The density function of $\theta_1, \cdots, \theta_n$ for the COE is $f(\theta_1, \cdots, \theta_n|\beta)$ with $\beta = 1$, and that for the CSE is $f(\theta_1, \cdots, \theta_n|\beta)$ with $\beta = 4$.

The purpose of this paper is to study the analogues of (1.1) and (1.2) for the circular β-ensembles with density function $f(\theta_1, \cdots, \theta_n|\beta)$ in (1.3) for any $\beta > 0$. Before stating the main results, we next introduce some background about the circular β-ensembles.

The circular ensembles were first introduced by physicist Dyson [6, 7, 8] for the study of nuclear scattering data. They are modifications of the Gaussian matrix ensembles. In fact, as studied in [6], Dyson shows that the consideration of time reversal symmetry leading to the three Gaussian ensembles behaves equally well to unitary matrices. A time reversal symmetry requires that $U = U^T$, no time reversal symmetry has no constraint, and a time reversal symmetry for a system with an odd number of spin 1/2 particles requires $U = U^D$, where D denotes the quaternion dual. Choosing such matrices with a uniform probability then gives COE, CUE and CSE, respectively (see, e.g., [9, 19]). The entries of COE and CUE are asymptotically complex normal random variables when the sizes of the matrices are large [13, 15, 16].

Let U be an $n \times n$ Haar-invariant unitary matrix. As mentioned earlier, U is also a CUE; the matrix $U^T U$ gives a COE. A similar but a bit more involved construction of CSE can be found in Ch. 9 from [19]. For the relations among the zonal polynomials, the Schur functions, the zonal spherical functions, the Gelfand pairs and the three circular ensembles, see, e.g., Chap. VII in [17] or Section 2.7 in [2] for reference.

Now we consider the moments in (1.1) and (1.2) for the circular β-ensembles. Taking $\beta = 1$ in (1.3), that is, choosing W_n such that it is an $n \times n$ COE, by an elementary check in Lemma 4.1, we have

$$\mathbb{E}\big[|\mathrm{Tr}(W_n)|^2\big] = \frac{2n}{n+1} \tag{1.4}$$

for all $n \geq 2$. This suggests that, unlike the right hand sides of (1.1) or (1.2) that are free of n, the moments for the general circular β-ensemble may depend on n for $\beta \neq 2$. In fact, by using the Jack functions, we will soon see from (1.8) below that the second moment in (1.4) does depend on n except $\beta = 2$, in which case W_n is an $n \times n$ CUE.

In this paper, we will prove some inequalities on the moments in (1.1) and (1.2) for the circular β-ensembles with arbitrary $\beta > 0$. In particular, some of our inequalities for $\beta = 2$ recover the equality in (1.1) by Diaconis and Evans [4]. Further, we evaluate the limiting behavior by letting $n \to \infty$ for the left hand side in (1.1) and $k \to \infty$ for the left hand side in (1.2) respectively. Their limits exist and look quite similar to the right hand sides of (1.1) and (1.2).

Now we state our main results. Let $\lambda = (\lambda_1, \lambda_2, \cdots)$ be a partition, that is, the sequence is in non-increasing order and only finite of λ_i's are non-zero. The weight of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Denote by $m_i(\lambda)$ the multiplicity of i in $(\lambda_1, \lambda_2, \cdots)$ for each i, and $l(\lambda)$ the length of λ: $l(\lambda) = m_1(\lambda) + m_2(\lambda) + \cdots$. Recall the convention $0! = 1$. Set

$$z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!. \tag{1.5}$$

Let $\rho = (\rho_1, \rho_2, \cdots)$ be a partition, and

$$p_\rho = \prod_{i=1}^{l(\rho)} p_{\rho_i}, \text{ where } p_k(x_1, x_2, \cdots) = x_1^k + x_2^k + \cdots \tag{1.6}$$

for integer $k \geq 1$ and indeterminates x_i's. The function p_ρ is called the power-sum symmetric function. For real number $\alpha > 0$, integers $K \geq 1$ and $n \geq 1$, define two constants $A = A(n, K, \alpha)$ and $B = B(n, K, \alpha)$ by

$$A = \Big(1 - \frac{|\alpha - 1|}{n - K + \alpha}\, \delta(\alpha \geq 1)\Big)^K \quad \text{and} \quad B = \Big(1 + \frac{|\alpha - 1|}{n - K + \alpha}\, \delta(\alpha < 1)\Big)^K, \tag{1.7}$$

where $\delta(\alpha \geq 1) = 1 - \delta(\alpha < 1)$ is 1 if $\alpha \geq 1$, or 0 otherwise. With these notation, our main result in this paper is stated as follows.

Theorem 1 *Let $\beta > 0$ and $\theta_1, \cdots, \theta_n$ have density $f(\theta_1, \cdots, \theta_n|\beta)$ as in (1.3). Set $Z_n = (e^{i\theta_1}, \cdots, e^{i\theta_n})$ and $\alpha = 2/\beta$. For partitions μ and ν, the following hold.*

(a) If $n \geq K = |\mu|$, then

$$A \leq \frac{\mathbb{E}\big[|p_\mu(Z_n)|^2\big]}{\alpha^{l(\mu)} z_\mu} \leq B.$$

(b) If $|\mu| \neq |\nu|$, then $\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] = 0$. If $\mu \neq \nu$ and $n \geq K = |\mu| \vee |\nu|$, then

$$\left|\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]\right| \leq \max\{|A-1|, |B-1|\} \cdot \alpha^{(l(\mu)+l(\nu))/2}(z_\mu z_\nu)^{1/2}.$$

(c) There exists a constant C depending only on β such that for any $m \geq 1$ and $n \geq 2$, we have

$$\Big|\mathbb{E}\big[|p_m(Z_n)|^2\big] - n\Big| \leq C\frac{n^3 2^{n\beta}}{m^{1\wedge\beta}}.$$

Take $\beta = 2$ in (a) and (b) of Theorem 1, then $A = 1$ and $B = 1$. The two results recover the result of Diaconis and Evans in (1.1). Further, letting $n \to \infty$ in (a) and (b) of Theorem 1, we see that A and B (depending on n) converge to 1; letting $m \to \infty$ in (c) of the theorem, then the last term in (c) goes to 0. So we obviously have the following results.

COROLLARY 1.1 *Let the conditions be as in Theorem 1. Then, for any $\beta > 0$,*

$$(a) \quad \lim_{n\to\infty} \mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] = \delta_{\mu\nu}\left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu;$$

$$(b) \quad \lim_{m\to\infty} \mathbb{E}\big[|p_m(Z_n)|^2\big] = n \quad \textit{for any } n \geq 2.$$

Part (b) of the above corollary says that, as $m \to \infty$, the limit of $\mathbb{E}\big[|p_m(Z_n)|^2\big]$ does not depend on parameter β, which is consistent with (1.2). By studying A and B in (1.7), we have the following corollary from Theorem 1.

COROLLARY 1.2 *Let $\beta > 0$ and $f(\theta_1, \cdots, \theta_n|\beta)$ be as in (1.3). Set $\alpha = 2/\beta$ and $Z_n = (e^{i\theta_1}, \cdots, e^{i\theta_n})$. Let μ and ν be partitions with $\mu \neq \nu$ and $K = |\mu| \vee |\nu|$. If $n \geq 2K$, then*

$$(a) \quad \Big|\frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1\Big| \leq \frac{6|1-\alpha|K}{n};$$

$$(b) \quad \Big|\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]\Big| \leq \frac{6|1-\alpha|K}{n} \cdot \alpha^{(l(\mu)+l(\nu))/2}(z_\mu z_\nu)^{1/2}.$$

The above results are in the forms of inequalities or limits. We actually derive an exact formula in (2.11) to compute $\mathbb{E}\big[|p_\mu(Z_n)|^2\big]$ for every partition μ. In general it is not easy to evaluate this quantity for arbitrary μ, however, we are able to do so when μ is special. For instance, by using the exact formula we calculate the moment in (1.4) for any $\beta > 0$ as follows.

Example For any $n \geq 1$,

$$\mathbb{E}[|p_1(Z_n)|^2] = \frac{2}{\beta}\frac{n}{n-1+2\beta^{-1}} = \begin{cases} \frac{2n}{n+1}, & \text{if } \beta = 1; \\ 1, & \text{if } \beta = 2; \\ \frac{n}{2n-1}, & \text{if } \beta = 4. \end{cases} \tag{1.8}$$

The verification of this formula through (2.11) is provided in Appendix. We also give $\mathbb{E}[|p_1(Z_n)|^4]$, $\mathbb{E}[|p_2(Z_n)|^2]$ and $\mathbb{E}[p_2(Z_n)\overline{p_1(Z_n)^2}]$ in closed forms in Appendix.

The main tool used in our proofs is the Jack functions. Diaconis and Evans [4] and Diaconis and Shahshahani [5] use the group representation theory to study (1.1) and (1.2) because $U(n)$ is a compact Lie group. The situations for the Circular Orthogonal Ensembles ($\beta = 1$) and the Circular Symplectic Ensembles ($\beta = 4$) are different. In fact, the two ensembles are not groups.

The proofs of (1.1) and (1.2), however, involve with the Schur functions, which have a strong relationship with the unitary groups. Looking at Figure 1, an Haar-invariant unitary matrix is also a

CUE. From the perspective of symmetric functions, the COE is connected to the zonal polynomials, and the CSE to zonal spherical functions. The three functions are special cases of the Jack polynomial $J_\lambda^{(\alpha)}$ with $\alpha = 1, 2$ and $1/2$, respectively, where λ is a partition. See Section 2 next for this or [17] for general properties of the Jack polynomials. By using the Jack functions, we are able to prove (a) and (b) in Theorem 1. Part (c) in the theorem is proved by evaluating the expectation/integral with respect to $f(\theta_1, \cdots, \theta_n|\beta)$ in (1.3) directly.

Remark Treating n as a variable, the bound $n^3 2^{n\beta} m^{-(1\wedge\beta)}$ in (c) of Theorem 1 seems quite large. It is likely to be improved.

The organization of the rest of the paper is as follows. We provide some necessary backgrounds of the Jack functions, the proofs of (a) and (b) of Theorem 1 and Corollary 1.2 in Section 2. The proof of part (c) is given in Section 3. In Appendix we prove (1.4) by two ways different from the method of the Jack functions. Some other explicit formulas of moments are also given in the same section.

2 Proofs of (a) and (b) of Theorem 1 and Corollary 1.2 by Using Jack Functions

For a partition λ, the notation $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ represents the conjugate partition of λ, whose Young diagram is obtained by transposing the Young diagram of λ. Given $\alpha > 0$ and partition λ, we denote by $J_\lambda^{(\alpha)}$ the Jack polynomial, see, e.g., Chapter VI from [17] or [9].

Recall power-symmetric function p_ρ in (1.6). Let $\theta_\rho^\lambda(\alpha)$ denote the coefficient of p_ρ in $J_\lambda^{(\alpha)}$, that is,

$$J_\lambda^{(\alpha)} = \sum_{\rho:|\rho|=|\lambda|} \theta_\rho^\lambda(\alpha) p_\rho. \tag{2.1}$$

The $\theta_\rho^\lambda(\alpha)$ are real numbers. Inversely, let $\Theta_\rho^\lambda(\alpha)$ be the coefficient of $J_\lambda^{(\alpha)}$ in p_ρ, that is,

$$p_\rho = \sum_{\lambda:|\lambda|=|\rho|} \Theta_\rho^\lambda(\alpha) J_\lambda^{(\alpha)}. \tag{2.2}$$

Set

$$C_\lambda(\alpha) = \prod_{(i,j)\in\lambda} \Big\{(\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha)\Big\}, \tag{2.3}$$

where (i, j) runs over all cells of the Young diagram of λ.

Lemma 2.1 *Recalling $\theta_\rho^\lambda(\alpha)$ in (2.1) and $\Theta_\rho^\lambda(\alpha)$ in (2.2). Then, for any partitions λ and ρ with $|\lambda| = |\rho|$, we have*

$$\Theta_\rho^\lambda(\alpha) = \frac{\alpha^{l(\rho)} z_\rho}{C_\lambda(\alpha)}\, \theta_\rho^\lambda(\alpha). \tag{2.4}$$

Proof. A scalar product relevant to the Jack functions is defined by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu}\, \alpha^{l(\lambda)} z_\lambda \tag{2.5}$$

for any partitions λ and μ, where z_λ is as in (1.5), see Section 10 of Chapter VI from [17]. Following from (10.22) in [17] and the definition of Jack functions, we see that

$$\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = \delta_{\lambda\mu}\, C_\lambda(\alpha). \tag{2.6}$$

It follows from (2.1) and (2.5) that

$$\langle J_\lambda^{(\alpha)}, p_\rho \rangle_\alpha = \langle \sum_v \theta_v^\lambda(\alpha) p_v, p_\rho \rangle_\alpha = \sum_v \theta_v^\lambda(\alpha) \langle p_v, p_\rho \rangle_\alpha = \theta_\rho^\lambda(\alpha) \alpha^{l(\rho)} z_\rho.$$

Similarly, by (2.2) and (2.6),

$$\langle J_\lambda^{(\alpha)}, p_\rho \rangle_\alpha = \langle J_\lambda^{(\alpha)}, \sum_v \Theta_\rho^v(\alpha) J_v^{(\alpha)} \rangle_\alpha = \sum_v \Theta_\rho^v(\alpha) \langle J_\lambda^{(\alpha)}, J_v^{(\alpha)} \rangle_\alpha = \Theta_\rho^\lambda(\alpha) C_\lambda(\alpha).$$

These two equalities lead to (2.4). ■

The coefficients θ_ρ^λ's satisfy the following orthogonality relations ((10.31) and (10.32) from [17]):

$$\sum_\rho z_\rho \alpha^{l(\rho)} \theta_\rho^\lambda(\alpha) \theta_\rho^\mu(\alpha) = \delta_{\lambda\mu} C_\lambda(\alpha); \tag{2.7}$$

$$\sum_\lambda \frac{1}{C_\lambda(\alpha)} \theta_\rho^\lambda(\alpha) \theta_\sigma^\lambda(\alpha) = \delta_{\rho\sigma} z_\rho^{-1} \alpha^{-l(\rho)}. \tag{2.8}$$

In other words, if $a_{\lambda\rho} := \big(z_\rho \alpha^{l(\rho)} / C_\lambda(\alpha)\big)^{1/2}\, \theta_\rho^\lambda(\alpha)$, then $A_m = (a_{\lambda\rho})_{|\lambda|=|\rho|=m}$ is an orthogonal matrix of size $p(m)$ for $m \geq 1$. Here $p(m)$ is the number of partitions of m. The following are some special cases of the Jack polynomials.

Example Let $\alpha = 1$, s_λ be the Schur polynomial and χ_μ^λ the character value for the irreducible representation of the symmetric groups. It is well known that $J_\lambda^{(1)} = h(\lambda) s_\lambda$ with $h(\lambda) = \sqrt{C_\lambda(1)}$ as the hook-length product. Further, by (7.8) from Chapter VI of [17] and (2.2) that

$$\theta_\mu^\lambda(1) = \frac{h(\lambda)\chi_\mu^\lambda}{z_\mu} \quad \text{and} \quad \Theta_\mu^\lambda(1) = \frac{\chi_\mu^\lambda}{h(\lambda)}.$$

Example Let $\alpha = 2$. Then $J_\lambda^{(2)}$ coincides with the zonal polynomial Z_λ. By (2.13) and (2.16) from Chapter VII of [17], we have

$$\theta_\mu^\lambda(2) = \frac{2^k k!}{2^{l(\mu)} z_\mu} \omega_\mu^\lambda \quad \text{and} \quad \Theta_\mu^\lambda(2) = \frac{2^k k!}{h(2\lambda)} \omega_\mu^\lambda,$$

with $k = |\lambda| = |\mu|$, where $h(2\lambda) = C_\lambda(2)$ is the hook-length product of $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ and ω_μ^λ is the value of the zonal spherical function of a Gelfand pair $(\mathfrak{S}_{2k}, \mathfrak{B}_k)$. Here $\mathfrak{S}_{2k}$ is the symmetric group and $\mathfrak{B}_k$ is the hyperoctahedral group in $\mathfrak{S}_{2k}$.

Example (Example 1(a) on p. 383 from [17]) For each partition ρ of k, we have

$$\theta_\rho^{(k)}(\alpha) = \frac{k!}{z_\rho}\alpha^{k-l(\rho)} \quad \text{and} \quad \theta_\rho^{(1^k)}(\alpha) = \frac{k!}{z_\rho}(-1)^{k-l(\rho)}. \tag{2.9}$$

For each partition λ with $l(\lambda) \le n$, we define

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j)\in\lambda} \frac{n+(j-1)\alpha-(i-1)}{n+j\alpha-i},$$

which is a positive real number.

LEMMA 2.2 *Let λ and μ be two partitions. Let $\alpha > 0$ and $n \ge 2$. Then*

$$\begin{aligned}&\frac{1}{(2\pi)^n}\int_{[0,2\pi)^n} J_\lambda^{(\alpha)}(e^{i\theta_1},\dots,e^{i\theta_n})J_\mu^{(\alpha)}(e^{-i\theta_1},\dots,e^{-i\theta_n})\prod_{1\le p<q\le n}|e^{i\theta_p}-e^{i\theta_q}|^{2/\alpha}\,d\theta_1\cdots d\theta_n\\ =\ & \delta_{\lambda\mu}\cdot\delta(l(\lambda)\le n)\cdot\frac{\Gamma(\frac{n}{\alpha}+1)}{\Gamma(1+\frac{1}{\alpha})^n}C_\lambda(\alpha)\mathcal{N}_\lambda^\alpha(n).\end{aligned}$$

Proof. Since $J_\lambda^{(\alpha)}(x_1,\dots,x_n)=0$ if $l(\lambda)>n$, we assume $l(\lambda)\le n$ in the following discussion. It is known (e.g., Theorem 12.1.1 from [19]) that

$$\frac{1}{(2\pi)^n}\int_{[0,2\pi)^n}\prod_{1\le p<q\le n}|e^{i\theta_p}-e^{i\theta_q}|^{2/\alpha}d\theta_1\cdots d\theta_n = \frac{\Gamma(\frac{n}{\alpha}+1)}{\Gamma(1+\frac{1}{\alpha})^n}. \tag{2.10}$$

From (10.22), (10.35) and (10.37) in [17], we see that

$$\begin{aligned}&\frac{1}{(2\pi)^n n!\,C_\lambda(\alpha)}\int_{[0,2\pi)^n} J_\lambda^{(\alpha)}(e^{i\theta_1},\dots,e^{i\theta_n})J_\mu^{(\alpha)}(e^{-i\theta_1},\dots,e^{-i\theta_n})\prod_{1\le p<q\le n}|e^{i\theta_p}-e^{i\theta_q}|^{2/\alpha}d\theta_1\cdots d\theta_n\\ =\ & c_n\mathcal{N}_\lambda^\alpha(n)\end{aligned}$$

where

$$c_n := \frac{1}{(2\pi)^n n!}\int_{[0,2\pi)^n}\prod_{1\le p<q\le n}|e^{i\theta_p}-e^{i\theta_q}|^{2/\alpha}d\theta_1\cdots d\theta_n = \frac{1}{n!}\cdot\frac{\Gamma(\frac{n}{\alpha}+1)}{\Gamma(1+\frac{1}{\alpha})^n}$$

by (2.10). Hence the desired conclusion follows. ■

PROPOSITION 2.1 *Let $\beta > 0$ be a constant. Suppose $\theta_1,\cdots,\theta_n$ have a joint density as in (1.3). Let $Z_n=(e^{i\theta_1},\cdots,e^{i\theta_n})$. Given partitions μ and ν of weight K, then*

$$\begin{aligned}&\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]\\ =\ & \alpha^{l(\mu)+l(\nu)}z_\mu z_\nu\sum_{\lambda\vdash K:\,l(\lambda)\le n}\frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)}\mathcal{N}_\lambda^\alpha(n).\end{aligned} \tag{2.11}$$

Proof. Reviewing (1.3), by (2.2) and Lemma 2.2, we have

$$\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] = \sum_{\lambda\vdash K:\,l(\lambda)\le n}\Theta_\mu^\lambda(\alpha)\Theta_\nu^\lambda(\alpha)C_\lambda(\alpha)\mathcal{N}_\lambda^\alpha(n)$$

where $\alpha = 2/\beta$. By Lemma 2.1, the above is identical to

$$\alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K:\, l(\lambda) \le n} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n).$$

The proof is completed. ■

For positive integers n and K and real number $\alpha > 0$. Define

$$\Gamma_{n,K}^\alpha = \max_{\lambda \vdash K:\, l(\lambda) \le n} \mathcal{N}_\lambda^\alpha(n) = \max_{\lambda \vdash K:\, l(\lambda) \le n} \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}; \tag{2.12}$$

$$\gamma_{n,K}^\alpha = \min_{\lambda \vdash K:\, l(\lambda) \le n} \mathcal{N}_\lambda^\alpha(n) = \min_{\lambda \vdash K:\, l(\lambda) \le n} \prod_{(i,j)\in\lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}. \tag{2.13}$$

Lemma 2.3 *Let $\alpha > 0$, $K \ge 1$ and $\Gamma_{n,K}^\alpha$ be as in (2.12) and $\gamma_{n,K}^\alpha$ be as in (2.13). If $n \ge K$, then $A \le \gamma_{n,K}^\alpha \le \Gamma_{n,K}^\alpha \le B$ where A and B are as in (1.7). Further, if $n \ge K$, then*

$$\max_{\lambda \vdash K} |\mathcal{N}_\lambda^\alpha(n) - 1| \le \max\{|A-1|, |B-1|\}. \tag{2.14}$$

Proof. For $\lambda \vdash K$ such that $l(\lambda) \le n$ and $(i,j) \in \lambda$, it is easy to check that

$$1 \le i \le \min\{n, K\} \quad \text{and} \quad 1 \le j \le K. \tag{2.15}$$

Thus, $n + (j-1)\alpha - (i-1) \ge n - i + 1 > 0$ and $n + j\alpha - i \ge j\alpha > 0$. It follows that

$$b_{i,j}(\alpha) := \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i} > 0. \tag{2.16}$$

Write

$$b_{i,j}(\alpha) = 1 + \frac{1-\alpha}{n + j\alpha - i}. \tag{2.17}$$

Case 1: $\alpha \ge 1$. By (2.16) and (2.17), we see that $b_{i,j}(\alpha) \in [0,1]$ for all $\lambda \vdash K$ such that $l(\lambda) \le n$ and $(i,j) \in \lambda$, which concludes $\Gamma_{n,K}^\alpha \le 1$.

Further, by (2.15), $n + j\alpha - i \ge n - K + \alpha > 0$ for all $\lambda \vdash K$ such that $l(\lambda) \le n$ and $(i,j) \in \lambda$. Thus, noticing $1 - \alpha \le 0$, we get

$$b_{i,j}(\alpha) \ge 1 + \frac{1-\alpha}{n-K+\alpha} = 1 - \frac{|1-\alpha|}{n-K+\alpha} > 0$$

for all $n \ge K$. This yields

$$\gamma_{n,K}^\alpha \ge \left(1 - \frac{|1-\alpha|}{n-K+\alpha}\right)^K.$$

The above two conclusions lead to that

$$\left(1 - \frac{|1-\alpha|}{n-K+\alpha}\right)^K \le \gamma_{n,K}^\alpha \le \Gamma_{n,K}^\alpha \le 1 \tag{2.18}$$

for all $n \geq K$ and $\alpha \geq 1$.

Case 2: $\alpha \in (0, 1]$. By (2.17), $b_{i,j}(\alpha) \geq 1$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$, which shows $\gamma_{n,K}^{\alpha} \geq 1$.

Moreover, by (2.15) again, $n + j\alpha - i \geq n - K + \alpha$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$. Thus, with $1 - \alpha > 0$, we have from (2.17) that

$$b_{i,j}(\alpha) \leq 1 + \frac{1-\alpha}{n-K+\alpha}.$$

By the definition of $\Gamma_{n,K}^{\alpha}$ and the earlier conclusion, we get

$$1 \leq \gamma_{n,K}^{\alpha} \leq \Gamma_{n,K}^{\alpha} \leq \left(1 + \frac{1-\alpha}{n-K+\alpha}\right)^{K}$$

for all $n \geq K$ and $\alpha \in (0, 1]$. This and (2.18) prove the first part of the lemma.

Finally, by the definitions in (2.12) and (2.13),

$$\gamma_{n,K}^{\alpha} \leq \mathcal{N}_{\lambda}^{\alpha}(n) = \prod_{(i,j)\in\lambda} b_{i,j}(\alpha) \leq \Gamma_{n,K}^{\alpha}$$

for all $\lambda \vdash K$ since $l(\lambda) \leq n$ holds automatically if $n \geq K$. By the proved conclusion,

$$A - 1 \leq \mathcal{N}_{\lambda}^{\alpha}(n) - 1 \leq B - 1$$

for all $\lambda \vdash K$. This implies (2.14). ■

Proof of (a) and (b) of Theorem 1. (a) By Proposition 2.1, take $\mu = \nu$ with weight K to have

$$\mathbb{E}\big[|p_{\mu}(Z_n)|^2\big] = \alpha^{2l(\mu)} z_{\mu}^2 \sum_{\lambda \vdash K : l(\lambda) \leq n} \frac{\theta_{\mu}^{\lambda}(\alpha)^2}{C_{\lambda}(\alpha)} \mathcal{N}_{\lambda}^{\alpha}(n). \tag{2.19}$$

Lemma 2.3 says that $\Gamma_{n,K}^{\alpha} > 0$ and $\gamma_{n,K}^{\alpha} > 0$ for all $n \geq K$. By the definitions of $\Gamma_{n,K}^{\alpha}$ in (2.12) and $\gamma_{n,K}^{\alpha}$ in (2.13), since $C_{\lambda}(\alpha) > 0$ for any partition λ and $\alpha > 0$,

$$\begin{aligned} \gamma_{n,K}^{\alpha} \cdot \alpha^{2l(\mu)} z_{\mu}^2 \sum_{\lambda \vdash K : l(\lambda) \leq n} \frac{\theta_{\mu}^{\lambda}(\alpha)^2}{C_{\lambda}(\alpha)} &\leq \mathbb{E}\big[|p_{\mu}(Z_n)|^2\big] \\ &\leq \Gamma_{n,K}^{\alpha} \cdot \alpha^{2l(\mu)} z_{\mu}^2 \sum_{\lambda \vdash K} \frac{\theta_{\mu}^{\lambda}(\alpha)^2}{C_{\lambda}(\alpha)}. \end{aligned} \tag{2.20}$$

From assumption $n \geq K$, if $\lambda \vdash K$, we know $l(\lambda) \leq n$ automatically. Therefore, from (2.8) the two sums in (2.20) are both equal to $z_{\mu}^{-1} \alpha^{-l(\mu)}$. Consequently,

$$\gamma_{n,K}^{\alpha} \leq \frac{\mathbb{E}\big[|p_{\mu}(Z_n)|^2\big]}{\alpha^{l(\mu)} z_{\mu}} \leq \Gamma_{n,K}^{\alpha}.$$

The conclusion (a) then follows from the first part of Lemma 2.3.

(b) First, assume $|\mu| \neq |\nu|$. Notice

$$\begin{aligned}
&\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] \\
= \quad & Const \cdot \int_0^{2\pi} \cdots \int_0^{2\pi} p_\mu(e^{i\theta_1}, \cdots, e^{i\theta_n})\overline{p_\nu(e^{i\theta_1}, \cdots, e^{i\theta_n})} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta \, d\theta_1 \cdots d\theta_n.
\end{aligned}$$

For an integrable function $h(x)$, we know $\int_0^{2\pi} h(e^{ix})\,dx = \int_b^{b+2\pi} h(e^{ix})\,dx$ for any $b \in \mathbb{R}$. Using the induction and the Fubini theorem, we see that

$$\begin{aligned}
&\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] \\
= \quad & Const \cdot \int_b^{b+2\pi} \cdots \int_b^{b+2\pi} p_\mu(e^{i\theta_1}, \cdots, e^{i\theta_n})\overline{p_\nu(e^{i\theta_1}, \cdots, e^{i\theta_n})} \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^\beta \, d\theta_1 \cdots d\theta_n.
\end{aligned}$$

Making transform $\eta_j = \theta_j - b$ for $1 \le j \le n$, noting that $p_\mu(e^{ib+i\eta_1}, \cdots, e^{ib+i\eta_n}) = e^{ib|\mu|}p_\mu(e^{i\eta_1}, \cdots, e^{i\eta_n})$ for any $b \in \mathbb{R}$, we obtain that

$$\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] = e^{ib(|\mu|-|\nu|)}\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]$$

for any $b \in \mathbb{R}$. If $|\mu| \neq |\nu|$, since b is arbitrary, we then conclude

$$\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] = 0$$

for all $n \ge 2$.

To prove the second part of (b), by the first part, it suffices to prove the conclusion for $n \ge |\mu| = |\nu| = K$. Observe that $l(\lambda) \le n$ if $\lambda \vdash K$. Thus, it follows from Proposition 2.1 that

$$\begin{aligned}
&\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] \\
= \quad & \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda\vdash K} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)} \mathcal{N}^\alpha_\lambda(n) \\
= \quad & \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \left[\sum_{\lambda\vdash K} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)} + \sum_{\lambda\vdash K} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)}\Big(\mathcal{N}^\alpha_\lambda(n) - 1\Big)\right] \\
= \quad & \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda\vdash K} \frac{\theta^\lambda_\mu(\alpha)\theta^\lambda_\nu(\alpha)}{C_\lambda(\alpha)}\Big(\mathcal{N}^\alpha_\lambda(n) - 1\Big) \qquad (2.21)
\end{aligned}$$

where the last identity comes from the orthogonal property in (2.8). Therefore,

$$\begin{aligned}
&\Big|\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]\Big| \\
\le \quad & \max_{\lambda\vdash K} |\mathcal{N}^\alpha_\lambda(n) - 1| \cdot \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda\vdash K} \frac{|\theta^\lambda_\mu(\alpha)| \cdot |\theta^\lambda_\nu(\alpha)|}{C_\lambda(\alpha)}.
\end{aligned}$$

Now, by the Cauchy-Schwartz inequality the sum above is bounded by

$$\Big(\sum_{\lambda\vdash K} \frac{|\theta^\lambda_\mu(\alpha)|^2}{C_\lambda(\alpha)}\Big)^{1/2} \cdot \Big(\sum_{\lambda\vdash K} \frac{|\theta^\lambda_\nu(\alpha)|^2}{C_\lambda(\alpha)}\Big)^{1/2} = z_\mu^{-1/2} z_\nu^{-1/2} \alpha^{-(l(\mu)+l(\nu))/2}$$

by (2.8). The above two inequalities imply

$$\begin{aligned}\Big|\mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big]\Big| &\le \max_{\lambda\vdash K}|\mathcal{N}_\lambda^\alpha(n)-1|\cdot \alpha^{(l(\mu)+l(\nu))/2}(z_\mu z_\nu)^{1/2}\\ &\le \max\{|A-1|,|B-1|\}\cdot \alpha^{(l(\mu)+l(\nu))/2}(z_\mu z_\nu)^{1/2}\end{aligned}$$

by (2.14). ■

Lemma 2.4 *Let A and B be as in (1.7) with $\beta>0$. Set $\alpha=2/\beta$. If $n\ge 2K$, then*

$$\max\{|A-1|,|B-1|\}\le \frac{6|1-\alpha|K}{n}.$$

Proof. By the definitions of A and B, it suffices to show that, as $n\ge 2K$,

$$1-\Big(1-\frac{\alpha-1}{n-K+\alpha}\Big)^K \le \frac{6|1-\alpha|K}{n} \qquad \text{for } \alpha\ge 1; \tag{2.22}$$

$$\Big(1+\frac{1-\alpha}{n-K+\alpha}\Big)^K-1 \le \frac{6|1-\alpha|K}{n} \qquad \text{for } \alpha\in(0,1). \tag{2.23}$$

First, if $\alpha\ge 1$, then $(\alpha-1)/(n-K+\alpha)\in[0,1)$. Notice $(1+x)^K\ge 1+Kx$ for all $x\ge -1$ (see, e.g., Theorem 42 on p.40 from [12]), we have

$$1-\Big(1-\frac{\alpha-1}{n-K+\alpha}\Big)^K\le \frac{K(\alpha-1)}{n-K+\alpha}\le \frac{2K|1-\alpha|}{n}$$

since $n-K+\alpha\ge n/2$ as $n\ge 2K$. This proves (2.22).

Second, for $\alpha\in(0,1)$, it is easy to verify that $(1-\alpha)/(n-K+\alpha)\le 1/K$ provided $n\ge 2K$. By the fact that $(1+x)^K\le 1+3Kx$ for all $0\le x\le 1/K$, we obtain

$$\Big(1+\frac{1-\alpha}{n-K+\alpha}\Big)^K-1\le \frac{3(1-\alpha)K}{n-K+\alpha}\le \frac{6|1-\alpha|K}{n}$$

since $n-K+\alpha\ge n/2$ if $n\ge 2K$ as used earlier. This concludes (2.23). ■

Proof of Corollary 1.2. (a) By Theorem 1

$$A-1\le \frac{\mathbb{E}\big[|p_\mu(Z_n)|^2\big]}{\alpha^{l(\mu)}z_\mu}-1\le B-1$$

Thus,

$$\Big|\frac{\mathbb{E}\big[|p_\mu(Z_n)|^2\big]}{\alpha^{l(\mu)}z_\mu}-1\Big|\le \max\{|A-1|,|B-1|\}.$$

The conclusion (a) then follows from Lemma 2.4.

(b) The conclusion obviously holds if $|\mu|\ne|\nu|$ by (b) of Theorem 1. If $|\mu|=|\nu|=K$, by (b) of Theorem 1 and Lemma 2.4, we get the desired result. ■

3 Proof of (c) of Theorem 1

We start the proof through a series of lemmas.

Lemma 3.1 *Let a be a real number. Then*

$$\frac{\Gamma(x+a)}{\Gamma(x)} \sim x^a$$

as $x \to +\infty$, where $\Gamma(x)$ is the gamma function.

Proof. Recall the Stirling formula (see, e.g., p.368 from [11] or (37) on p.204 from [1]):

$$\log\Gamma(z) = z\log z - z - \frac{1}{2}\log z + \log\sqrt{2\pi} + \frac{1}{12z} + O\left(\frac{1}{x^3}\right)$$

as $x = \mathrm{Re}\,(z) \to +\infty$. Then

$$\log\frac{\Gamma(x+a)}{\Gamma(x)} = (x+a)\log(x+a) - x\log x - a - \frac{1}{2}\left(\log(x+a) - \log x\right) + O\left(\frac{1}{x}\right) \tag{3.1}$$

as $x \to \infty$. First, use the fact that $\log(1+t) \sim t + O(t^2)$ as $t \to 0$ to get

$$\begin{aligned}(x+a)\log(x+a) - x\log x &= (x+a)\left(\log x + \log\left(1+\frac{a}{x}\right)\right) - x\log x\\ &= (x+a)\Big(\log x + \frac{a}{x} + O(x^{-2})\Big) - x\log x\\ &= a\log x + a + O\left(\frac{1}{x}\right)\end{aligned}$$

as $x \to \infty$. Evidently, $\log(x+a) - \log x = \log(1+ax^{-1}) = O(1/x)$ as $x \to \infty$. Plugging these two assertions into (3.1), we have

$$\log\frac{\Gamma(x+a)}{\Gamma(x)} = a\log x + O\left(\frac{1}{x}\right)$$

as $x \to \infty$. The proof is complete. ■

Lemma 3.2 *Let $\beta > 0$. For positive integers m and k and real numbers $a_1, \cdots, a_k$, define*

$$D = \int_0^{\pi} \cos(2mt)\Big|\prod_{i=1}^{k}\sin\big(t+a_i\big)\Big|^{\beta}\,dt.$$

Then $|D| \le 6(1+\beta)(\frac{k}{m})^{1\wedge\beta}$.

Proof. First, since $|D| \le \int_0^{\pi} 1\,dt = \pi$, the conclusion obviously holds for $m = 1$. Now we assume $m \ge 2$. Set $s = mt$. Then

$$\begin{aligned}D &= \frac{1}{m}\int_0^{m\pi}\cos(2s)\,\Big|\prod_{i=1}^{k}\sin\big(\frac{s}{m}+a_i\big)\Big|^{\beta}\,ds\\ &= \frac{1}{m}\sum_{j=0}^{m-1}\int_{j\pi}^{(j+1)\pi}\cos(2s)\,\Big|\prod_{i=1}^{k}\sin\big(\frac{s}{m}+a_i\big)\Big|^{\beta}\,ds\\ &= \frac{1}{m}\sum_{j=0}^{m-1}\int_0^{\pi}\cos(2s)\,\Big|\prod_{i=1}^{k}\sin\big(\frac{s+j\pi}{m}+a_i\big)\Big|^{\beta}\,ds\\ &= \int_0^{\pi} L_m(s)\cos(2s)\,ds\end{aligned} \tag{3.2}$$

where we make a transform: $s \to s - j\pi$ in the second identity to get the third one, and

$$L_m(s) = \frac{1}{m}\sum_{j=0}^{m-1}\Big|\prod_{i=1}^{k}\sin\big(b_{ij} + \frac{s}{m}\big)\Big|^{\beta} \tag{3.3}$$

for $0 \le s \le \pi$ and $b_{ij} = a_i + \frac{j\pi}{m}$. Note that the inequality $\big|\,|x|^{\beta} - |y|^{\beta}\big| \le (1+\beta)|x-y|^{1\wedge\beta}$ for any $\beta > 0$ and $x, y \in [-1, 1]$, it follows that

$$\begin{aligned}
&\Big|L_m(s) - \frac{1}{m}\sum_{j=0}^{m-1}\Big|\prod_{i=1}^{k}\sin b_{ij}\Big|^{\beta}\Big| \\
\le\ & \frac{1}{m}\sum_{j=0}^{m-1}\Big|\,\Big|\prod_{i=1}^{k}\sin\big(b_{ij} + \frac{s}{m}\big)\Big|^{\beta} - \Big|\prod_{i=1}^{k}\sin b_{ij}\Big|^{\beta}\Big| \\
\le\ & \frac{1+\beta}{m}\sum_{j=0}^{m-1}\Big|\prod_{i=1}^{k}\sin\big(b_{ij} + \frac{s}{m}\big) - \prod_{i=1}^{k}\sin b_{ij}\Big|^{\beta\wedge 1}. \qquad (3.4)
\end{aligned}$$

Now, by the chain rule, $\Big(\prod_{i=1}^{k}\sin(b_{ij}+t)\Big)' = \sum_{l=1}^{k}\cos(b_{lj}+t)\prod_{1\le i\le k,\, i\ne l}\sin(b_{ij}+t)$ for any $t \in \mathbb{R}$. Thus the absolute value of the derivative is bounded by k for ant $t \in \mathbb{R}$. By the mean-value theorem,

$$\Big|\prod_{i=1}^{k}\sin\big(b_{ij} + \frac{s}{m}\big) - \prod_{i=1}^{k}\sin b_{ij}\Big| \le \frac{ks}{m}.$$

This implies that the last term in (3.4) is controlled by

$$\frac{1+\beta}{m}\sum_{j=0}^{m-1}\Big(\frac{ks}{m}\Big)^{1\wedge\beta} = (1+\beta)\Big(\frac{ks}{m}\Big)^{1\wedge\beta}.$$

It follows from (3.4) that

$$\Big|L_m(s) - \frac{1}{m}\sum_{j=0}^{m-1}\Big|\prod_{i=1}^{k}\sin b_{ij}\Big|^{\beta}\Big| \le (1+\beta)\Big(\frac{ks}{m}\Big)^{1\wedge\beta}. \tag{3.5}$$

Set $C = \frac{1}{m}\sum_{j=0}^{m-1}\Big|\prod_{i=1}^{k}\sin b_{ij}\Big|^{\beta}$. Notice $\int_0^{\pi}\cos(2s)\,ds = 0$. From the above we use the simple fact that $|\cos(2s)| \le 1$ to have

$$\begin{aligned}
\Big|\int_0^{\pi} L_m(s)\cos(2s)\,ds\Big| &= \Big|\int_0^{\pi} C\cos(2s)\,ds + \int_0^{\pi}(L_m(s) - C)\cos(2s)\,ds\Big| \\
&\le (1+\beta)\Big(\frac{k}{m}\Big)^{1\wedge\beta}\int_0^{\pi} s^{1\wedge\beta}\,ds.
\end{aligned}$$

Now the last integral above is bounded by $\int_0^1 1\,ds + \int_1^{\pi} s\,ds = (\pi^2+1)/2 \le 6$. The proof is completedby using (3.2). ■

LEMMA 3.3 *For $\beta > 0$, let $f(\theta_1, \cdots, \theta_n|\beta)$ be as in (1.3). Define*

$$I(m,n) = \int_0^{2\pi}\cdots\int_0^{2\pi}\cos(m(\theta_2 - \theta_1))f(\theta_1, \cdots, \theta_n|\beta)\,d\theta_1\cdots d\theta_n, \quad m \ge 0,\ n \ge 2.$$

Then, for some constant $K = K(\beta)$, we have $|I(m,n)| \le (Kn2^{n\beta})m^{-(1\wedge\beta)}$ for all $m \ge 1$ and $n \ge 2$.

Proof. Evidently, since $f(\theta_1, \cdots, \theta_n|\beta)$ is a probability density function, we know

$$I(0,n) = 1 \tag{3.6}$$

for all $n \geq 2$. Since $|e^{ix} - e^{iy}|^2 = |1 - e^{i(x-y)}|^2 = (1 - \cos(x-y))^2 + \sin^2(x-y) = 2(1 - \cos(x-y)) = 4\sin^2((x-y)/2)$ for any $x, y \in \mathbb{R}$, the probability density function in (1.3) becomes

$$f(\theta_1, \cdots, \theta_n|\beta) = C_n \prod_{1 \leq j < k \leq n} \Big| \sin\Big(\frac{\theta_j - \theta_k}{2}\Big)\Big|^\beta \tag{3.7}$$

where $\theta_1, \cdots, \theta_n \in [0, 2\pi]$ and

$$C_n = 2^{n(n-1)\beta/2}(2\pi)^{-n} \cdot \frac{\Gamma(1+\beta/2)^n}{\Gamma(1+\beta n/2)}.$$

Now,

$$\begin{aligned} I(m,n) &= \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(m(\theta_2 - \theta_1)) f(\theta_1, \cdots, \theta_n|\beta)\, d\theta_1 \cdots d\theta_n & (3.8) \\ &= C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(m(\theta_2 - \theta_1)) \cdot \prod_{1 \leq j < k \leq n} \Big| \sin\Big(\frac{\theta_j - \theta_k}{2}\Big)\Big|^\beta d\theta_2 \cdots d\theta_n\, d\theta_1. \end{aligned}$$

Making transforms $x_i = \theta_i - \theta_1$ for $i = 2, 3, \cdots, n$, we obtain that

$$I(m,n) = C_n \int_0^{2\pi} \int_{-\theta_1}^{2\pi - \theta_1} \cdots \int_{-\theta_1}^{2\pi-\theta_1} \cos(mx_2) \cdot G_n(x)\, dx_2 \cdots dx_n\, d\theta_1 \tag{3.9}$$

where

$$G_n(x) = \prod_{i=2}^n \Big| \sin\Big(\frac{x_i}{2}\Big)\Big|^\beta \cdot \prod_{2 \leq j < k \leq n} \Big| \sin\Big(\frac{x_j - x_k}{2}\Big)\Big|^\beta \tag{3.10}$$

where the second product is understood to be 1 if $n = 2$. For a periodic and integrable function $h(x)$ with period 2π, we know that $\int_b^{b+2\pi} h(x)\, dx = \int_0^{2\pi} h(x)\, dx$. By induction and the Fubini theorem, we have

$$\begin{aligned} I(m,n) &= C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) \cdot G_n(x)\, dx_2 \cdots dx_n\, d\theta_1 \\ &= (2\pi) C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) \cdot G_n(x)\, dx_2 \cdots dx_n & (3.11) \\ &= (2\pi) C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) J_n(x) H_n(x)\, dx_2 \cdots dx_n & (3.12) \end{aligned}$$

where $G_n(x) = J_n(x)H_n(x)$ and

$$H_n(x) = \begin{cases} \prod_{i=3}^n \big|\sin\big(\frac{x_i}{2}\big)\big|^\beta \cdot \prod_{3 \leq j < k \leq n} \big|\sin\big(\frac{x_j - x_k}{2}\big)\big|^\beta, & \text{if } n \geq 4; \\ \big|\sin\big(\frac{x_3}{2}\big)\big|^\beta, & \text{if } n = 3; \\ 1, & \text{if } n = 2 \end{cases}$$

and

$$J_n(x)=\begin{cases}\left|\sin\left(\frac{x_2}{2}\right)\right|^{\beta}\prod_{i=3}^{n}\left|\sin\left(\frac{x_2-x_i}{2}\right)\right|^{\beta}, & \text{if } n\ge 3;\\ \left|\sin\left(\frac{x_2}{2}\right)\right|^{\beta}, & \text{if } n=2.\end{cases}$$

In particular,

$$I(m,2)=2\pi C_2\int_0^{2\pi}\cos(mx_2)J_2(x)\,dx_2. \tag{3.13}$$

Taking $m=0$ in (3.11), we know from (3.6) that

$$\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{i=2}^{n}\left|\sin\left(\frac{x_i}{2}\right)\right|^{\beta}\cdot\prod_{2\le j<k\le n}\left|\sin\left(\frac{x_j-x_k}{2}\right)\right|^{\beta}dx_2\,dx_3\cdots dx_n=\frac{1}{2\pi C_n}$$

for all $n\ge 2$, where the second product above is understood to be 1 if $n=2$. This implies

$$\int_0^{2\pi}\cdots\int_0^{2\pi}H_n(x)\,dx_3\cdots dx_n=\frac{1}{2\pi C_{n-1}} \tag{3.14}$$

for all $n\ge 3$. Now, recalling the definition of $J_n(x)$, let $t=x_2/2$, we have

$$\int_0^{2\pi}\cos(mx_2)J_n(x)\,dx_2=2\int_0^{\pi}\cos(2mt)\Big|\prod_{i=1}^{n-1}\sin\left(t+a_i\right)\Big|^{\beta}dt$$

for all $n\ge 2$, where $a_1=0$, $a_i=-x_{i+1}/2$ for $i=2,\cdots,n-1$. By Lemma 3.2,

$$\Big|\int_0^{2\pi}\cos(mx_2)J_n(x)\,dx_2\Big|\le 12(1+\beta)\left(\frac{n}{m}\right)^{1\wedge\beta} \tag{3.15}$$

for all $n\ge 2$. Therefore, this and (3.13) imply that for some constant $K_1=K_1(\beta)$,

$$|I(m,2)|\le\frac{K_1}{m^{(1\wedge\beta)}}. \tag{3.16}$$

Now assume $n\ge 3$. By (3.12) and (3.15), and then (3.14), we obtain

$$\begin{aligned}|I(m,n)| &\le 24\pi(1+\beta)C_n\Big(\frac{n}{m}\Big)^{1\wedge\beta}\int_0^{2\pi}\cdots\int_0^{2\pi}H_n(x)\,dx_3\cdots dx_n\\ &= 12(1+\beta)\Big(\frac{n}{m}\Big)^{1\wedge\beta}\frac{C_n}{C_{n-1}}\end{aligned} \tag{3.17}$$

for all $n\ge 3$. Now,

$$\frac{C_n}{C_{n-1}}=\frac{\Gamma(1+\beta/2)}{2\pi}\cdot\frac{\Gamma(1+\beta n/2-\beta/2)}{\Gamma(1+\beta n/2)}\cdot 2^{(n-1)\beta} \tag{3.18}$$

for all $n\ge 3$. By Lemma 3.1, there exists a constant $K_2=K_2(\beta)$ such that

$$\frac{\Gamma(1+\beta n/2-\beta/2)}{\Gamma(1+\beta n/2)}\le\frac{K_2}{n^{\beta/2}}$$

for all $n\ge 1$. This, (3.17) and (3.18) imply that there exists a constant $K=K(\beta)$ such that

$$|I(m,n)|\le K\cdot\Big(\frac{n}{m}\Big)^{1\wedge\beta}\cdot\frac{1}{n^{\beta/2}}\cdot 2^{n\beta}=Kn^{(1\wedge\beta)-\beta/2}\frac{2^{n\beta}}{m^{1\wedge\beta}}\le K\frac{n2^{n\beta}}{m^{1\wedge\beta}}$$

for all $n \geq 3$. This together with (3.16) proves the lemma. ■

Proof of (c) of Theorem 1. Observe that, for any real numbers $x_1, \cdots, x_n$,

$$\begin{aligned}\Big|\sum_{j=1}^{n} e^{ix_j}\Big|^2 &= \sum_{j=1}^{n} e^{ix_j} \cdot \sum_{j=1}^{n} e^{-ix_j}\\ &= n + \sum_{j \neq k} e^{i(x_j - x_k)} = n + \sum_{1 \leq j < k \leq n} \left(e^{i(x_j - x_k)} + e^{-i(x_j - x_k)}\right)\\ &= n + 2 \sum_{1 \leq j < k \leq n} \cos(x_j - x_k).\end{aligned}$$

Thus, by the symmetry of $f(\theta_1, \cdots, \theta_n|\beta)$,

$$\mathbb{E}\big[|p_m(Z_n)|^2\big] = \mathbb{E}\left[\Big|\sum_{j=1}^{n} e^{im\theta_j}\Big|^2\right] = n + n(n-1) \cdot \mathbb{E}[\cos\{m(\theta_1 - \theta_2)\}]. \tag{3.19}$$

The conclusion then follows from Lemma 3.3. ■

4 Appendix

In this section we calculate some moments for the circular β-ensembles. The first result below is an independent check of the second moment of the trace of a COE given in (1.4). The derivation does not depend on the Jack function as used in Section 2. It only uses the distribution of the entries of the COE.

LEMMA 4.1 *Let W_n be an $n \times n$ circular orthogonal ensemble (COE), that is, $W_n = U_n^T U_n$ for some Haar-invariant unitary matrix U_n. Then $\mathbb{E}\big[|\mathrm{Tr}(W_n)|^2\big] = 2n/(n+1)$ for all $n \geq 2$.*

First Proof of Lemma 4.1. We prove the lemma in three steps.

Step 1. Write $U_n = (u_{rs})$. First, we claim that

$$\mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = 0 \tag{4.1}$$

if $r \neq p$ or $s \neq q$. In fact, since U_n is Haar-invariant unitary, the distributions of UU_n and U_nU are the same as that of U_n for any unitary matrix U. In particular, take $U = \mathrm{diag}(e^{i\theta_k})_{1 \leq k \leq n}$ to obtain that

$$\mathcal{L}\left(\left(e^{i\theta_r} u_{rs}\right)_{1 \leq r,s \leq n}\right) = \mathcal{L}\left(\left(e^{i\theta_s} u_{rs}\right)_{1 \leq r,s \leq n}\right) = \mathcal{L}\left(\left(u_{rs}\right)_{1 \leq r,s \leq n}\right) \tag{4.2}$$

for any $\theta_1, \cdots, \theta_n \in \mathbb{R}$, where $\mathcal{L}(X)$ is the joint distribution of the entries of random matrix X. If $r \neq p$, taking $\theta_r - \theta_p = \pi/2$, then by (4.2), we have that

$$\mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = e^{2i(\theta_r - \theta_p)} \mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = -\mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2]$$

which means (4.1). The case for $s = q$ can be proved similarly.

Step 2. Recall notation $(2m-1)!! = (2m-1)(2m-3)\cdots 3\cdot 1$ for any integer $m \geq 1$, and $(-1)!! = 1$ by convention. We have the following fact (Lemma 2.4 from [14]):

$$\mathbb{E}[\xi_1^{a_1}\xi_2^{a_2}\cdots\xi_n^{a_n}] = \frac{\prod_{i=1}^n (2a_i-1)!!}{\prod_{i=1}^a (n+2i-2)}. \tag{4.3}$$

where $a_1, \cdots, a_n$ are non-negative integers with $a = \sum_{i=1}^n a_i$, $\xi_i = X_i^2/(X_1^2 + \cdots + X_n^2)$ and $X_1, \cdots, X_n$ are i.i.d. random variables with $X_1 \sim N(0,1)$.

Step 3. Evidently, $\mathrm{Tr}(W_n) = \sum_{1\leq i,j\leq n} u_{ij}^2$. Notice, from the invariant property, by exchanging some rows and some columns of U_n, we see that the distributions of u_{rs} and u_{11} are identical for any $1 \leq r, s \leq n$. By (4.1),

$$\mathbb{E}\big[|\mathrm{Tr}(W_n)|^2\big] = \mathbb{E}\Bigg[(\sum_{r,s} u_{rs}^2)(\sum_{p,q} \bar{u}_{p,q}^2)\Bigg] = \mathbb{E}\Bigg[\sum_{r,s} |u_{rs}|^4\Bigg] = n^2 E\big[|u_{11}|^4\big]. \tag{4.4}$$

It is known (e.g., Lemma 2.1 in [13, 14]) that the probability distribution of $|u_{11}|^2$ is the same as that of $(X_1^2 + X_2^2)/\sum_{i=1}^{2n} {X_i}^2$. By (4.3),

$$\mathbb{E}[\xi_1^2] = \frac{3}{2n(2n+2)} \quad \text{and} \quad \mathbb{E}[\xi_1\xi_2] = \frac{1}{2n(2n+2)}.$$

Then

$$\mathbb{E}[|u_{11}|^4] = \mathbb{E}[(\xi_1+\xi_2)^2] = 2\mathbb{E}[\xi_1^2] + 2\mathbb{E}[\xi_1\xi_2] = \frac{2}{n(n+1)}.$$

Substitute this into (4.4) to see that $\mathbb{E}\big[|\mathrm{Tr}(W_n)|^2\big] = 2n/(n+1)$. ■

Second Proof of Lemma 4.1. We use the following formula due to Collins [3] (see also [18]): let $(u_{ij})_{1\leq i,j\leq n}$ be an $n\times n$ CUE matrix (or equivalently, an Haar-distributed unitary matrix) and let $i_1, \ldots, i_k$, $j_1, \ldots, j_k$, $i'_1, \ldots, i'_k$, $j'_1 \ldots, j'_k$ be elements in $\{1, 2, \ldots, n\}$. Then

$$\mathbb{E}\big[u_{i_1j_1}\cdots u_{i_kj_k}\overline{u_{i'_1j'_1}\cdots u_{i'_kj'_k}}\big] = \sum_{\sigma,\tau\in\mathfrak{S}_k} \mathrm{Wg}_{n,k}(\sigma^{-1}\tau)\Big(\prod_{p=1}^k \delta_{i_p,i'_{\sigma(p)}}\Big)\Big(\prod_{q=1}^k \delta_{j_q,j'_{\tau(q)}}\Big). \tag{4.5}$$

Here $\mathfrak{S}_k$ is the symmetric group and $\mathrm{Wg}_{n,k}$ is a class function on $\mathfrak{S}_k$, called the Weingarten function for the unitary group. For our purpose, we do not need the explicit definition of $\mathrm{Wg}_{n,k}$ but use the case for $k=2$. In fact, for $n\geq 2$, we know (see (5.2) of [3])

$$\mathrm{Wg}_{n,2}(\mathrm{id}_2) = \frac{1}{n^2-1} \qquad \text{and} \qquad \mathrm{Wg}_{n,2}((1\ 2)) = -\frac{1}{n(n^2-1)}, \tag{4.6}$$

where id_2 and $(1\ 2)$ are the identity permutation and the transposition on $\{1,2\}$, respectively.

We have $|\mathrm{Tr}(W_n)|^2 = \sum_{r,s,p,q} u_{rs}^2\bar{u}_{pq}^2$. By (4.5), $\mathbb{E}[u_{rs}^2\bar{u}_{pq}^2]$ is zero unless $r=p$ and $s=q$. Moreover, $\mathbb{E}[u_{rs}^2\bar{u}_{rs}^2] = \mathbb{E}[|u_{11}|^4]$ for all $1\leq r$, $s\leq n$. Therefore, using (4.5) and (4.6), we obtain

$$\mathbb{E}\big[|\mathrm{Tr}(W_n)|^2\big] = n^2\mathbb{E}[|u_{11}|^4] = 2n^2\Big\{\mathrm{Wg}_{n,2}(\mathrm{id}_2) + \mathrm{Wg}_{n,2}((1\ 2))\Big\} = \frac{2n}{n+1}. \qquad ■$$

Lemma 4.1 corresponds to the conclusion for $\beta = 1$ in (1.8), which is derived through Proposition 2.1 by the Jack functions. Now we apply the same proposition to derive some other moments for the circular β-ensembles. Let p_k and Z_n be as in Theorem 1.

Example Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$\begin{aligned} \mathbb{E}[|p_1(Z_n)|^4] &= \frac{2n\alpha^2(n^2+2(\alpha-1)n-\alpha)}{(n+\alpha-1)(n+\alpha-2)(n+2\alpha-1)} \\ &= \begin{cases} \frac{8(n^2+2n-2)}{(n+1)(n+3)}, & \text{if } \beta = 1; \\ 2, & \text{if } \beta = 2; \\ \frac{2n^2-2n-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4. \end{cases} \end{aligned} \tag{4.7}$$

Example Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$\begin{aligned} \mathbb{E}[|p_2(Z_n)|^2] &= \frac{2\alpha n(n^2+2(\alpha-1)n+\alpha^2-3\alpha+1)}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)} \\ &= \begin{cases} \frac{4(n^2+2n-1)}{(n+1)(n+3)}, & \text{if } \beta = 1; \\ 2, & \text{if } \beta = 2; \\ \frac{4n^2-4n-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4. \end{cases} \end{aligned} \tag{4.8}$$

Example Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$\begin{aligned} &\mathbb{E}\Big[p_2(Z_n)\overline{p_1(Z_n)^2}\Big] = \mathbb{E}\Big[\overline{p_2(Z_n)}p_1(Z_n)^2\Big] \\ &= \frac{2\alpha^2(\alpha-1)n}{(n+\alpha-1)(n+2\alpha-1)(n+\alpha-2)} \\ &= \begin{cases} \frac{8}{(n+1)(n+3)}, & \text{if } \beta = 1; \\ 0, & \text{if } \beta = 2; \\ \frac{-1}{(2n-1)(2n-3)}, & \text{if } \beta = 4. \end{cases} \end{aligned} \tag{4.9}$$

In particular, if $\beta \neq 2$, as $n \to \infty$,

$$\mathbb{E}\Big[p_2(Z_n)\overline{p_1(Z_n)^2}\Big] \sim 2\alpha^2(\alpha-1)n^{-2}. \tag{4.10}$$

Proofs of (1.8), (4.7), (4.8) and (4.9). Let $n \geq 2$, μ and ν be partitions of 2. Set $\alpha = 2/\beta$. By Proposition 2.1 and (2.9), we have

$$\begin{aligned} \mathbb{E}\Big[p_\mu(Z_n)\overline{p_\nu(Z_n)}\Big] &= \alpha^{l(\mu)+l(\nu)}\Big(\frac{4\alpha^{2-l(\mu)}\alpha^{2-l(\nu)}}{2\alpha^2(\alpha+1)}\frac{n(n+\alpha)}{(n+\alpha-1)(n+2\alpha-1)} \\ &\quad + \frac{4(-1)^{2-l(\mu)}(-1)^{2-l(\nu)}}{2\alpha(\alpha+1)}\frac{n(n-1)}{(n+\alpha-1)(n+\alpha-2)}\Big) \\ &= \frac{2\alpha^{l(\mu)+l(\nu)}n}{\alpha^2(\alpha+1)(n+\alpha-1)}\Big(\frac{\alpha^{4-l(\mu)-l(\nu)}(n+\alpha)}{n+2\alpha-1} + \frac{(-1)^{l(\mu)+l(\nu)}\alpha(n-1)}{n+\alpha-2}\Big). \end{aligned} \tag{4.11}$$

(i) Take $\mu = \nu = (1)$ in Proposition 2.1. Since $\theta^{(1)}_{(1)}(\alpha) = 1$, $C_{(1)}(\alpha) = \alpha$ for any $\alpha > 0$, we obtain (1.8).

(ii) Taking $\mu = \nu = (1, 1)$ in (4.11), (4.7) follows.

(iii) Taking $\mu = \nu = (2)$ in (4.11), (4.8) follows.

(iv) Taking $\mu = (2)$ and $\nu = (1, 1)$ in (4.11), we get the identity for the first expectation in (4.9). Since the value of the expectation is real, the identity for the second expectation follows. With the earlier conclusion, (4.10) is obvious. ■

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References

[1] Ahlfors, L. V. (1979). Complex Analysis, 3rd. Edition. McGraw-Hill, Inc.

[2] Blower, G. (2009). Random Matrices: High Dimensional Phenomena (London Mathematical Society Lecture Note Series). Cambridge University Press, 1st edition.

[3] Collins, B. (2003). Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability. Int. Math. Res. Not. no. 17, 953–982.

[4] Diaconis, P. and Evans, S. (2001). Linear functionals of eigenvalues of random matrices. Transactions Amer. Math. Soc. 353, 2615-2633.

[5] Diaconis, P. and Shahshahani, M. (1994). On the eigenvalues of random matrices, J. Appl. Probab. 31A, 49-62.

[6] Dyson, F. J. (1962a). Statistical theory of energy levels of complex systems, I. J. Math. Phys. 3, 140-156.

[7] Dyson, F. J. (1962b). Statistical theory of energy levels of complex systems, II. J. Math. Phys. 3, 166-175.

[8] Dyson, F. J. (1962c). Statistical theory of energy levels of complex systems, III. J. Math. Phys. 3, 1191-1198.

[9] Forrester, P. (2010). Log-Gases and Random Matrices (London Mathematical Society Monographs). Princeton University Press.

[10] Forrester, P. and Warnaar, S. (2008). The importance of the Selberg integral. Bull. Amer. Math. Soc. 45(4), 489-534.

[11] Gamelin, T. W. (2001). Complex Analysis. 1st Ed., Springer.

[12] Hardy, G., Littlewood, J. E. and Pólya, G. (1988). Inequalities, 2nd Ed. Cambridge University Press.

[13] Jiang, T. (2009). The entries of Haar-invariant matrices from the classical compact groups. Journal of Theoretical Probability (Online with DOI 10.1007/s10959-009-0241-7).

[14] Jiang, T. (2009). A variance formula related to quantum conductance. Physics Letters A 373, 2117-2121.

[15] Jiang, T. (2009). The entries of circular orthogonal ensembles. Journal of Mathematical Physics 50(6), 063302.

[16] Jiang, T. (2006). How many entries of a typical orthogonal matrix can be approximated by independent normals? Ann. Probab. 34(4), 1497-1529.

[17] Macdonald, I. (1998). Symmertic Functions and Hall Polynomials, Oxford Mathematical Monographs, 2nd Edn, Oxford University Press.

[18] Matsumoto, S. and Novak, J. (2009). Jucys-Murphy elements and unitary matrix integrals. preprint: arXiv:0905.1992v2.

[19] Mehta, M. L. (2004). Random Matrices, 3rd edition. Pure and Applied Mathematics (Amsterdam), 142. Elsevier/Academic Press, Amsterdam.

[20] Pastur, L. and Vasilchuk, V. (2004). On the moments of traces of matrices of classical groups. Comm. Math. Phys. 252, 149-166.

[21] Rains, E. (1997). High powers of random elements of compact Lie groups, Probab. Theory Related Fields 107, 219-241.

[22] Stolz, M. (2005). On the Diaconis-Shashahani method in random matrix theory. Journal of Algebraic Combinatorics 22, 471-491.