Statistical Properties of Eigenvalues of Laplace-Beltrami Operators

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Abstract

We study the eigenvalues of a Laplace-Beltrami operator defined on the set of the symmetric polynomials, where the eigenvalues are expressed in terms of partitions of integers. To study the behaviors of these eigenvalues, we assign partitions with the restricted uniform measure, the restricted Jack measure, the uniform measure or the Plancherel measure. We first obtain a new limit theorem on the restricted uniform measure. Then, by using it together with known results on other three measures, we prove that the global distribution of the eigenvalues is asymptotically a new distribution μ, the Gamma distribution, the Gumbel distribution and the Tracy-Widom distribution, respectively. The Tracy-Widom distribution is obtained for a special case only due to a technical constraint. An explicit representation of μ is obtained by a function of independent random variables. Two open problems are also asked.

Keywords: Laplace-Beltrami operator, eigenvalue, random partition, Plancherel measure, uniform measure, restricted Jack measure, restricted uniform measure, Tracy-Widom distribution, Gumbel distribution, Gamma distribution.
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1 Introduction

Consider the Laplace-Beltrami operator

$$\Delta_\alpha = \frac{\alpha}{2}\sum_{i=1}^{m} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{1\le i\neq j\le m} \frac{1}{y_i - y_j}\cdot y_i^2 \frac{\partial}{\partial y_i} \tag{1.1}$$

defined on the set of symmetric and homogeneous polynomial $u(x_1, \cdots, x_m)$ of all degrees. There are two important quantities associated with the operator: its eigenfunctions and eigenvalues. The eigenfunctions are the α-Jack polynomials and the eigenvalues are given by

$$\lambda_\kappa = n(m-1) + a(\kappa')\alpha - a(\kappa) \tag{1.2}$$

where $\kappa = (k_1, k_2, \cdots k_m)$ with $k_m > 0$ is a partition of integer n, that is, $\sum_{i=1}^m k_i = n$ and $k_1 \ge \cdots \ge k_m$, and κ' is the transpose of κ and

$$a(\kappa) = \sum_{i=1}^{m}(i-1)k_i = \sum_{i\ge 1}\binom{k_i'}{2}; \tag{1.3}$$

see, for example, Theorem 3.1 from Stanley (1989) or p. 320 and p. 327 from Macdonald (1998).

The Jack polynomials are multivariate orthogonal polynomials (Macdonald, 1998). They consist of three special cases: the zonal polynomials with $\alpha = 2$ which appear frequently in multivariate analysis of statistics (e.g., Muirhead, 1982); the Schur polynomials with $\alpha = 1$ and the zonal spherical functions with $\alpha = \frac{1}{2}$ which have rich applications in the group representation theory, algebraic combinatorics, statistics and random matrix theory [e.g., Macdonald (1998), Fulton and Harris (1999), Forrester (2010)].

In this paper we consider the statistical behaviors of the eigenvalues λ_κ given in (1.2). That is, how does λ_κ look like if κ is picked randomly? For example, what are the sample mean and the sample variance of λ_κ's, respectively? In fact, even though the expression of λ_κ is explicit, it is non-trivial to answer the question. In particular, it is hard to use a software to analyze them because the size of $\{\kappa;\ \kappa \text{ is a partition of } n\}$ is of order $\frac{1}{n}e^{C\sqrt{n}}$ for some constant C; see (2.57).

The same question was asked for the eigenvalues of random matrices and the eigenvalues of Laplace operators defined on compact Riemannian manifolds. For instance, the typical behavior of the eigenvalues of a large Wigner matrix is the Wigner semi-circle law (Wigner, 1958), and that of a Wishart matrix is the Marchenko-Pastur law (Marchenko and Pastur, 1967). The Weyl law is obtained for the eigenvalues of a Laplace-Beltrami operator acting on functions with the Dirichlet condition which vanish at the boundary of a bounded domain in the Euclidean space (Weyl, 1911). For example, the Weyl asymptotic formula says that $\frac{\lambda_k}{k^{d/2}} \sim (4\pi)^{-d/2}\frac{vol(M)}{\Gamma(\frac{d}{2}+1)}$ as $k \to \infty$, where d is the dimension of M and $vol(M)$

is the volume of M. It is proved by analyzing the trace of a heat kernel; see, e.g., p. 13 from Borthwick (2012). Let Δ_S be the spherical Laplacian operator on the unit sphere in $\mathbb{R}^{n+1}$. It is known that the eigenvalues of $-\Delta_S$ are $k(k+n-1)$ for $k = 0, 1, 2, \cdots$ with multiplicity of $\binom{n+k}{n} - \binom{n+k-2}{n}$; see, e.g., ch. 2 from Shubin (2001). Some other types of Laplace-Beltrami operators appear in the Riemannian symmetric spaces; see, e.g., Méliot (2014). Their eigenvalues are also expressed in terms of partitions of integers. Similar to this paper, those eigenvalues can also be analyzed.

To study a typical property of λ_κ in (1.2), how do we pick a partition randomly? We will sample κ by using four popular probability measures: the restricted uniform measure, the restricted Jack measure, the uniform measure and the Plancherel measure. While studying λ_κ for fixed operator Δ_α with m variables, the two restricted measures are adopted to investigate λ_κ by letting n become large. Look at the infinite version of the operator Δ_α:

$$\Delta_{\alpha,\infty} := \frac{\alpha}{2} \sum_{i=1}^{\infty} y_i^2 \frac{\partial^2}{\partial y_i^2} + \sum_{1 \le i \ne j < \infty} \frac{1}{y_i - y_j} \cdot y_i^2 \frac{\partial}{\partial y_i}, \tag{1.4}$$

which acts on the set of symmetric and homogeneous polynomial $u(x_1, \cdots, x_m)$ of degree $m \ge 0$ being arbitrary; see, for example, page 327 from Macdonald (1998). Recall (1.2). At "level" n, the set of eigenvalues of $\Delta_{\alpha,\infty}$ is $\{\lambda_\kappa; \kappa \in \mathcal{P}_n\}$. In this situation, the partition length m depends on n, this is the reason that we employ the uniform measure and the Plancherel measure.

Under the four measures, we prove in this paper that the limiting distribution of random variable λ_κ is a new distribution μ, the Gamma distribution, the Gumbel distribution and the Tracy-Widom distribution, respectively. Due to a technical constraint, the Tracy-Widom distribution is obtained for the case $\alpha = 1$ only. For other $\alpha > 0$, see a less precise result in Theorem 5 and Conjecture 1. The distribution μ is characterized by a function of independent random variables. More specifically, μ is the push-forward of $\frac{\alpha}{2} \cdot \frac{\xi_1^2 + \cdots + \xi_m^2}{(\xi_1 + \cdots + \xi_m)^2}$ where ξ_i's are i.i.d. random variables with the density $e^{-x} I(x \ge 0)$. In the following we will present these results in this order. We will see, in addition to a tool on random partitions developed in this paper (Theorem 6), a fruitful of work along this direction has been used: the approximation result on random partitions under the uniform measure by Pittel (1997); the largest part of a random partition asymptotically following the Tracy-Widom law by Baik *et al.* (1999), Borodin *et al.* (2000), Okounkov (2000) and Johannson (2001); Kerov's central limit theorem (Ivanov and Olshanski, 2001); the Stein method on random partitions by Fulman (2004); the limit law of random partitions under restricted Jack measure by Matsumoto (2008).

A consequence of our theory provides an answer at (1.6) for the size of the sample mean and sample variance of λ_κ aforementioned.

We do not pursue applications of our results in this paper. They may be useful in Migdal's formula for the partition functions of the 2D Yang-Mills theory [e.g., Witten

(1991) and Woodward (2005)]. Further possibilities can be seen, e.g., in the papers by Okounkov (2003) and Borodin and Gorin (2012).

We study the eigenvalues of the Laplace-Beltrami operator in terms of four different measures. This can also be continued by other probability measures on random partitions, for example, the q-analog of the Plancherel measure [e.g., Kerov (1992) and Féray and Méliot (2012)], the multiplicative measures [e.g., Vershik (1996)], the β-Plancherel measure (Baik and Rains, 2001), the Jack measure and the Schur measure [e.g., Okounkov (2003)].

Organization of the paper: We present our limit laws by using the four measures in Sections 1.1, 1.2, 1.3 and 1.4, respectively. Four figures corresponding to the four theorems are provided to show that curves based on data and the limiting curves match very well. In Section 1.5, we state a new result on random partitions. In Section 2, we prove all of the results. In Section 3 (Appendix), we compute the sample mean and sample variance of λ_κ mentioned in (1.6), calculate a non-trivial integral used earlier and derive the density function in Theorem 1 for two cases.

Notation: $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$. We assume that n is large and asymptotic notation such as $o(\cdot), O(\cdot)$ will be used under the assumption that $n \to \infty$. Let $\{X_n;\, n \geq 1\}$ be random variables and $\{w_n;\, n \geq 1\}$ be non-zero constants. If $\{X_n/w_n;\, n \geq 1\}$ is bounded in probability, i.e., $\lim_{K\to\infty} \sup_{n\geq 1} P(|X_n/w_n| \geq K) = 0$, we then write $X_n = O_p(w_n)$ as $n \to \infty$. If X_n/w_n converges to 0 in probability, we write $X_n = o_p(w_n)$. We write "cdf" for "cumulative distribution function" and "pdf" for "probability density function". We use $\kappa \vdash n$ if κ is a partition of n. The notation $[x]$ stands for the largest integer less than or equal to x.

Graphs: The convergence in Theorems 1, 2, 3 and 4 are illustrated in Figures 1-4: we compare the empirical pdfs, also called histograms in statistics literature, with their limiting pdfs in the left columns. The right columns compare the empirical cdfs with their limiting cdfs. These graphs suggest that the empirical ones and their limits match very well.

1.1 Limit under restricted uniform distribution

Let $\mathcal{P}_n$ denote the set of all partitions of n. Now we consider a subset of $\mathcal{P}_n$. Let $\mathcal{P}_n(m)$ and $\mathcal{P}'_n(m)$ be the sets of partitions of n with lengths at most m and with lengths exactly equal to m, respectively. Note that $\mathcal{P}_n(n) = \mathcal{P}_n$. Our limiting laws of λ_κ under the two measures are derived as follows. A simulation is shown in Figure 1.

THEOREM 1. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with $\alpha > 0$. Let $m \geq 2$, $\{\xi_i;\, 1 \leq i \leq m\}$ be i.i.d. random variables with density $e^{-x}I(x \geq 0)$ and μ be the measure induced by $\frac{\alpha}{2} \cdot \frac{\xi_1^2+\cdots+\xi_m^2}{(\xi_1+\cdots+\xi_m)^2}$. Then, under the uniform measure on $\mathcal{P}_n(m)$ or $\mathcal{P}'_n(m)$, $\frac{\lambda_\kappa}{n^2} \to \mu$ weakly as $n \to \infty$.*

By the definition of $\mathcal{P}'_n(m)$, the above theorem gives the typical behavior of the eigenvalues of the Laplace-Beltrami operator for fixed m. We will prove this theorem in Section 2.2. In Section 3.2, we compute the pdf $f(t)$ of $\frac{\xi_1^2+\cdots+\xi_m^2}{(\xi_1+\cdots+\xi_m)^2}$, which is different from μ by a multiplicative scalar, for $m=2,3$. It shows that $f(t)=\frac{1}{\sqrt{2t-1}}I_{[\frac{1}{2},1]}(t)$ for $m=2$; for $m=3$, the support of μ is $[\frac{1}{3},1]$ and

$$f(t)=\begin{cases}\frac{2}{\sqrt{3}}\pi, & \text{if } \frac{1}{3}\le t<\frac{1}{2};\\ \frac{2}{\sqrt{3}}\big(\pi-3\arccos\frac{1}{\sqrt{6t-2}}\big), & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$

From our computation, it does not seem easy to derive an explicit formula for the density function as $m\ge 4$. It would be interesting to explore this. The proof of Theorem 1 relies on a new result on random partitions from $\mathcal{P}_n(m)$ and $\mathcal{P}_n'(m)$ with the uniform distributions, which is of independent interest. We postpone it until Section 1.5.

Given numbers $x_1,\cdots,x_r$, the average and dispersion/fluctation of the data are usually measured by the sample mean $\bar{x}$ and the sample variance s^2, respectively, where

$$\bar{x}=\frac{1}{r}\sum_{i=1}^r x_i \quad\text{and}\quad s^2=\frac{1}{r-1}\sum_{i=1}^r (x_i-\bar{x})^2. \tag{1.5}$$

Replacing x_i's by λ_κ's as in (1.2) for all $\kappa\in\mathcal{P}_n(m)'$, then $r=|\mathcal{P}_n(m)'|$. We will prove in Section 3.1 that, by Theorem 1 and the bounded convergence theorem, we have

$$\frac{\bar{x}}{n^2}\to\frac{\alpha}{m+1}\quad\text{and}\quad\frac{s^2}{n^4}\to\frac{(m-1)\alpha^2}{(m+1)^2(m+2)(m+3)} \tag{1.6}$$

as $n\to\infty$. The proof is given in Section 3.1. The moment $(1/r)\sum_{i=1}^r x_i^j$ with x_i's replaced by λ_κ's can be analyzed similarly for other $j\ge 3$.

Comments. By a standard characterization of spacings of i.i.d. random variables with the uniform distribution on $[0,1]$ through exponential random variables [see, e.g., Sec 2.5.3 from Rubinstein and Kroese (2007) and Chapter 5 from Devroye (1986)], the limiting distribution μ in Theorem 1 is identical to any of the following:
(i) $\frac{\alpha}{2}\cdot\sum_{i=1}^m y_i^2$, where $y:=(y_1,\ldots,y_m)$ uniformly sits on $\{y\in[0,1]^m;\sum_{i=1}^m y_i=1\}$.
(ii) $\frac{\alpha}{2}\cdot\sum_{i=1}^m(U_{(i)}-U_{(i-1)})^2$ where $U_{(1)}\le\ldots\le U_{(m-1)}$ are the order statistics of i.i.d. random variables $\{U_i;\,1\le i\le m\}$ with uniform distribution on $[0,1]$ and $U_{(0)}=0$, $U_{(m)}=1$.

1.2 Limit under restricted Jack distribution

The Jack measure with parameter α chooses a partition $\kappa\in\mathcal{P}_n$ with probability

$$P(\kappa)=\frac{\alpha^n n!}{c_\kappa(\alpha)c_\kappa'(\alpha)}, \tag{1.7}$$

where

$$c_\kappa(\alpha)=\prod_{(i,j)\in\kappa}(\alpha(\kappa_i-j)+(\kappa_j'-i)+1)\quad\text{and}\quad c_\kappa'(\alpha)=\prod_{(i,j)\in\kappa}(\alpha(\kappa_i-j)+(\kappa_j'-i)+\alpha).$$

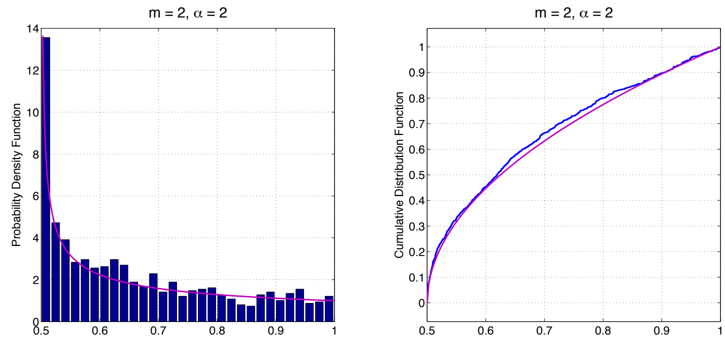

Figure 1: The histogram/empirical cdf of λ_κ/n^2 for $\alpha = m = 2$ is compared with pdf/cdf of μ in Theorem 1 at $n = 2000$. We independently sampled 1000 points according to μ.

The Jack measure naturally appears in the Atiyah-Bott formula from the algebraic geometry; see an elaboration in the notes by Okounkov (2013).

In this section, we consider the random restricted Jack measure studied by Matsumoto (2008). Let m be a fixed positive integer. Recall $\mathcal{P}_n(m)$ is the set of integer partitions of n with at most m parts. The induced restricted Jack distribution with parameter α on $\mathcal{P}_n(m)$ is defined by [we follow the notation by Matsumoto (2008)]

$$P^\alpha_{n,m}(\kappa) = \frac{1}{C_{n,m}(\alpha)} \frac{1}{c_\kappa(\alpha) c'_\kappa(\alpha)}, \quad \kappa \in \mathcal{P}_n(m), \tag{1.8}$$

with the normalizing constant

$$C_{n,m}(\alpha) = \sum_{\mu \in \mathcal{P}_n(m)} \frac{1}{c_\mu(\alpha) c'_\mu(\alpha)}.$$

Similarly, replacing $\mathcal{P}_n(m)$ above with "$\mathcal{P}'_n(m)$", we get the restricted Jack measure on $\mathcal{P}'_n(m)$. We call it $Q^\alpha_{n,m}$. The following is our result under the two measures.

THEOREM 2. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with parameter $\alpha > 0$. Set $\beta = 2/\alpha$. Then, for given $m \geq 2$, if κ is chosen according to $P^\alpha_{n,m}$ or $Q^\alpha_{n,m}$, then*

$$\frac{\lambda_\kappa - a_n}{b_n} \to \text{Gamma distribution with pdf } h(x) = \frac{1}{\Gamma(v)\,(2/\beta)^v} x^{v-1} e^{-\beta x/2} \text{ for } x \geq 0$$

weakly as $n \to \infty$, where

$$a_n = \frac{m - \alpha - 1}{2} n + \frac{\alpha}{2m} n^2, \quad b_n = \frac{n}{2m}, \quad v = \frac{1}{4}(m-1)\cdot(m\beta + 2).$$

By the definition of $\mathcal{P}'_n(m)$, the above theorem gives the typical behavior of the eigenvalues of the Laplace-Beltrami operator for fixed m under the restricted Jack measure.

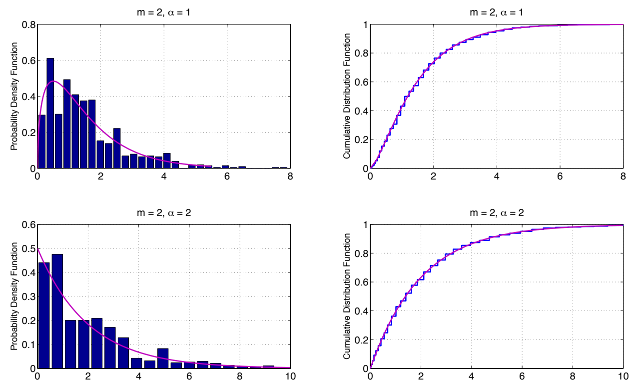

Figure 2: Top row compares histogram/empirical cdf of $(\lambda_n - a_n)/b_n$ in Theorem 2 for $m = 2$, $\alpha = 1$ with Gamma pdf/cdf at $n = 1000$. The quantity "$(\lambda_n - a_n)/b_n$" is independently sampled for 800 times. Similar interpretation applies to the bottom row for $m = \alpha = 2$.

Write $v = \frac{1}{2}\cdot\frac{1}{2}(m-1)(m\beta+2)$. Then the limiting distribution becomes a χ^2 distribution with (integer) degree of freedom $\frac{1}{2}(m-1)(m\beta+2)$ for $\beta = 1, 2$ or 4. See Figure 2 for numerical simulation.

We will prove Theorem 2 in Section 2.3. Indeed, since $\mathcal{P}_m(n)$ and $\mathcal{P}'_m(n)$ have asymptotically the same size, and neither the uniform measure nor the restricted Jack measure is concentrated on any set in $\mathcal{P}_m(n)$ or $\mathcal{P}'_m(n)$, for the proofs of Theorem 1 and 2, it suffices to prove the results on $\mathcal{P}_m(n)$.

1.3 Limit under uniform distribution

Let $\mathcal{P}_n$ denote the set of all partitions of n and $p(n)$ the number of such partitions. Recall the operator $\Delta_{\alpha,\infty}$ in (1.4) and the eigenvalues in (1.2). At "level" n, the set of eigenvalues is $\{\lambda_\kappa; \kappa \in \mathcal{P}_n\}$. The parameter "$m$" appearing in Theorems 1 and 2 is irrelevant here. Now we choose κ according to the uniform distribution on $\mathcal{P}_n$. The limiting distribution of λ_κ is given below. Denote $\zeta(x)$ the Riemman's zeta function.

Theorem 3. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with parameter $\alpha > 0$. If κ is chosen uniformly from the set $\mathcal{P}_n$, then*

$$cn^{-3/2}\lambda_\kappa - \log\frac{\sqrt{n}}{c} \to \text{Gumbel distribution with cdf } G(x) = \exp\big(-e^{-(x+K)}\big)$$

weakly as $n \to \infty$, where $c = \frac{\pi}{\sqrt{6}}$ and $K = \frac{6\zeta(3)}{\pi^2}(1-\alpha)$.

In Figure 3, we simulate the distribution of λ_κ at $n = 4000$ and compare with the Gumbel distribution $G(x)$ as in Theorem 3. Its proof will be given at Section 2.4. Comparing Figure 1 and Figure 3, we see the limiting behaviours of λ_κ differ significantly under uniform measures on $\mathcal{P}_n(m)$ with m fixed and $\mathcal{P}_n(n)$ with $m = n \to \infty$ respectively.

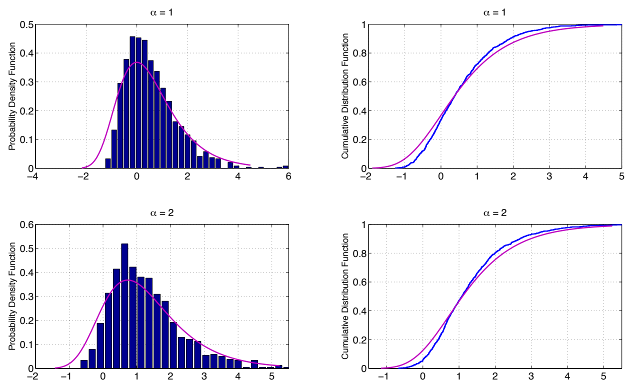

Figure 3: Top row compares histogram/empirical cdf of "$cn^{-3/2}\lambda_\kappa - \log\frac{\sqrt{n}}{c}$" for $\alpha = 1$ with the pdf $G'(x)$/cdf $G(x)$ in Theorem 3 at $n = 4000$. The quantity "$cn^{-3/2}\lambda_\kappa - \log\frac{\sqrt{n}}{c}$" is independently sampled for 1000 times. Similar interpretation applies to the bottom row for $\alpha = 2$.

1.4 Limit under Plancherel distribution

Review the operator $\Delta_{\alpha,\infty}$ in (1.4) and the eigenvalues in (1.2). At "level" n, the set of eigenvalues is $\{\lambda_\kappa; \kappa \in \mathcal{P}_n\}$. There is no parameter "m" appearing in Theorems 1 and 2. We now apply the Plancherel measure to understand this set of eigenvalues.

A random partition κ of n has the Plancherel measure if it is chosen from $\mathcal{P}_n$ with probability

$$P(\kappa) = \frac{\dim(\kappa)^2}{n!}, \tag{1.9}$$

where $\dim(\kappa)$ is the dimension of irreducible representations of the symmetric group $\mathcal{S}_n$ associated with κ. It is given by

$$\dim(\kappa) = \frac{n!}{\prod_{(i,j)\in\kappa}(k_i - j + k'_j - i + 1)}.$$

See, e.g., Frame *et al.* (1954). This measure is a special case of the α-Jack measure defined in (1.7) with $\alpha = 1$. The Tracy-Widom distribution is defined by

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right),\ s \in \mathbb{R}, \tag{1.10}$$

where $q(x)$ is the solution to the Painléve II differential equation

$$\begin{aligned} &q''(x) = xq(x) + 2q(x)^3 \ \text{ with boundary condition} \\ &q(x) \sim \mathrm{Ai}(x) \text{ as } x \to +\infty \end{aligned}$$

and $\mathrm{Ai}(x)$ denotes the Airy function. Replacing the uniform measure in Theorem 3 with the Plancherel measure, we get the following result.

Theorem 4. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with parameter $\alpha = 1$. If κ follows the Plancherel measure, then*

$$\frac{\lambda_\kappa - 2 \cdot n^{3/2}}{n^{7/6}} \to F_2$$

weakly as $n \to \infty$, where F_2 is as in (1.10).

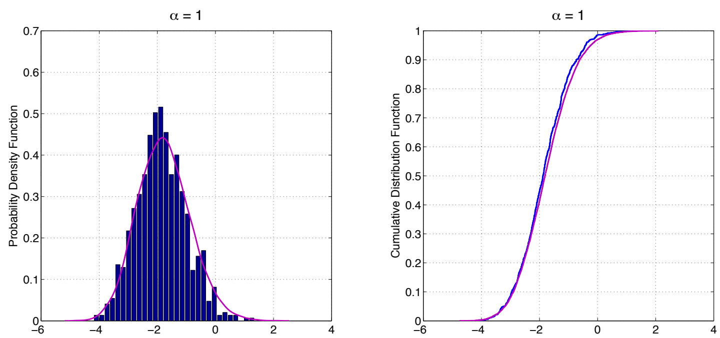

Figure 4: The histogram/empirical cdf of $T := (\lambda_\kappa - 2 \cdot n^{3/2})n^{-7/6}$ for $\alpha = 1$ is compared with pdf/cdf of F_2 in Theorem 4 at $n = 5000$. The value of T is independently sampled for 800 times.

The proof of this theorem will be presented in Section 2.5. In Figure 4, we simulate the limiting distribution of λ_κ with $\alpha = 1$ and compare it with F_2. For any $\alpha \neq 1$, we prove a weak result as follows.

Theorem 5. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with parameter $\alpha > 0$. If κ follows the Plancherel measure, then for any sequence of real numbers $\{a_n > 0\}$ with $\lim_{n\to\infty} a_n = \infty$,*

$$\frac{\lambda_\kappa - \left(2 + \frac{128}{27\pi^2}(\alpha - 1)\right) n^{3/2}}{n^{5/4} \cdot a_n} \to 0$$

in probability as $n \to \infty$.

The proof of Theorem 5 will be given in Section 2.6. We provide a conjecture on the limiting distribution for λ_κ with arbitrary $\alpha > 0$ under Plancherel measure.

Conjecture 1. *Let $\kappa \vdash n$ and λ_κ be as in (1.2). If κ has the Plancherel measure, then*

$$\frac{\lambda_\kappa - \left(2 + \frac{128}{27\pi^2}(\alpha - 1)\right) \cdot n^{3/2}}{n^{7/6}} \to (3 - 2\alpha)F_2$$

weakly as $n \to \infty$, where F_2 is as in (1.10).

The quantities "$3 - 2\alpha$" and "$n^{7/6}$" can be seen from the proofs of Theorems 4 and 5. The conjecture will be confirmed if there is a stronger version of the central limit theorem by Kerov [Theorem 5.5 by Ivanov and Olshanski (2001)]: the central limit theorem still holds if the Chebyshev polynomials are replaced by smooth functions.

One can also consider the same quantity under the α-Jack measure as in (1.7), a generalization of the Plancherel measure. However, under this measure, the limiting distribution of the largest part of a random partition is not known. There is only a conjecture made by Dolega and Féray (2014). In virtue of this and our proof of Theorem 4, we give a conjecture on λ_κ studied in this paper.

Conjecture 2. *Let $\kappa \vdash n$ and λ_κ be as in (1.2) with parameter $\alpha > 0$. If κ follows the α-Jack measure [the "α" here is the same as that in (1.2)], then*

$$\frac{\lambda_\kappa - 2\alpha^{-1/2} n^{3/2}}{n^{7/6}} \to F_\alpha$$

weakly as $n \to \infty$, and F_α is the α-analogue of the Tracy-Widom distribution F_2 in (1.10). The law F_α is equal to Λ_0 stated in Theorem 1.1 from Ramírez et al. (2011).

1.5 A new result on random partitions

At the same time as proving Theorem 1, we find the following result on the restricted random partitions, which is also interesting on its own merits.

Theorem 6. *Given $m \geq 2$. Let $\mathcal{P}_n(m)$ and $\mathcal{P}_n(m)'$ be as in Theorem 1. Let $(k_1, \cdots, k_m) \vdash n$ follow the uniform distribution on $\mathcal{P}_n(m)$ or $\mathcal{P}_n(m)'$. Then, as $n \to \infty$, $\frac{1}{n}(k_1, \cdots, k_m)$ converges weakly to the uniform distribution on the ordered simplex*

$$\Delta := \Big\{(x_1, \cdots, x_m) \in [0,1]^m;\ x_1 > \cdots > x_m \ \textit{and} \ \sum_{i=1}^m x_i = 1\Big\}. \tag{1.11}$$

It is known from Rabinowitz (1989) that the volume of $\Delta = \frac{\sqrt{m}}{m!(m-1)!}$. So the density function of the uniform distribution on Δ is equal to $\frac{m!(m-1)!}{\sqrt{m}}$.

If one picks a random partition $\kappa = (k_1, k_2, \cdots) \vdash n$ under the uniform measure, that is, under the uniform measure on $\mathcal{P}_n$, put the Young diagram of κ in the first quadrant, and shrink the curve by a factor of $n^{-1/2}$, Vershik (1996) proves that the new random curve converges to the curve $e^{-cx} + e^{-cy} = 1$ for $x, y > 0$, where $c = \pi/\sqrt{6}$. For the Plancherel measure, Logan and Shepp (1977) and Vershik and Kerov (1977) prove that, for a rotated and shrunk Young diagram κ, its boundary curve (see the "zig-zag" curve in Figure 5) converges to $\Omega(x)$, where

$$\Omega(x) = \begin{cases} \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4 - x^2}), & |x| \leq 2; \\ |x|, & |x| > 2. \end{cases} \tag{1.12}$$

As m is no longer fixed but equal to n, the above law differs from the one presented in Theorem 6. We will prove this result in Section 2.1.

2 Proofs

In this section we will prove the theorems stated earlier. Theorem 6 will be proved first because it will be used later.

2.1 Proof of Theorem 6

The following conclusion is based on the fact that $\mathcal{P}_n(m)$ and $\mathcal{P}_n(m)'$ have asymptotically the same size, and is not difficult to prove. We skip its proof.

Lemma 2.1. *Review the notation in Theorem 6. Assume, under $\mathcal{P}_n(m)$, $\frac{1}{n}(k_1, \cdots, k_m)$ converges weakly to the uniform distribution on Δ as $n \to \infty$. Then the same convergence also holds true under $\mathcal{P}_n(m)'$.*

We now introduce the equivalence of two uniform distributions.

Lemma 2.2. *Let $m \geq 2$ and $X_1 > \cdots > X_m \geq 0$ be random variables. Recall (1.11). Set*

$$W = \Big\{(x_1, \cdots, x_{m-1}) \in [0,1]^{m-1};\ x_1 > \cdots > x_m \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1\Big\}. \tag{2.1}$$

Then $(X_1, \cdots, X_m)$ follows the uniform distribution on Δ if and only if $(X_1, \cdots, X_{m-1})$ follows the uniform distribution on W.

Proof of Lemma 2.2. First, assume that $(X_1, \cdots, X_m)$ follows the uniform distribution on Δ. Then $(X_1, \cdots, X_{m-1})^T = A(X_1, \cdots, X_m)^T$ where A is the projection matrix with $A = (I_{m-1}, \mathbf{0})$ where $\mathbf{0}$ is a $(m-1)$-dimensional zero vector. Since a linear transform sends a uniform distribution to another uniform distribution [see p. 158 from Fristedt and Gray (1997)], and since $A\Delta = W$, we get that $(X_1, \cdots, X_{m-1})$ is uniformly distributed on W.

Now, assume $(X_1, \cdots, X_{m-1})$ is uniform on W. First, it is well known that

$$\text{the volume of } \Big\{(x_1, \cdots, x_m) \in [0,1]^m;\ \sum_{i=1}^{m} x_i = 1\Big\} = \frac{\sqrt{m}}{(m-1)!}; \tag{2.2}$$

see, e.g., Rabinowitz (1989). Thus, by symmetry,

$$\text{the volume of } \Delta = \frac{\sqrt{m}}{m!(m-1)!}. \tag{2.3}$$

Therefore, to show that $(X_1, \cdots, X_m)$ has the uniform distribution on Δ, it suffices to prove that, for any bounded measurable function φ defined on $[0,1]^m$,

$$E\varphi(X_1, \cdots, X_m) = \frac{m!(m-1)!}{\sqrt{m}} \int_{\Delta} \varphi(x_1, \cdots, x_m)\, dS \tag{2.4}$$

where the right hand side is a surface integral. Seeing that $\mathcal{A} : (x_1, \cdots, x_{m-1}) \in W \to (x_1, \cdots, x_{m-1}, 1-\sum_{i=1}^{m-1} x_i) \in \Delta$ is a one-to-one and onto map, then by a change of variables formula [see, e.g., Proposition 6.6.1 from Berger and Gostiaux (1988)],

$$\int_{\Delta} \varphi(x_1, \cdots, x_m)\, dS = \int_W \varphi\Big(x_1, \cdots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i\Big) \cdot \det(B^T B)^{1/2}\, dx_1 \cdots dx_{m-1}$$

where

$$B := \frac{\partial(x_1, \cdots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i)}{\partial(x_1, \cdots, x_{m-1})} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ -1 & -1 & \cdots -1 & -1 \end{pmatrix}_{m\times(m-1)}.$$

Trivially, $B^T B = I_{m-1} + ee^T$, where $e = (1, \cdots, 1)^T \in \mathbb{R}^{m-1}$, which has eigenvalues 1 with $m-2$ folds and eigenvalue m with one fold. Hence, $\det(B^T B) = m$. Thus, the right hand side of (2.4) is identical to

$$m!(m-1)! \int_W \varphi\Big(x_1, \cdots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i\Big)\, dx_1 \cdots dx_{m-1}. \tag{2.5}$$

It is well known that

$$\text{the volume of } \Big\{(x_1, \cdots, x_{m-1}) \in [0,1]^{m-1};\ \sum_{i=1}^{m-1} x_i \le 1\Big\} = \frac{1}{(m-1)!};$$

see, e.g., Stein (1966). Thus, by symmetry,

$$\text{the volume of } W = \frac{1}{m!(m-1)!}. \tag{2.6}$$

This says that the density of the uniform distribution on W is identical to $m!(m-1)!$. Consequently, the left hand side of (2.4) is equal to

$$m!(m-1)! \int_W \varphi\Big(x_1, \cdots, x_{m-1}, 1 - \sum_{i=1}^{m-1} x_i\Big)\, dx_1 \cdots dx_{m-1},$$

which together with (2.5) leads to (2.4). □

Fix $m \ge 2$. Let $\mathcal{P}_n(m)$ be the set of partitions of n with lengths at most m. It is known from Erdös and Lehner (1941) that

$$|\mathcal{P}_n(m)| \sim \frac{\binom{n-1}{m-1}}{m!} \sim \frac{n^{m-1}}{m!(m-1)!} \tag{2.7}$$

as $n \to \infty$.

Let us comment on the proof of Theorem 6 first. To show the weak convergence, for any bounded continuous function f defined on $\overline{W}$, the closure of W, it suffices to prove

$$\frac{1}{|\mathcal{P}_n(m)|} \sum_{(k_1,\cdots,k_m)\vdash n} f(\frac{k_1}{n},\dots,\frac{k_{m-1}}{n}) \to \frac{1}{\mathrm{Vol}(W)} \int_W f(x_1,\dots,x_{m-1})\, d\mathbf{x} \tag{2.8}$$

as $n \to \infty$. At first sight, it seems (2.8) can be obtained easily by using the convergence of a multi-dimensional Riemann sum to the corresponding integral. However, the interaction among the parts $k_1,\dots,k_m$ are complicated. The difficulty lies in controlling the LHS of (2.8) on the boundary of $\mathcal{P}_n(m)$ (that is, either two parts are equal or a certain part is zero), together with the restriction $\sum_{i=1}^m k_i = n$. Therefore, we need to make extra efforts. The main proof of this section is given below.

Proof of Theorem 6. By Lemma 2.1, it is enough to prove that, under $\mathcal{P}_n(m)$, $\frac{1}{n}(k_1,\cdots,k_m)$ converges weakly to the uniform distribution on Δ as $n \to \infty$.

We first prove the case for $m = 2$. In fact, since $k_1 + k_2 = n$ and $k_1 \ge k_2$, we have $\frac{1}{2}n \le k_1 \le n$. Recall W in (2.1). We know W is the interval $(\frac{1}{2}, 1)$. So it is enough to check that k_1 has the uniform distribution on $(\frac{1}{2}, 1)$. Indeed, for any $x \in (\frac{1}{2}, 1)$, the distribution function of $\frac{k_1}{n}$ is given by

$$\begin{aligned} P\Big((k_1, n-k_1);\ \frac{k_1}{n} \le x\Big) &= P\Big((k, n-k);\ \frac{n}{2} \le k_1 \le [nx]\Big) \\ &= \frac{nx - \frac{1}{2}n + O(1)}{\frac{1}{2}n + O(1)} \to 2x - 1 \end{aligned}$$

as $n \to \infty$, which is exactly the cdf of the uniform distribution on $(1/2, 1)$.

As per (2.6), the volume of W in (2.1) equals $\frac{1}{m!(m-1)!}$. Thus the density of the uniform distribution on W has the constant value of $m!(m-1)!$ on W. To prove the conclusion, it suffices to show the convergence of their moment generating functions, that is,

$$Ee^{(t_1k_1+\cdots+t_mk_m)/n} \to Ee^{t_1\xi_1+\cdots+t_m\xi_m} \tag{2.9}$$

as $n \to \infty$ for all $(t_1,\cdots,t_m) \in \mathbb{R}^m$, where $(\xi_1,\cdots,\xi_{m-1})$ has the uniform distribution on W by Lemma 2.2. We prove this by several steps.

Step 1: Estimate of LHS of (2.9). From (2.9), we know that the left hand side of (2.9) is identical to

$$\begin{aligned} &\frac{1}{|\mathcal{P}_n(m)|} \sum_{(k_1,\cdots,k_m)} e^{(t_1k_1+\cdots+t_mk_m)/n} \\ = &\frac{1}{|\mathcal{P}_n(m)|} \sum_{k_1>\cdots>k_m} e^{(t_1k_1+\cdots+t_mk_m)/n} + \frac{1}{|\mathcal{P}_n(m)|} \sum_{k\in Q_n} e^{(t_1k_1+\cdots+t_mk_m)/n} \end{aligned} \tag{2.10}$$

where all of the sums above are taken over $\mathcal{P}_n(m)$ with the corresponding restrictions, and

$$Q_n := \{k = (k_1, \cdots, k_m) \vdash n;\ k_i = k_j \text{ for some } 1 \le i < j \le m\}.$$

Let us first estimate the size of Q_n. Observe

$$Q_n = \cup_{i=1}^{m-1}\{k = (k_1, \cdots, k_m) \vdash n;\ k_i = k_{i+1}\}.$$

For any $\kappa = (k_1, \cdots, k_m) \vdash n$ with $k_i = k_{i+1}$, we know $k_1 + \cdots + 2k_i + k_{i+2} + \cdots + k_m = n$, which is a non-negative integer solutions of $j_1 + \cdots + j_{m-1} = n$. It is easily seen that the number of non-negative integer solutions of the equation $j_1 + \cdots + j_{m-1} = n$ is equal to $\binom{n+m-2}{m-2}$. Therefore,

$$|Q_n| \le (m-1)\binom{n+m-2}{m-2} \sim (m-1)\frac{n^{m-2}}{(m-2)!} \tag{2.11}$$

as $n \to \infty$. Also, by (2.7), $|\mathcal{P}_n(m)| \sim \frac{n^{m-1}}{m!(m-1)!}$. For $e^{(t_1k_1+\cdots+t_mk_m)/n} \le e^{|t_1|+\cdots+|t_m|}$ for all k_i's, we see that the last term in (2.10) is of order $O(n^{-1})$. Furthermore, we can assume all the k_i's are positive since $|\mathcal{P}_n(m-1)| = o(|\mathcal{P}_n(m)|)$. Consequently,

$$Ee^{(t_1k_1+\cdots+t_mk_m)/n} \quad \sim \quad \frac{m!(m-1)!}{n^{m-1}}\sum e^{(t_1k_1+\cdots+t_mk_m)/n} \tag{2.12}$$

where $(k_1, \cdots, k_m) \vdash n$ in the last sum runs over all positive integers such that $k_1 > \cdots > k_m > 0$.

Step 2: Estimate of RHS of (2.9). For a set $\mathcal{A}$, let $I_{\mathcal{A}}$ or $I(\mathcal{A})$ denote the indicator function of $\mathcal{A}$ which takes value 1 on the set A and 0 otherwise. Review that the density function on W is equal to the constant $m!(m-1)!$. For $\xi_1 + \cdots + \xi_m = 1$, we have

$$\begin{aligned} & Ee^{t_1\xi_1+\cdots+t_m\xi_m} \\ = \ & m!(m-1)!e^{t_m}\int_{[0,1]^{m-1}} e^{(t_1-t_m)x_1+\cdots+(t_{m-1}-t_m)x_{m-1}} I_{\mathcal{A}}\, dx_1 \ldots dx_{m-1} \\ = \ & m!(m-1)!e^{t_m}\int_{[0,1]^{m-1}} f(x_1, \cdots, x_{m-1}) I_{\mathcal{A}}\, dx_1 \ldots dx_{m-1}, \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} & \mathcal{A} = \Big\{(x_1, \cdots, x_{m-1}) \in [0,1]^{m-1};\ x_1 > \cdots > x_{m-1} > 1 - \sum_{i=1}^{m-1} x_i \ge 0\Big\}; \\ & f(x_1, \cdots, x_{m-1}) := e^{(t_1-t_m)x_1+\cdots+(t_{m-1}-t_m)x_{m-1}}. \end{aligned} \tag{2.14}$$

Step 3: Difference between LHS and RHS of (2.9). Denote

$$\begin{aligned} & \mathcal{A}_n := \Big\{(k_1, \cdots, k_{m-1}) \in \{1, \cdots, n\}^{m-1};\ \frac{k_1}{n} > \cdots > \frac{k_{m-1}}{n} > 1 - \sum_{i=1}^{m-1}\frac{k_i}{n} > 0\Big\}; \\ & f_n(k_1, \cdots, k_{m-1}) := e^{(t_1-t_m)k_1/n+\cdots+(t_{m-1}-t_m)k_{m-1}/n} \end{aligned}$$

for all $(k_1, \cdots, k_{m-1}) \in \mathcal{A}_n$. From (2.12), we obtain

$$
\begin{aligned}
& Ee^{(t_1k_1+\cdots+t_mk_m)/n} \\
\sim\ & e^{t_m}\frac{m!(m-1)!}{n^{m-1}} \sum_{k_1>\cdots>k_m>0} e^{(t_1-t_m)k_1/n+\cdots+(t_{m-1}-t_m)k_{m-1}/n} \\
=\ & m!(m-1)!e^{t_m}\sum_{k_1=1}^{n}\cdots\sum_{k_{m-1}=1}^{n}\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}}\cdots\int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} f_n(k_1,\cdots,k_m)I_{\mathcal{A}_n}\ dx_1\ldots dx_{m-1}.
\end{aligned}
$$

Writing the integral in (2.13) similar to the above, we get that

$$
\begin{aligned}
& Ee^{t_1\xi_1+\cdots+t_m\xi_m} - Ee^{(t_1k_1+\cdots+t_mk_m)/n} \\
\sim\ & m!(m-1)!e^{t_m}\sum_{k_1=1}^{n}\cdots\sum_{k_{m-1}=1}^{n}\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}}\cdots\int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
& \big(f(x_1,\cdots,x_{m-1})I_{\mathcal{A}} - f_n(k_1,\cdots,k_m)I_{\mathcal{A}_n}\big)\ dx_1\ldots dx_{m-1}
\end{aligned}
$$

which again is identical to

$$
\begin{aligned}
& m!(m-1)!e^{t_m}\sum_{k_1=1}^{n}\cdots\sum_{k_{m-1}=1}^{n}\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}}\cdots\int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
& \quad f(x_1,\cdots,x_{m-1})\left(I_{\mathcal{A}} - I_{\mathcal{A}_n}\right)\ dx_1\ldots dx_{m-1} \qquad (2.15) \\
& +m!(m-1)!e^{t_m}\sum_{k_1=1}^{n}\cdots\sum_{k_{m-1}=1}^{n}\int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}}\cdots\int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} \\
& \quad \left(f(x_1,\cdots,x_{m-1}) - f_n(k_1,\cdots,k_{m-1})\right)I_{\mathcal{A}_n}\ dx_1\ldots dx_{m-1} \qquad (2.16) \\
=\ & m!(m-1)!e^{t_m}\left(\mathcal{S}_1+\mathcal{S}_2\right),
\end{aligned}
$$

where $\mathcal{S}_1$ stands for the sum in (2.15) and $\mathcal{S}_2$ stands for the sum in (2.16). The next step is to show both $\mathcal{S}_1 \to 0$ and $\mathcal{S}_2 \to 0$ as $n \to \infty$ and this completes the proof.

Step 4: Proof of that $\mathcal{S}_2 \to 0$. First, for the term $\mathcal{S}_2$, given that

$$\frac{k_1-1}{n} \le x_1 \le \frac{k_1}{n}, \cdots, \frac{k_{m-1}-1}{n} \le x_{m-1} \le \frac{k_{m-1}}{n},$$

we have

$$|f(x_1,\cdots,x_{m-1}) - f_n(k_1,\cdots,k_{m-1})| \le \frac{1}{n}\exp\Big\{\sum_{i=1}^{m-1}|t_i-t_m|\Big\}\cdot\sum_{i=1}^{m-1}|t_i-t_m|.$$

Indeed, the above follows from the mean value theorem by considering $|g(1)-g(0)|$, where

$$g(s) := \exp\Big\{\sum_{i=1}^{m-1}(t_i-t_m)[sx_i+(1-s)\frac{k_i}{n}]\Big\}.$$

Thus

$$|\mathcal{S}_2| \le \Big(\frac{1}{n}\Big)^{m-1} n^{m-1} \frac{\exp\big\{\sum_{i=1}^{m-1}|t_i-t_m|\big\}\cdot\sum_{i=1}^{m-1}|t_i-t_m|}{n} \to 0$$

as $n\to\infty$.

Step 5. Proof of that $\mathcal{S}_1\to 0$. From (2.14), we immediately see that

$$\|f\|_\infty := \sup_{(x_1,\cdots,x_{m-1})\in[0,1]^{m-1}} |f(x_1,\cdots,x_{m-1})| \le e^{|t_1-t_m|+\cdots|t_{m-1}-t_m|}. \tag{2.17}$$

By definition, as k_i ranges from 1 to n for $i=1,\dots,m-1$, the function $I_{\mathcal{A}_n}$ equals 1 only when the followings hold

$$\frac{k_1}{n} > \frac{k_2}{n},\cdots,\frac{k_{m-2}}{n} > \frac{k_{m-1}}{n}, \frac{k_1+\cdots k_{m-2}+2k_{m-1}}{n} > 1, \frac{k_1+\cdots+k_{m-1}}{n} < 1. \tag{2.18}$$

Similarly, $I_{\mathcal{A}}$ equals 1 only when

$$x_1 > x_2,\cdots,x_{m-2} > x_{m-1}, x_1+\cdots+x_{m-2}+2x_{m-1} > 1, x_1+\cdots+x_{m-1} < 1. \tag{2.19}$$

Let $\mathcal{B}_n$ be a subset of $\mathcal{A}_n$ such that

$$\mathcal{B}_n = \mathcal{A}_n \cap \Big\{(k_1,\cdots,k_{m-1})\in\{1,2,\cdots,n\}^{m-1};\ \frac{k_{m-1}}{n}+\sum_{i=1}^{m-1}\frac{k_i}{n} > \frac{m}{n}+1\Big\}.$$

Given $(k_1,\cdots,k_{m-1})\in\mathcal{B}_n$, for any

$$\frac{k_1-1}{n} < x_1 < \frac{k_1}{n},\cdots,\frac{k_{m-1}-1}{n} < x_{m-1} < \frac{k_{m-1}}{n}, \tag{2.20}$$

it is easy to verify from (2.18) and (2.19) that $I_{\mathcal{A}}=1$. Hence,

$$\begin{aligned} & I_{\mathcal{A}_n} = I_{\mathcal{B}_n} + I_{\mathcal{A}_n\setminus\mathcal{B}_n} \\ \le\ & I_{\mathcal{A}} + I\Big\{(k_1,\cdots,k_{m-1})\in\{1,\cdots,n\}^{m-1};\ 1 < \frac{k_{m-1}}{n}+\sum_{i=1}^{m-1}\frac{k_i}{n} \le \frac{m}{n}+1\Big\} \\ =\ & I_{\mathcal{A}} + \sum_{j=n+1}^{n+m} I_{E_j} \end{aligned} \tag{2.21}$$

where

$$E_j := \Big\{(k_1,\cdots,k_{m-1})\in\{1,\cdots,n\}^{m-1};\ k_1+\cdots+k_{m-2}+2k_{m-1}=j\Big\}$$

for $n+1\le j\le m+n$. Similar to the argument as in *Step 1*,

$$\max_{n\le j\le m+n}|E_j| = O(n^{m-2}) \tag{2.22}$$

as $n \to \infty$. On the other hand, consider a subset of $\mathcal{A}_n^c := \{1, \cdots, n\}^{m-1} \backslash \mathcal{A}_n$ defined by

$$\begin{aligned}\mathcal{C}_n \quad := \quad & \Big\{(k_1, \cdots, k_{m-1}) \in \{1, 2, \cdots, n\}^{m-1};\ \text{either } k_i \le k_{i+1} - 1 \text{ for some } 1 \le i \le m-2, \\ & \text{or } k_1 + \cdots + k_{m-2} + 2k_{m-1} \le n,\ \text{ or } k_1 + \cdots + k_{m-1} \ge m + n - 1\Big\}.\end{aligned}$$

Set $\mathcal{A}^c = [0,1]^{m-1} \backslash \mathcal{A}$. Given $(k_1, \cdots, k_{m-1}) \in \mathcal{C}_n$, for any k_i's and x_i's satisfying (2.20), it is not difficult to check that $I_{\mathcal{A}^c} = 1$. Consequently,

$$\begin{aligned} I_{\mathcal{A}_n^c} \quad &= \quad I_{\mathcal{C}_n} + I\Big\{(k_1, \cdots, k_{m-1}) \in \mathcal{A}_n^c;\ k_i > k_{i+1} - 1 \text{ for all } \ 1 \le i \le m-2, \\ & \qquad\qquad k_1 + \cdots + k_{m-2} + 2k_{m-1} > n,\ \text{ and } k_1 + \cdots + k_{m-1} < m + n - 1\Big\} \\ &\le \quad I_{\mathcal{A}^c} + I(\mathcal{D}_{n,1}) + I(\mathcal{D}_{n,2}), \end{aligned}$$

or equivalently,

$$I_{\mathcal{A}_n} \ge I_{\mathcal{A}} - I(\mathcal{D}_{n,1}) - I(\mathcal{D}_{n,2}), \tag{2.23}$$

where

$$\begin{aligned} \mathcal{D}_{n,1} &:= \bigcup_{i=1}^{m-2} \big\{(k_1, \cdots, k_{m-1}) \in \{1, 2, \cdots, n\}^{m-1};\ k_i = k_{i+1}\big\}; \\ \mathcal{D}_{n,2} &:= \bigcup_{i=n}^{n+m-2} \big\{(k_1, \cdots, k_{m-1}) \in \{1, 2, \cdots, n\}^{m-1};\ k_1 + \cdots + k_{m-1} = i\big\}. \end{aligned}$$

By the same argument as in (2.11), we have $\max_{1 \le i \le 2} |\mathcal{D}_{n,i}| = O(n^{m-2})$ as $n \to \infty$. Joining (2.21) and (2.23), and assuming (2.20) holds, we arrive at

$$|I_{\mathcal{A}_n} - I_{\mathcal{A}}| \le I(\mathcal{D}_{n,1}) + I(\mathcal{D}_{n,2}) + \sum_{i=n+1}^{n+m} I_{E_i}$$

and $\sum_{i=1}^{2} |\mathcal{D}_{n,i}| + \sum_{i=n+1}^{n+m} |E_i| = O(n^{m-2})$ as $n \to \infty$ by (2.22). Review $\mathcal{S}_1$ in (2.15). Observe that $\mathcal{D}_{n,i}$'s and E_i's do not depend on x, we obtain from (2.17) that

$$\begin{aligned} \mathcal{S}_1 \quad &\le \quad \|f\|_\infty \cdot \sum_{k_1=1}^{n} \cdots \sum_{k_{m-1}=1}^{n} \Big[\sum_{i=1}^{2} I(\mathcal{D}_{n,i}) + \sum_{i=n}^{n+m} I_{E_i}\Big] \int_{\frac{k_1-1}{n}}^{\frac{k_1}{n}} \cdots \int_{\frac{k_{m-1}-1}{n}}^{\frac{k_{m-1}}{n}} 1 \, dx_1 \dots dx_{m-1} \\ &= \quad \|f\|_\infty \cdot \Big(\sum_{i=1}^{2} |\mathcal{D}_{n,i}| + \sum_{i=n}^{n+m} |E_i|\Big) \cdot \frac{1}{n^{m-1}} \\ &= \quad O(n^{-1}) \end{aligned}$$

as $n \to \infty$. The proof is completed.

□

2.2 Proof of Theorem 1

We first rewrite the eigenvalues of the Laplace-Beltrami operator given in (1.2) in terms of κ instead of a mixing of κ and κ'. A similar expression, which is essentially the same as ours, can be found on p. 596 from Dumitriu *et al.* (2007). So we skip the proof.

LEMMA 2.3. *Let $\alpha > 0$. Let λ_κ be as in (1.2). For $\kappa = (k_1, \cdots, k_m) \vdash n$, we have*

$$\lambda_\kappa = (m - \frac{\alpha}{2})n + \sum_{i=1}^{m} (\frac{\alpha}{2} k_i - i) k_i. \tag{2.24}$$

Let η follow the chi-square distribution $\chi^2(v)$ with density function

$$(2^{v/2}\Gamma(v/2))^{-1} x^{\frac{v}{2}-1} e^{-x/2}, \qquad x > 0. \tag{2.25}$$

The following lemma is on p. 486 from Kotz *et al.* (2000).

LEMMA 2.4. *Let $m \geq 2$ and $\eta_1, \cdots, \eta_m$ be independent random variables with $\eta_i \sim \chi^2(v_i)$ for each i. Set $X_i = \eta_i/(\eta_1 + \cdots + \eta_m)$ for each i. Then $(X_1, \cdots, X_{m-1})$ has density*

$$f(x_1, \cdots, x_{m-1}) = \frac{\Gamma(\frac{1}{2}\sum_{j=1}^{m} v_j)}{\prod_{j=1}^{m} \Gamma(\frac{1}{2} v_j)} \Big[\prod_{j=1}^{m-1} x_j^{(v_j/2)-1} \Big] \Big(1 - \sum_{j=1}^{m-1} x_j \Big)^{(v_m/2)-1}$$

on the set $U = \{(x_1, \cdots, x_{m-1}) \in [0,1]^{m-1}; \sum_{i=1}^{m-1} x_i \leq 1\}$.

Proof of Theorem 1. By Lemma 2.3, for m is fixed and $k_1 \leq n$, we have

$$\frac{\lambda_\kappa}{n^2} = \frac{\alpha}{2} \cdot \sum_{i=1}^{m} \Big(\frac{k_i}{n} \Big)^2 + o(1)$$

as $n \to \infty$. By Theorem 6, under the uniform distribution on either $\mathcal{P}_n(m)$ or $\mathcal{P}_n(m)'$, $\frac{1}{n}(k_1, \cdots, k_m)$ converges weakly to $(Z_1, \cdots, Z_m)$, which has the uniform measure on Δ. Note that Δ is the ordered simplex, hence we can not get the desired conclusion by directly applying (i) or (ii) from the Comments after the statement of Theorem 1. We will resolve this issue next.

Let $\xi_1, \cdots, \xi_m$ be independent random variables with the common density $e^{-x} I(x \geq 0)$. Set

$$S_m = \xi_1 + \cdots + \xi_m \quad \text{and} \quad X_i = \frac{\xi_{(i)}}{S_m}, \quad 1 \leq i \leq m$$

where $\xi_{(1)} > \cdots > \xi_{(m)}$ are the order statistics. By the continuous mapping theorem and the fact $\sum_{i=1}^{m} \xi_{(i)}^2 = \sum_{i=1}^{m} \xi_i^2$, we only need to show that $(Z_1, \cdots, Z_m)$ has the same distribution as that of $(X_1, \cdots, X_m)$. Review W in Lemma 2.2. Recall that the volume of the convex body W (as per (2.6)) is $(m!(m-1)!)^{-1}$. Therefore, by Lemma 2.2, it suffices to prove that

$$E\varphi(X_1, \cdots, X_{m-1}) = m!(m-1)! \int_W \varphi(x_1, \cdots, x_{m-1})\, dx_1 \cdots dx_{m-1} \tag{2.26}$$

for any bounded and measurable function φ defined on $[0,1]^{m-1}$. Recalling (2.25), we know $\chi^2(2)/2$ has the exponential density function $e^{-x} I(x \geq 0)$. Taking $v_1 = v_2 = \cdots = v_m = 2$ in Lemma 2.4, we see that the density function of $\big(\frac{\xi_1}{S_m}, \cdots, \frac{\xi_{m-1}}{S_m}\big)$ on U is equal to the constant $\Gamma(m) = (m-1)!$. Furthermore,

$$E\varphi(X_1, \cdots, X_{m-1}) = \sum_{\pi} E\Big[\varphi\Big(\frac{\xi_{\pi(1)}}{S_m}, \cdots, \frac{\xi_{\pi(m-1)}}{S_m}\Big) I(\xi_{\pi(1)} > \cdots > \xi_{\pi(m)})\Big],$$

where the sum is taken over every permutation π of m. Write $S_m = \xi_{\pi(1)} + \cdots + \xi_{\pi(m)}$. By the i.i.d. property of ξ_i's, we get

$$\begin{aligned} & E\varphi(X_1, \cdots, X_{m-1}) \\ = \;& m! \cdot E\Big[\varphi\Big(\frac{\xi_{(1)}}{S_m}, \cdots, \frac{\xi_{(m-1)}}{S_m}\Big) I\Big(\frac{\xi_{(1)}}{S_m} > \cdots > \frac{\xi_{(m-1)}}{S_m} > 1 - \frac{\sum_{i=1}^{m-1} \xi_{(i)}}{S_m}\Big)\Big] \\ = \;& m!(m-1)! \int_U \varphi(x_1, \cdots, x_{m-1}) I\Big(x_1 > \cdots > x_{m-1} > 1 - \sum_{i=1}^{m-1} x_i\Big)\, dx_1 \cdots dx_{m-1} \end{aligned}$$

for $\big(\frac{\xi_{(1)}}{S_m}, \cdots, \frac{\xi_{(m-1)}}{S_m}, 1 - \frac{\sum_{i=1}^{m-1} \xi_{(i)}}{S_m}\big)$ is a function of $\big(\frac{\xi_1}{S_m}, \cdots, \frac{\xi_{m-1}}{S_m}\big)$ which has a constant density $(m-1)!$ on U as shown earlier. Easily, the last term above is equal to the right hand side of (2.26). The proof is then completed. $\square$

2.3 Proof of Theorem 2

We start with a result on the restricted Jack probability measure $P^{\alpha}_{n,m}$ as in (1.8).

Lemma 2.5. *(Matsumoto, 2008). Let $\alpha > 0$ and $\beta = 2/\alpha$. For a given integer $m \geq 2$, let $\kappa = (k_{n,1}, \cdots, k_{n,m}) \vdash n$ be chosen with probability $P^{\alpha}_{n,m}(\kappa)$. Then, as $n \to \infty$,*

$$\Big(\sqrt{\frac{\alpha m}{n}}\big(k_{n,i} - \frac{n}{m}\big)\Big)_{1 \leq i \leq m}$$

converges weakly to a limiting distribution with density function

$$g(x_1, \cdots, x_m) = const \cdot e^{-\frac{\beta}{2}\sum_{i=1}^m x_i^2} \cdot \prod_{1 \leq j < k \leq m} |x_j - x_k|^{\beta} \tag{2.27}$$

for all $x_1 \geq x_2 \geq \cdots \geq x_m$ such that $x_1 + \cdots + x_m = 0$.

The idea of the proof of Theorem 2 below lies in that, by virtue of Lemma 2.5, we are able to write λ_κ in (1.2) in terms of the trace of a Wishart matrix. Due to this we get the Gamma density by evaluating the moment generating function (or the Laplace transform) of the trace through (2.27).

Proof of Theorem 2. Let

$$Y_{n,i} = \sqrt{\frac{\alpha m}{n}}\big(k_{n,i} - \frac{n}{m}\big)$$

for $1 \le i \le m$. By Lemma 2.5, under $P^{\alpha}_{n,m}$, we know $(Y_{n,1}, \cdots, Y_{n,m})$ converges weakly to a random vector $(X_1, \cdots, X_m)$ with density function $g(x_1, \cdots, x_m)$ as in (2.27). Checking the proof of Lemma 2.5, it is easy to see that its conclusion still holds for $Q^{\alpha}_{n,m}$ without changing its proof. Solve for $k_{n,i}$'s to have

$$k_{n,i} = \frac{n}{m} + \sqrt{\frac{n}{\alpha m}}Y_{n,i}$$

for $1 \le i \le m$. Substitute these for the corresponding terms in (2.24) to see that

$$\begin{aligned}
&\lambda_\kappa - \big(m - \frac{\alpha}{2}\big)n \\
= \ & \sum_{i=1}^{m}\Big[\frac{\alpha}{2}\big(\frac{n}{m} + \sqrt{\frac{n}{m\alpha}}Y_{n,i}\big) - i\Big]\cdot\big(\frac{n}{m} + \sqrt{\frac{n}{m\alpha}}Y_{n,i}\big) \\
= \ & \frac{\alpha}{2}\sum_{i=1}^{m}\big(\frac{n}{m} + \sqrt{\frac{n}{m\alpha}}Y_{n,i}\big)^2 - \sum_{i=1}^{m} i\big(\frac{n}{m} + \sqrt{\frac{n}{m\alpha}}Y_{n,i}\big) \\
= \ & \frac{\alpha}{2}\cdot\frac{n^2}{m} + \sqrt{\alpha}\cdot\big(\frac{n}{m}\big)^{3/2}\sum_{i=1}^{m}Y_{n,i} + \frac{n}{2m}\sum_{i=1}^{m}Y_{n,i}^2 - \frac{n(m+1)}{2} - \sqrt{\frac{n}{m\alpha}}\sum_{i=1}^{m} iY_{n,i} \\
= \ & \frac{\alpha}{2}\cdot\frac{n^2}{m} - \frac{n(m+1)}{2} + \frac{n}{2m}\sum_{i=1}^{m}Y_{n,i}^2 - \sqrt{\frac{n}{m\alpha}}\sum_{i=1}^{m} iY_{n,i}
\end{aligned}$$

since $\sum_{i=1}^{m} Y_{n,i} = 0$. According to the notation of a_n and b_n,

$$\frac{\lambda_\kappa - a_n}{b_n} = \sum_{i=1}^{m}Y_{n,i}^2 - \frac{2}{\sqrt{\alpha}}\sqrt{\frac{m}{n}}\sum_{i=1}^{m} iY_{n,i}.$$

Since $(Y_{n,1}, \cdots, Y_{n,m})$ converges weakly to the random vector $(X_1, \cdots, X_m)$, taking

$$h_1(y_1, \cdots, y_m) = \sum_{i=1}^{m} iy_i \quad \text{and} \quad h_2(y_1, \cdots, y_m) = \sum_{i=1}^{m} y_i^2,$$

respectively, by the continuous mapping theorem,

$$\sum_{i=1}^{m} iY_{n,i} \to \sum_{i=1}^{m} iX_i \quad \text{and} \quad \sum_{i=1}^{m} Y_{n,i}^2 \to \sum_{i=1}^{m} X_i^2$$

weakly as $n \to \infty$. By the Slutsky lemma,

$$\frac{\lambda_\kappa - a_n}{b_n} = \sum_{i=1}^m Y_{n,i}^2 + O_p\big(n^{-1/2}\big) \to \sum_{i=1}^m X_i^2$$

weakly as $n \to \infty$. Now let us calculate the moment generating function of $\sum_{i=1}^m X_i^2$. Recall (2.27). Let C_n be the normalizing constant such that

$$g(x_1, \cdots, x_m) = C_m \cdot e^{-\frac{\beta}{2}\sum_{i=1}^m x_i^2} \cdot \prod_{1 \le j < k \le m} |x_j - x_k|^\beta$$

is a probability density function on the subset of $\mathbb{R}^m$ such that $x_1 \ge x_2 \ge \cdots \ge x_m$ and $x_1 + \cdots + x_m = 0$. We then have

$$\begin{aligned}
Ee^{t\sum_{i=1}^m X_i^2} &= C_m \int_{\mathbb{R}^{m-1}} e^{t\sum_{i=1}^m x_i^2} g(x_1, \cdots, x_m)\, dx_1, \cdots, dx_{m-1} \\
&= C_m \int_{\mathbb{R}^{m-1}} e^{-\frac{\beta}{2}\sum_{i=1}^m (1-\frac{2t}{\beta})x_i^2} \prod_{1 \le j < k \le m} |x_j - x_k|^\beta\, dx_1, \cdots, dx_{m-1} \\
&= \Big(1 - \frac{2t}{\beta}\Big)^{-\frac{1}{2}\cdot(\frac{m(m-1)}{2}\beta + (m-1))} \cdot \int_{\mathbb{R}^{m-1}} g(y_1, \cdots, y_m)\, dy_1, \cdots, dy_{m-1} \\
&= \Big(1 - \frac{2t}{\beta}\Big)^{-\frac{1}{4}(m-1)\cdot(m\beta+2)} \qquad (2.28)
\end{aligned}$$

for $t < \frac{\beta}{2}$, where a transform $y_i = (1 - \frac{2t}{\beta})^{1/2} x_i$ is taken in the third step for $i = 1, \cdots, m-1$. It is easy to check that the term in (2.28) is also the generating function of the Gamma distribution with density function $h(x) = \frac{1}{\Gamma(v)\,(2/\beta)^v} x^{v-1} e^{-\beta x/2}$ for all $x \ge 0$, where $v = \frac{1}{4}(m-1)\cdot(m\beta+2)$. By the uniqueness theorem, we know the conclusion holds. □

2.4 Proof of Theorem 3

The following lemma is Theorem 2 from Pittel (1997).

LEMMA 2.6. *Let $\kappa = (k_1, \cdots, k_m)$ be a partition of n chosen according to the uniform measure on $\mathcal{P}(n)$. Then*

$$k_j = \begin{cases} \big(1 + O_p((\log n)^{-1})\big)E(j) & \text{if } 1 \le j \le \log n; \\ E(j) + O_p(nj^{-1}\log n)^{1/2} & \text{if } \log n \le j \le n^{1/2}; \\ E(j) + O_p(e^{-cjn^{-1/2}} n^{1/2}\log n)^{1/2} & \text{if } n^{1/2} \le j \le \kappa_n; \\ (1 + O_p(a_n^{-1}))E(j) & \text{if } \kappa_n \le j \le k_n \end{cases}$$

uniformly as $n \to \infty$, where $c = \pi/\sqrt{6}$,

$$E(x) = \frac{\sqrt{n}}{c} \log \frac{1}{1 - e^{-cxn^{-1/2}}} \quad \text{for } x > 0,$$

$$\kappa_n = \Big[\frac{\sqrt{n}}{4c}\log n\Big] \quad \text{and} \quad k_n = \Big[\frac{\sqrt{n}}{2c}(\log n - 2\log\log n - a_n)\Big]$$

with $a_n \to \infty$ *and* $a_n = o(\log\log n)$ *as* $n \to \infty$.

Based on Lemma 2.6, we get the following law of large numbers. This is a key estimate in the proof of Theorem 3.

Lemma 2.7. *Let* $\kappa = (k_1, \cdots, k_m)$ *be a partition of* n *chosen according to the uniform measure on* $\mathcal{P}(n)$. *Then* $n^{-3/2}\sum_{j=1}^m k_j^2 \to a$ *in probability as* $n \to \infty$, *where*

$$a = c^{-3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt \tag{2.29}$$

and $c = \pi/\sqrt{6}$. *The above conclusion also holds if "*$\sum_{j=1}^m k_j^2$*" is replaced by "*$2\sum_{j=1}^m jk_j$*".*

Proof of Lemma 2.7. Define

$$F(x) = \log\frac{1}{1-e^{-cxn^{-1/2}}}$$

for $x > 0$. Note that both $E(x)$ and $F(x)$ are decreasing in $x \in (0, \infty)$.

Step 1. We first claim that

$$\max_{1\le j\le \frac{1}{6}\sqrt{n}\log n}\Big|\frac{k_j}{E(j)} - 1\Big| \to 0 \tag{2.30}$$

in probability as $n \to \infty$. (The choice of $1/6$ is rather arbitrary here. Actually, any number strictly less than $1/2c$ would work). We prove this next.

Notice

$$\begin{aligned}\max_{x\ge 1} E(x) = E(1) &= -\frac{\sqrt{n}}{c}\log\left(1-e^{-cn^{-1/2}}\right)\\ &\sim -\frac{\sqrt{n}}{c}\log\left(cn^{-1/2}\right) \sim \frac{1}{2c}\sqrt{n}\log n\end{aligned}$$

as $n \to \infty$ since $1 - e^{-x} \sim x$ as $x \to 0$. Observe

$$\frac{\sqrt{nj^{-1}\log n}}{E(j)} = -c\sqrt{\log n}\cdot\frac{j^{-1/2}}{\log\left(1-e^{-cjn^{-1/2}}\right)}.$$

Therefore,

$$\max_{\log n\le j\le (\log n)^2}\frac{\sqrt{nj^{-1}\log n}}{E(j)} \le \frac{c}{F(\log^2 n)} \to 0$$

and

$$\max_{\log^2 n\le j\le n^{1/2}}\frac{\sqrt{nj^{-1}\log n}}{E(j)} \le c\frac{(\log n)^{-1/2}}{F(n^{1/2})} \to 0$$

as $n \to \infty$. By Lemma 2.6,

$$\max_{\log n \le j \le \sqrt{n}} \Big| \frac{k_j}{E(j)} - 1 \Big| = o_p(1) \tag{2.31}$$

as $n \to \infty$. Now we consider the case for $n^{1/2} \le j \le \kappa_n$ where κ_n is as in Lemma 2.6. Trivially, $\frac{1}{4c} > \frac{1}{6}$. Notice that

$$\begin{aligned}
\max_{n^{1/2} \le j \le (1/6)\sqrt{n}\log n} \frac{(e^{-cjn^{-1/2}} n^{1/2} \log n)^{1/2}}{E(j)} &\le \frac{(e^{-c} n^{1/2} \log n)^{1/2}}{E((1/6)\sqrt{n}\log n)} \\
&= \frac{(ce^{-c/2}) n^{-1/4} (\log n)^{1/2}}{F((1/6)\sqrt{n}\log n)}.
\end{aligned}$$

Evidently,

$$F\big(\frac{1}{6}\sqrt{n}\log n\big) = -\log\big(1 - e^{-(c/6)\log n}\big) \sim \frac{1}{n^{c/6}} \tag{2.32}$$

as $n \to \infty$. This says

$$\max_{n^{1/2} \le j \le (1/6)\sqrt{n}\log n} \Big| \frac{k_j}{E(j)} - 1 \Big| = o_p(1)$$

as $n \to \infty$ by Lemma 2.6. This together with (2.31) and the first expression of k_j in Lemma 2.6 concludes (2.30), which is equivalent to that

$$k_j = E(j) + \epsilon_{n,j} E(j) \tag{2.33}$$

uniformly for all $1 \le j \le (1/6)\sqrt{n}\log n$, where $\epsilon_{n,j}$'s satisfy

$$H_n := \sup_{1 \le j \le (1/6)\sqrt{n}\log n} |\epsilon_{n,j}| \to 0 \tag{2.34}$$

in probability as $n \to \infty$.

Step 2. We approximate the two sums in (2.35) and (2.36) below by integrals in this step. The assertions (2.33) and (2.34) imply that

$$\sum_{1 \le j \le (1/6)\sqrt{n}\log n} k_j^2 = \Big(\sum_{1 \le j \le (1/6)\sqrt{n}\log n} E(j)^2 \Big)(1 + o_p(1)); \tag{2.35}$$

$$\sum_{1 \le j \le (1/6)\sqrt{n}\log n} j k_j = \Big(\sum_{1 \le j \le (1/6)\sqrt{n}\log n} jE(j) \Big)(1 + o_p(1)) \tag{2.36}$$

as $n \to \infty$. For $E(x)$ is decreasing in x we have

$$\int_1^m E(x)^2\,dx = \sum_{j=1}^{m-1} \int_j^{j+1} E(x)^2\,dx \le \sum_{j=1}^{m-1} E(j)^2$$

for any $m \geq 2$. Consequently,

$$\sum_{1 \leq j \leq (1/6)\sqrt{n}\log n} E(j)^2 \geq \int_1^{m_1} E(x)^2\, dx$$

with $m_1 = \left[\frac{1}{6}\sqrt{n}\log n\right]$. Similarly,

$$\int_0^{m+1} E(x)^2\, dx = \sum_{j=0}^{m} \int_j^{j+1} E(x)^2\, dx \geq \sum_{j=1}^{m+1} E(j)^2$$

for any $m \geq 1$. The two inequalities imply

$$\int_1^{m_1} E(x)^2\, dx \leq \sum_{1 \leq j \leq (1/6)\sqrt{n}\log n} E(j)^2 \leq \int_0^{\infty} E(x)^2\, dx. \tag{2.37}$$

By the same argument,

$$\int_1^{m_1} E(x)\, dx \leq \sum_{1 \leq j \leq (1/6)\sqrt{n}\log n} E(j) \leq \int_0^{\infty} E(x)\, dx. \tag{2.38}$$

Now we estimate $\sum_{1 \leq j \leq (1/6)\sqrt{n}\log n} jE(j)$. Use the inequality

$$jE(j+1) \leq \int_j^{j+1} xE(x)\, dx \leq (j+1)E(j)$$

to have

$$(j+1)E(j+1) - E(j+1) \leq \int_j^{j+1} xE(x)\, dx \leq jE(j) + E(j)$$

for all $j \geq 0$. Sum the inequalities over j and use (2.38) to get

$$\begin{aligned} \int_1^{m_1} xE(x)\, dx - \int_0^{\infty} E(x)\, dx \quad &\leq \sum_{1 \leq j \leq (1/6)\sqrt{n}\log n} jE(j) \\ &\leq \int_0^{\infty} xE(x)\, dx + \int_0^{\infty} E(x)\, dx. \end{aligned} \tag{2.39}$$

Step 3. In this step, we evaluate integrals $\int E(x)\, dx$, $\int E(x)^2\, dx$ and $\int xE(x)\, dx$. First,

$$\int_0^{\infty} E(x)\, dx = \frac{\sqrt{n}}{c} \int_0^{\infty} \log \frac{1}{1 - e^{-cxn^{-1/2}}}\, dx.$$

Set

$$t = e^{-cxn^{-1/2}} \quad \text{then} \quad x = \frac{\sqrt{n}}{c} \log \frac{1}{t} \quad \text{and} \quad dx = -\frac{\sqrt{n}}{ct} dt. \tag{2.40}$$

Hence

$$\int_0^\infty E(x)\,dx = \frac{n}{c^2}\int_0^1 \frac{\log(1-t)}{-t}\,dt = O(n) \tag{2.41}$$

as $n \to \infty$ considering the second integral above is finite. Using the same discussion, we have

$$\int_0^\infty E(x)^2\,dx = \frac{n^{3/2}}{c^3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt;$$
$$\int_0^\infty xE(x)\,dx = \frac{n^{3/2}}{c^3}\int_0^1 \frac{1}{t}\log\frac{1}{t}\log\frac{1}{1-t}\,dt.$$

By the two identities above (3.44) from Pittel (1997), we have

$$\int_0^1 \frac{\log^2(1-t)}{t}\,dt = 2\int_0^1 \frac{1}{t}\log\frac{1}{t}\log\frac{1}{1-t}\,dt. \tag{2.42}$$

From the same calculation as in (2.40), we see that

$$\int_1^{m_1} E(x)^2\,dx = \frac{n^{3/2}}{c^3}\int_{e^{-cm_1n^{-1/2}}}^{e^{-cn^{-1/2}}} \frac{\log^2(1-t)}{t}\,dt \sim \frac{n^{3/2}}{c^3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt$$

as $n \to \infty$ since $m_1 = \left[\frac{1}{6}\sqrt{n}\log n\right]$. By the same reasoning,

$$\int_1^{m_1} xE(x)\,dx \sim \frac{n^{3/2}}{2c^3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt.$$

The above two integrals and that in (2.41) join (2.37), (2.38) and (2.39) to conclude

$$\sum_{1\le j\le (1/6)\sqrt{n}\log n} E(j)^2 \sim \frac{n^{3/2}}{c^3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt; \tag{2.43}$$
$$\sum_{1\le j\le (1/6)\sqrt{n}\log n} jE(j) \sim \frac{n^{3/2}}{2c^3}\int_0^1 \frac{\log^2(1-t)}{t}\,dt \tag{2.44}$$

as $n \to \infty$.

Step 4. We will get the desired conclusion in this step. Now connecting (2.43) and (2.44) with (2.35) and (2.36) we obtain

$$\sum_{1\le j\le (1/6)\sqrt{n}\log n} k_j^2 = an^{3/2}(1+o_p(1)); \tag{2.45}$$
$$\sum_{1\le j\le (1/6)\sqrt{n}\log n} jk_j = \frac{a}{2}n^{3/2}(1+o_p(1)) \tag{2.46}$$

as $n \to \infty$, where "a" is as in (2.29). For the number of parts of $\kappa = (k_1, \cdots, k_m)$, Erdös and Lehner (1941) obtain that

$$\frac{\pi}{\sqrt{6n}}m - \log\frac{\sqrt{6n}}{\pi} \to \mu \tag{2.47}$$

weakly as $n \to \infty$ where μ is a probability measure with cdf $F_\mu(v) = e^{-e^{-v}}$ for every $v \in \mathbb{R}$. See also Fristedt (1993). This implies that

$$P\big(m > \frac{1}{c}\sqrt{n}\log n\big) \to 0 \tag{2.48}$$

as $n \to \infty$. Now, for any $\epsilon > 0$, by (2.45),

$$\begin{aligned}
& P\Big(\big|a - n^{-3/2}\sum_{j=1}^{m} k_j^2\big| \geq \epsilon\Big) \\
\leq\ & P\Big(\big|a - n^{-3/2}\sum_{1\leq j\leq \frac{1}{6}\sqrt{n}\log n} k_j^2\big| \geq \epsilon/2\Big) + P\Big(n^{-3/2}\sum_{\frac{1}{6}\sqrt{n}\log n\leq j\leq m} k_j^2 \geq \epsilon/2\Big) \\
\leq\ & P\Big(m > \frac{1}{c}\sqrt{n}\log n\Big) + P\Big(n^{-3/2}\sum_{\frac{1}{6}\sqrt{n}\log n\leq j\leq m} k_j^2 \geq \epsilon/2, m \leq \frac{1}{c}\sqrt{n}\log n\Big) + o(1) \\
\leq\ & P\Big(n^{-3/2}\sum_{\frac{1}{6}\sqrt{n}\log n\leq j\leq \frac{1}{c}\sqrt{n}\log n} k_j^2 \geq \epsilon/2\Big) + o(1)
\end{aligned} \tag{2.49}$$

as $n \to \infty$. Denote by l_n the least integer greater than or equal to $\frac{1}{6}\sqrt{n}\log n$. Seeing that k_j is decreasing in j, it is seen from (2.33) and then (2.32) that

$$\begin{aligned}
k_j \leq k_{l_n} &= E(l_n)(1 + o_p(1)) \\
&\leq E\big(\frac{1}{6}\sqrt{n}\log n\big)(1 + o_p(1)) \\
&\sim c^{-1}n^{(1/2)-(c/6)}(1 + o_p(1))
\end{aligned} \tag{2.50}$$

for all $\frac{1}{6}\sqrt{n}\log n \leq j \leq \frac{1}{c}\sqrt{n}\log n$ as $n \to \infty$. This implies

$$\begin{aligned}
n^{-3/2}\sum_{\frac{1}{6}\sqrt{n}\log n\leq j\leq \frac{1}{c}\sqrt{n}\log n} k_j^2 &\leq C \cdot n^{-3/2}\sqrt{n}(\log n)\big(n^{1/2-c/6}\big)^2(1 + o_p(1)) \\
&\sim Cn^{-c/3}(\log n)(1 + o_p(1)) = o_p(1)
\end{aligned}$$

as $n \to \infty$, where C is a constant. This together with (2.49) yields the first conclusion of the lemma. Similarly, by (2.46) and (2.48), for any $\epsilon > 0$,

$$\begin{aligned}
& P\Big(\big|\frac{a}{2} - n^{-3/2}\sum_{j=1}^{m} jk_j\big| \geq \epsilon\Big) \\
\leq\ & P\Big(\big|\frac{a}{2} - n^{-3/2}\sum_{1\leq j\leq \frac{1}{6}\sqrt{n}\log n} jk_j\big| \geq \epsilon/2\Big) \\
& \quad + P\Big(n^{-3/2}\sum_{\frac{1}{6}\sqrt{n}\log n\leq j\leq \frac{1}{c}\sqrt{n}\log n} jk_j \geq \epsilon/2\Big) + P\Big(m > \frac{1}{c}\sqrt{n}\log n\Big) \to 0
\end{aligned}$$

as $n \to \infty$ considering

$$\begin{aligned} n^{-3/2} \sum_{\frac{1}{6}\sqrt{n}\log n \le j \le \frac{1}{c}\sqrt{n}\log n} jk_j &\le C \cdot n^{-3/2} \cdot n^{(1/2)-(c/6)} (\sqrt{n}\log n)^2 (1 + o_p(1)) \\ &= Cn^{-c/6} (\log n)^2 (1 + o_p(1)) \to 0 \end{aligned}$$

in probability as $n \to \infty$ by (2.50) again. We then get the second conclusion of the lemma. □

Finally we are ready to prove Theorem 3.

Proof of Theorem 3. Let a be as in (2.29). Set

$$U_n = \frac{\pi}{\sqrt{6n}} m - \log \frac{\sqrt{6n}}{\pi};$$

$$V_n = a - n^{-3/2} \sum_{j=1}^{m} k_j^2; \quad W_n = \frac{a}{2} - n^{-3/2} \sum_{j=1}^{m} jk_j.$$

By (2.47) and Lemma 2.7, U_n converges weakly to cdf $F_\mu(v) = e^{-e^{-v}}$ as $n \to \infty$, and both V_n and W_n converge to 0 in probability. Solving m, $\sum_{j=1}^{m} k_j^2$ and $\sum_{j=1}^{m} jk_j$ in terms of U_n, V_n and W_n, respectively, and substituting them for the corresponding terms of λ_κ in Lemma 3, we get

$$\begin{aligned} \lambda_\kappa &= -\frac{\alpha}{2} n + nm + \sum_{j=1}^{m} (\frac{\alpha}{2} k_j - j) k_j \\ &= -\frac{\alpha}{2} n + n\big(U_n + \log \frac{\sqrt{6n}}{\pi}\big) \cdot \frac{\sqrt{6n}}{\pi} + \frac{\alpha}{2} (a - V_n) n^{3/2} - (\frac{a}{2} - W_n) n^{3/2}. \end{aligned}$$

Therefore,

$$c \frac{\lambda_\kappa}{n^{3/2}} - \log \frac{\sqrt{n}}{c} = U_n + \big(\frac{\alpha - 1}{2}\big) ac - \frac{c\alpha}{2} V_n + cW_n + o(1) \tag{2.51}$$

as $n \to \infty$. We finally evaluate a in (2.29). Indeed, by (2.42), the Taylor expansion and integration by parts,

$$\begin{aligned} (ac) \cdot c^2 &= \int_0^1 \frac{\log^2(1-t)}{t}\, dt \\ &= 2 \int_0^1 \frac{1}{t} \log t \log(1-t)\, dt \\ &= -2 \int_0^1 \frac{1}{t} \log t \sum_{n=1}^{\infty} \frac{t^n}{n}\, dt = -2 \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 t^{n-1} \log t\, dt \\ &= 2 \sum_{n=1}^{\infty} \frac{1}{n^3} = 2\zeta(3). \end{aligned}$$

This and (2.51) prove the theorem by the Slutsky lemma. □

2.5 Proof of Theorems 4

Proof of Theorem 4. For a partition κ and its conjugate κ', Frobenius (1900) shows that

$$\frac{a(\kappa') - a(\kappa)}{\binom{n}{2}} = \frac{\chi^{\kappa}_{(2,1^{n-2})}}{\dim(\kappa)},$$

where $\chi^{\kappa}_{(2,1^{n-2})}$ is the value of χ^{κ}, the irreducible character of $\mathcal{S}_n$ associated to κ, on the conjugacy class indexed by $(2, 1^{n-2}) \vdash n$.

By Theorem 6.1 from Ivanov and Olshanski (2001) for the special case

$$p_2^{\#(n)}(\kappa) := n(n-1)\frac{\chi^{\kappa}_{(2,1^{n-2})}}{\dim(\kappa)}$$

or Theorem 1.2 from Fulman (2004), we have

$$\frac{a(\kappa') - a(\kappa)}{n} \to N\big(0, \frac{1}{2}\big)$$

weakly as $n \to \infty$. It is known from Baik *et al.* (1999), Borodin *et al.* (2000), Johannson (2001) and Okounkov (2000) that

$$\frac{k_1 - 2\sqrt{n}}{n^{1/6}} \to F_2 \quad \text{and} \quad \frac{m - 2\sqrt{n}}{n^{1/6}} \to F_2 \tag{2.52}$$

weakly as $n \to \infty$, where F_2 is as in (1.10). The k_1 and m have the same limiting distribution in (2.52), since k_1 and m are duals under transposition, and the distribution stays the same under transposition. Therefore, by using (1.2) for the case $\alpha = 1$,

$$\begin{aligned}\frac{\lambda_\kappa - 2n^{3/2}}{n^{7/6}} &= \frac{n(m-1) + a(\kappa') - a(\kappa) - 2n^{3/2}}{n^{7/6}} \\ &= \frac{m - 2\sqrt{n}}{n^{1/6}} - n^{-1/6} + \frac{a(\kappa') - a(\kappa)}{n^{7/6}}\end{aligned}$$

converges weakly to F_2 as $n \to \infty$, where F_2 is as in (1.10). □

2.6 Proof of Theorem 5

The proof of Theorem 5 is involved. The reason is that, when $\alpha = 1$, the term $a(\kappa') - a(\kappa)$ is negligible as shown in the proof of Theorem 4 . When $\alpha \neq 1$, reviewing (1.2), it will be seen next that the term $a(\kappa')\alpha - a(\kappa)$, under the Plancherel measure, is much larger and contributes to λ_κ essentially.

We first recall some notation. Let $\kappa = (k_1, k_2, \cdots, k_m)$ with $k_m \geq 1$ be a partition of n. Set coordinates u and v by

$$u = \frac{j - i}{\sqrt{n}} \quad \text{and} \quad v = \frac{i + j}{\sqrt{n}}. \tag{2.53}$$

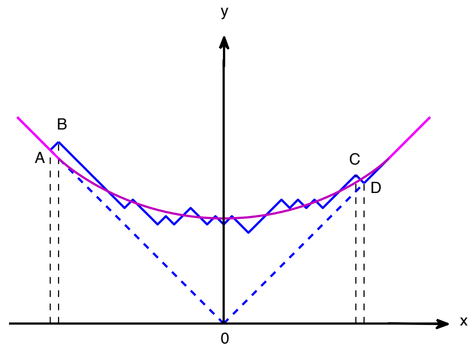

Figure 5: The "zig-zag" curve is the graph of $y = g_\kappa(x)$ and the smooth one is $y = \Omega(x)$. Facts: $A = (-\frac{m}{\sqrt{n}}, \frac{m}{\sqrt{n}})$, $D = (\frac{k_1}{\sqrt{n}}, \frac{k_1}{\sqrt{n}})$, and $g_\kappa(x) = \Omega(x)$ if $x \geq \max\{\frac{k_1}{\sqrt{n}}, 2\}$ or $x \leq -\max\{\frac{m}{\sqrt{n}}, 2\}$.

This is the same as flipping and then rotating the diagram of κ counter clockwise $135°$ and scaling it by a factor of $\sqrt{n/2}$ so that the area of the new diagram is equal to 2. Denote by $g_\kappa(x)$ the boundary curve of the new Young diagram. See such a graph as in Figure 5. It follows that $g_\kappa(x)$ is a Lipschitz function for all $x \in \mathbb{R}$.

For a piecewise smooth and compactly supported function $h(x)$ defined on $\mathbb{R}$, its Sobolev norm is given by

$$\|h\|_\theta^2 = \iint_{\mathbb{R}^2} \Big(\frac{h(s) - h(t)}{s - t}\Big)^2 \, dsdt. \tag{2.54}$$

Let $\kappa = (k_1, k_2, \cdots, k_m)$ with $k_m \geq 1$ be a partition of n. For $x \geq 0$, the notation $\lceil x \rceil$ stands for the least positive integer greater than or equal to x. Define

$$f_\kappa(x) = \frac{1}{\sqrt{n}} k_{\lceil \sqrt{n} x \rceil}, \qquad x \geq 0. \tag{2.55}$$

Recall from (1.12) that $\Omega(x) = \frac{2}{\pi}(x \arcsin \frac{x}{2} + \sqrt{4 - x^2})$ for $|x| \leq 2$ and $|x|$ otherwise. The following is a large deviation bound on a rare event under the Plancherel measure.

Lemma 2.8. *Define $L_\kappa(x) = \frac{1}{2} g_\kappa(2x)$ and $\bar{\Omega}(x) = \frac{1}{2}\Omega(2x)$ for $x \in \mathbb{R}$. Then for any $n \geq 2$ and any subset $\mathcal{F}$ of the partitions of n,*

$$P(\mathcal{F}) \leq \exp\big\{C\sqrt{n} - n \inf_{\kappa \in \mathcal{F}} I(\kappa)\big\},$$

where $C > 0$ is an absolute constant and

$$I(\kappa) = \|L_\kappa - \bar{\Omega}\|_\theta^2 - 4 \int_{|s|>1} (L_\kappa(s) - \bar{\Omega}(s)) \cosh^{-1}|s| \, ds. \tag{2.56}$$

Proof of Lemma 2.8. For any non-increasing function $F(x)$ defined on $(0, \infty)$ such that $\int_{\mathbb{R}} F(x)\, dx = 1$, define

$$\theta_F = 1 + 2\int_0^\infty \int_0^{F(x)} \log\big(F(x) + F^{-1}(y) - x - y\big)\, dy\, dx$$

where $F^{-1}(y) = \inf\{x \in \mathbb{R};\, F(x) \le y\}$. According to (1.8) from Logan and Shepp (1977), $P(\kappa) \le C\sqrt{n}\cdot \exp\big\{-n\theta_{f_\kappa}\big\}$ for all $n \ge 2$, where C is a numerical constant and f_κ is defined as in (2.55). By the Euler-Hardy-Ramanujan formula, $p(n)$, the total number of partitions of n, satisfies that

$$p(n) \sim \frac{1}{4\sqrt{3}\,n}\cdot \exp\Big\{\frac{2\pi}{\sqrt{6}}\sqrt{n}\Big\} \tag{2.57}$$

as $n \to \infty$. Thus, for any subset $\mathcal{F}$ of the partitions of n, we have

$$\begin{aligned} P(\mathcal{F}) &\le Cp(n)\cdot\sqrt{n}\exp\Big\{-n\inf_{\kappa\in\mathcal{F}}\theta_{f_\kappa}\Big\} \\ &\le C'\exp\Big\{C'\sqrt{n} - n\inf_{\kappa\in\mathcal{F}}\theta_{f_\kappa}\Big\} \end{aligned}$$

where C' is another numerical constant independent of n. For the curve $y = f_\kappa(x)$ in (2.55), consider the following transform

$$X = \frac{x-y}{2} \quad\text{and}\quad Y = \frac{x+y}{2}.$$

We name the new curve by $y = L_{f_\kappa}(x)$. By (2.53) and the definition $L_\kappa(x) = \frac{1}{2}g_\kappa(2x)$, we have $L_{f_\kappa}(x) = L_\kappa(-x)$ for all $x \in \mathbb{R}$. By Lemmas 2, 3 and 4 from Kerov (2003),

$$\begin{aligned} \theta_{f_\kappa} &= \|L_{f_\kappa} - \bar{\Omega}\|_\theta^2 + 4\int_{|s|>1}(L_{f_\kappa}(s) - \bar{\Omega}(s))\cosh^{-1}|s|\,ds \\ &= \|L_\kappa - \bar{\Omega}\|_\theta^2 - 4\int_{|s|>1}(L_\kappa(s) - \bar{\Omega}(s))\cosh^{-1}|s|\,ds \end{aligned}$$

considering $\Omega(x)$ is an even function. We then get the desired result. □

The next lemma says that the second term on the right hand side of (2.56) is small for almost all partitions.

Lemma 2.9. *Let $L_\kappa(x)$ and $\bar{\Omega}(x)$ be as in Lemma 2.8. Let $\{t_n > 0;\, n \ge 1\}$ satisfy $t_n \to \infty$ and $t_n = o(n^{1/3})$ as $n \to \infty$. Set $H_n = \{\kappa = (k_1, \cdots, k_m) \vdash n;\, k_m \ge 1,\, 2\sqrt{n} - t_n n^{1/6} \le m,\, k_1 \le 2\sqrt{n} + t_n n^{1/6}\}$. Then, as $n \to \infty$, $P(H_n) \to 1$ and*

$$\int_{|s|>1}(L_\kappa(s) - \bar{\Omega}(s))\cosh^{-1}|s|\,ds \cdot I_{H_n} = O(n^{-2/3}t_n^2). \tag{2.58}$$

Proof of Lemma 2.9. Since m and k_1 have the same probability distribution under the Plancherel measure, by (2.52), $\lim_{n\to\infty} P(H_n) = 1$. Review the definitions of L_κ and $\bar{\Omega}$ in Lemma 2.8. Trivially,

$$\text{LHS of (2.58)} = \frac{1}{4}\int_{|x|>2}(g_\kappa(x) - \Omega(x))\cosh^{-1}\frac{|x|}{2}\,dx\cdot I_{H_n}.$$

By definition, $g_\kappa(x) = \Omega(x)$ if $x \geq \frac{k_1}{\sqrt{n}} \vee 2$ or $x \leq -\big(\frac{m}{\sqrt{n}} \vee 2\big)$. It follows that

$$\begin{aligned}
& \text{LHS of (2.58)} \\
\leq\ & C_n \cdot \Big[\int_2^{2+n^{-1/3}t_n} \big|g_\kappa(x) - \Omega(x)\big|\, dx + \int_{-2-n^{-1/3}t_n}^{-2} \big|g_\kappa(x) - \Omega(x)\big|\, dx \Big] \qquad (2.59)
\end{aligned}$$

where

$$\begin{aligned}
C_n &= \sup\Big\{\cosh^{-1}\frac{|x|}{2};\ -(\frac{m}{\sqrt{n}} \vee 2) \leq x \leq \frac{k_1}{\sqrt{n}} \vee 2\Big\} \cdot I_{H_n} \\
&\leq \sup\Big\{\cosh^{-1}\frac{|x|}{2};\ -3 \leq x \leq 3\Big\} < \infty
\end{aligned}$$

as n is sufficiently large. Now

$$\begin{aligned}
& \int_2^{2+n^{-1/3}t_n} \big|g_\kappa(x) - \Omega(x)\big|\, dx \cdot I_{H_n} \\
\leq\ & n^{-1/3}t_n \cdot \max\Big\{\big|g_\kappa(x) - \Omega(x)\big|;\ 2 \leq x \leq 2 + n^{-1/3}t_n\Big\} \cdot I_{H_n}. \qquad (2.60)
\end{aligned}$$

By the triangle inequality, the Liptschitz property of $g_\kappa(x)$ and the fact $\Omega(x) = |x|$ for $|x| \geq 2$, we see

$$\begin{aligned}
\big|g_\kappa(x) - \Omega(x)\big| &\leq \big|g_\kappa(x) - g_\kappa(2 + 2n^{-1/3}t_n)\big| + \big|g_\kappa(2 + 2n^{-1/3}t_n) - \Omega(x)\big| \\
&\leq \big|x - (2 + 2n^{-1/3}t_n)\big| + \big|2 + 2n^{-1/3}t_n - x\big| \\
&\leq 2\big[(2 + 2n^{-1/3}t_n) - x\big] \leq 4n^{-1/3}t_n
\end{aligned}$$

for $2 \leq x \leq 2 + n^{-1/3}t_n$ and $\kappa \in H_n$ whence $g_\kappa(2 + 2n^{-1/3}t_n) = 2 + 2n^{-1/3}t_n$. This and (2.60) imply that the first integral in (2.59) is dominated by $O(n^{-2/3}t_n^2)$. By the same argument, the second integral in (2.59) has the same upper bound. Then the conclusion follows. □

To prove Lemma 2.10, we need to examine $g_\kappa(x)$ more closely. For $(k_1, k_2, \ldots, k_m) \vdash n$, assume

$$\begin{aligned}
& k_1 = \cdots = k_{l_1} > k_{l_1+1} = \cdots = k_{l_2} > \cdots > k_{l_{p-1}+1} = \cdots = k_m \geq 1 \ \text{ with} \\
& 0 = l_0 < l_1 < \cdots < l_p = m \qquad (2.61)
\end{aligned}$$

for some $p \geq 1$. To ease notation, let $\bar{k}_i = k_{l_i}$ for $i = 1, 2, \cdots, p$ and $\bar{k}_{p+1} = 0$. So the partition κ is determined by $\{\bar{k}_i, l_i\}$'s. It is easy to see that the corners (see, e.g., points A, B, C, D in Figure 5) sitting on the curve of $y = g_\kappa(x)$ listed from the leftmost to the rightmost in order are

$$\big(-\frac{l_p}{\sqrt{n}}, \frac{l_p}{\sqrt{n}}\big), \cdots, \big(\frac{\bar{k}_i - l_i}{\sqrt{n}}, \frac{\bar{k}_i + l_i}{\sqrt{n}}\big), \big(\frac{\bar{k}_{i+1} - l_i}{\sqrt{n}}, \frac{\bar{k}_{i+1} + l_i}{\sqrt{n}}\big), \cdots, \big(\frac{\bar{k}_1}{\sqrt{n}}, \frac{\bar{k}_1}{\sqrt{n}}\big)$$

for $i = 1, 2, \cdots, p$. As a consequence,

$$g_\kappa(x) = \begin{cases} \frac{2\bar{k}_i}{\sqrt{n}} - x, & \text{if } \frac{\bar{k}_i - l_i}{\sqrt{n}} \le x \le \frac{\bar{k}_i - l_{i-1}}{\sqrt{n}}; \\ \frac{2l_i}{\sqrt{n}} + x, & \text{if } \frac{\bar{k}_{i+1} - l_i}{\sqrt{n}} \le x \le \frac{\bar{k}_i - l_i}{\sqrt{n}} \end{cases} \tag{2.62}$$

for all $1 \le i \le p$, and $g_\kappa(x) = |x|$ for other $x \in \mathbb{R}$. In particular, taking $i = 1$ and p, respectively, we get

$$g_\kappa(x) = \begin{cases} \frac{2k_1}{\sqrt{n}} - x, & \text{if } \frac{k_1 - l_1}{\sqrt{n}} \le x \le \frac{k_1}{\sqrt{n}}; \\ \frac{2m}{\sqrt{n}} + x, & \text{if } -\frac{m}{\sqrt{n}} \le x \le \frac{k_m - m}{\sqrt{n}} \end{cases}$$

for $l_0 = 0$, $l_p = m$, $\bar{k}_1 = k_1$, and $\bar{k}_p = k_m$.

We need to estimate $\sum_{i=1}^m ik_i$ in the proof of Theorem 5. The following lemma links it to $g_\kappa(x)$. We will then be able to evaluate the sum through Kerov's central limit theorem (Ivanov and Olshanski, 2001).

Lemma 2.10. *Let $\kappa = (k_1, k_2, \cdots, k_m) \vdash n$ with $k_m \ge 1$ and $g_\kappa(x)$ be as in (2.62). Then*

$$\sum_{i=1}^m ik_i = \frac{1}{8}n^{3/2}\int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\,dx - \frac{1}{6}m^3 + \frac{1}{2}n.$$

Proof of Lemma 2.10. Easily,

$$\begin{aligned} &\int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\,dx \\ = \ &\sum_{i=1}^p \int_{(\bar{k}_i - l_i)/\sqrt{n}}^{(\bar{k}_i - l_{i-1})/\sqrt{n}} (g_\kappa(x) - x)^2\,dx + \sum_{i=1}^p \int_{\frac{\bar{k}_{i+1} - l_i}{\sqrt{n}}}^{\frac{\bar{k}_i - l_i}{\sqrt{n}}} (g_\kappa(x) - x)^2\,dx. \end{aligned} \tag{2.63}$$

By (2.62), the slopes of $g_\kappa(x)$ in the first sum of (2.63) are equal to -1. Hence, it is equal to

$$\begin{aligned} 4\sum_{i=1}^p \int_{(\bar{k}_i - l_i)/\sqrt{n}}^{(\bar{k}_i - l_{i-1})/\sqrt{n}} \Big(\frac{\bar{k}_i}{\sqrt{n}} - x\Big)^2\,dx &= 4\sum_{i=1}^p \int_{l_{i-1}/\sqrt{n}}^{l_i/\sqrt{n}} t^2\,dt \\ &= 4\int_{l_0/\sqrt{n}}^{l_p/\sqrt{n}} t^2\,dt = \frac{4m^3}{3n^{3/2}} \end{aligned}$$

because $l_0 = 0$ and $l_p = m$. In the second sum in (2.63), $g_\kappa(x)$ has slopes equal to 1. As a consequence, it is identical to

$$\sum_{i=1}^p \int_{\frac{\bar{k}_{i+1} - l_i}{\sqrt{n}}}^{\frac{\bar{k}_i - l_i}{\sqrt{n}}} \frac{4l_i^2}{n}\,dx = \frac{4}{n^{3/2}}\sum_{i=1}^p (\bar{k}_i - \bar{k}_{i+1})l_i^2.$$

In summary,

$$\int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\, dx = \frac{4m^3}{3n^{3/2}} + \frac{4}{n^{3/2}} \sum_{i=1}^{p} (\bar{k}_i - \bar{k}_{i+1}) l_i^2. \tag{2.64}$$

Now, let us evaluate the sum. Set $k_j = 0$ for $j > m$ for convenience and $\Delta_i = k_i - k_{i+1}$ for $i = 1, 2, \cdots$. Then $\Delta_i = 0$ unless $i = l_1, \cdots, l_p$. Observe

$$\begin{aligned} \sum_{i=1}^{\infty} i k_i = \sum_{i=1}^{\infty} i \sum_{j=i}^{\infty} \Delta_j &= \sum_{j=1}^{\infty} \Delta_j \sum_{i=1}^{j} i \\ &= \frac{1}{2} \sum_{j=1}^{\infty} j^2 \Delta_j + \frac{1}{2} \sum_{j=1}^{\infty} j \Delta_j. \end{aligned}$$

Furthermore,

$$\sum_{j=1}^{\infty} j \Delta_j = \sum_{j=1}^{\infty} \sum_{i=1}^{j} \Delta_j = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \Delta_j = \sum_{i=1}^{\infty} k_i = n.$$

The above two assertions say that $\sum_{j=1}^{\infty} j^2 \Delta_j = -n + 2 \sum_{i=1}^{\infty} i k_i$. Now,

$$\sum_{j=1}^{\infty} j^2 \Delta_j = \sum_{i=1}^{p} l_i^2 (k_{l_i} - k_{l_i+1}) = \sum_{i=1}^{p} l_i^2 (\bar{k}_i - \bar{k}_{i+1})$$

by the fact $k_{l_i+1} = k_{l_{i+1}} = \bar{k}_{i+1}$ from (2.61). This together with (2.64) shows

$$\int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\, dx = \frac{4m^3}{3n^{3/2}} + \frac{4}{n^{3/2}} \Big(-n + 2 \sum_{i=1}^{\infty} i k_i \Big).$$

Solve this equation to get

$$\sum_{i=1}^{\infty} i k_i = \frac{1}{8} n^{3/2} \int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\, dx - \frac{1}{6} m^3 + \frac{1}{2} n.$$

The proof is complete. □

Under the Plancherel measure, both $m/\sqrt{n}$ and $k_1/\sqrt{n}$ go to 2 in probability. In lieu of this fact, the next lemma writes the integral in Lemma 2.10 in a slightly cleaner form. The main tools of the proof are the Tracy-Widom law of the largest part of a random partition, the large deviations and Kerov's cental limit theorem.

Lemma 2.11. *Let $g_\kappa(x)$ be as in (2.62) and set*

$$Z_n = \int_{-m/\sqrt{n}}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2\, dx - \int_{-2}^{2} (\Omega(x) - x)^2\, dx$$

where $\Omega(x)$ is as in (1.12). Then, for any $\{a_n > 0; n \geq 1\}$ with $\lim_{n\to\infty} a_n = \infty$, we have

$$\frac{n^{1/4}}{a_n} Z_n \to 0$$

in probability as $n \to \infty$.

Proof of Lemma 2.11. Without loss of generality, we assume

$$a_n = o(n^{1/4}) \tag{2.65}$$

as $n \to \infty$. Set

$$Z'_n = \int_{-2}^{2} (g_\kappa(x) - x)^2 \, dx - \int_{-2}^{2} (\Omega(x) - x)^2 \, dx.$$

Write

$$\frac{n^{1/4}}{a_n} Z_n = \frac{n^{1/4}}{a_n} Z'_n + \frac{1}{n^{1/12} a_n} R_{n,1} + \frac{1}{n^{1/12} a_n} R_{n,2}, \tag{2.66}$$

where

$$R_{n,1} := n^{1/3} \int_{-m/\sqrt{n}}^{-2} (g_\kappa(x) - x)^2 \, dx;$$
$$R_{n,2} := n^{1/3} \int_{2}^{k_1/\sqrt{n}} (g_\kappa(x) - x)^2 \, dx.$$

We will show the three terms on the right hand side of (2.66) go to zero in probability.

Step 1. We will prove a stronger result that both $R_{n,1}$ and $R_{n,2}$ are of order of $O_p(1)$ as $n \to \infty$. We start with $R_{n,1}$. The proof essentially bounds the integrand of $R_{n,1}$ for $-m/\sqrt{n} \leq x \leq -2$, which can be achieved via (2.52) and the following result. By Theorem 5.5 from Ivanov and Olshanski (2001),

$$\delta_n := \sup_{x\in\mathbb{R}} |g_\kappa(x) - \Omega(x)| \to 0 \tag{2.67}$$

in probability as $n \to \infty$, where $\Omega(x)$ is defined in (1.12). Observe that

$$\frac{1}{2}|g_\kappa(x) - x|^2 \leq \delta_n^2 + (\Omega(x) - x)^2$$

for each $x \in \mathbb{R}$. Denote $C = \sup_{-3\leq x\leq 0}(\Omega(x) - x)^2$ and

$$C_n = \sup_{-m/\sqrt{n}\leq x\leq -2} (\Omega(x) - x)^2.$$

Then $P(C_n > 2C) \leq P(\frac{m}{\sqrt{n}} > 3) \to 0$ by (2.52). Therefore, $C_n = O_p(1)$. It follows that

$$|R_{n,1}| \leq 2n^{1/3} \Big|\frac{m}{\sqrt{n}} - 2\Big| \cdot (\delta_n^2 + C_n) = O_p(1) \tag{2.68}$$

by (2.52) again. Similarly, $R_{n,2} = O_p(1)$ as $n \to \infty$.

In the rest of the proof, we only need to show $\frac{n^{1/4}}{a_n} Z'_n$ goes to zero in probability. This again takes several steps.

Step 2. In this step we will reduce Z'_n to a workable form. By the same argument as the one used in proving (2.68), we have

$$\begin{aligned} Z'_n &= \int_{-2}^{2} (g_\kappa(x) - \Omega(x))(g_\kappa(x) - \Omega(x) + 2(\Omega(x) - x))\,dx \\ &= \int_{-2}^{2} |g_\kappa(x) - \Omega(x)|^2\,dx + \int_{-2}^{2} f_1(x)(g_\kappa(x) - \Omega(x))\,dx \\ &\le \int_{-2}^{2} |g_\kappa(x) - \Omega(x)|^2\,dx + \sqrt{\int_{-2}^{2} f_1(x)\,dx} \cdot \sqrt{\int_{-2}^{2} |g_\kappa(x) - \Omega(x)|^2\,dx} \end{aligned}$$

where $f_1(x) := 2(\Omega(x) - x)$ for all $x \in \mathbb{R}$, and the last inequality above follows from the Cauchy-Schwartz inequality. To show $\frac{n^{1/4}}{a_n} Z'_n$ goes to zero in probability, since $f_1(x)$ is a bounded function on $\mathbb{R}$, it suffices to prove

$$Z''_n : = \frac{n^{1/2}}{a_n^2} \int_{-2}^{2} |g_\kappa(x) - \Omega(x)|^2\,dx \to 0 \tag{2.69}$$

in probability by (2.65). Set

$$\begin{aligned} H_n = \Big\{ &\kappa = (k_1, \cdots, k_m) \vdash n;\ 2\sqrt{n} - n^{1/6}\log n \le m,\ k_1 \le 2\sqrt{n} + n^{1/6}\log n \text{ and} \\ &\Big| n^{1/3} \int_{-2}^{2} (g_\kappa(x) - \Omega(x))\,ds \Big| \le 1 \Big\}. \end{aligned} \tag{2.70}$$

Step 3 . We prove in this step that

$$\lim_{n\to\infty} P(H_n^c) = 0. \tag{2.71}$$

Note that $g_\kappa(s) = \Omega(s) = |s|$ if $s \ge \max\{\frac{k_1}{\sqrt{n}}, 2\}$ or $s \le -\max\{\frac{m}{\sqrt{n}}, 2\}$. Also, the areas encircled by $t = |s|$ and $t = g_\kappa(s)$ and that by $t = |s|$ and $t = \Omega(s)$ are both equal to 2; see Figure 5. It is trivial to see that $\int_a^b (g_\kappa(s) - \Omega(s))\,du = \int_{\mathbb{R}} (g_\kappa(s) - \Omega(s))\,du = 0$ for $a := -\max\{\frac{m}{\sqrt{n}}, 2\}$ and $b := \max\{\frac{k_1}{\sqrt{n}}, 2\}$. Define

$$h_\kappa(s) = g_\kappa(s) - \Omega(s).$$

We see

$$-\int_{-2}^{2} h_\kappa(s)\,ds = \int_a^{-2} h_\kappa(s)\,ds + \int_2^b h_\kappa(s)\,ds.$$

Thus,

$$\begin{aligned} \Big| n^{1/3} \int_{-2}^{2} h_\kappa(s)\,ds \Big| &\le \Big| n^{1/3} \int_a^{-2} h_\kappa(s)\,ds \Big| + \Big| n^{1/3} \int_2^b h_\kappa(s)\,ds \Big| \\ &\le 2n^{1/3} \max_{s\in\mathbb{R}} |h_\kappa(s)| \cdot (|a+2| + |b-2|). \end{aligned}$$

From (2.67), $\max_{s\in\mathbb{R}} |h_\kappa(s)| \to 0$ in probability. Further $|a+2| \leq |\frac{m}{\sqrt{n}} - 2|$ and $|b-2| \leq |\frac{k_1}{\sqrt{n}} - 2|$. By (2.52) again, we obtain $n^{1/3}\int_{-2}^2 h_\kappa(s)\,du \to 0$ in probability. This and the first conclusion of Lemma 2.9 imply that $\lim_{n\to\infty} P(H_n^c) = 0$.

Step 4. Review H_n in (2.70) and the limit in (2.71). From the bound $P(Z_n'' > \epsilon) \leq P(H_n \cap \{Z_n'' > \epsilon\}) + P(H_n^c)$, we apply Lemma 2.8 for the set $\mathcal{F} = H_n \cap \{Z_n'' > \epsilon\}$ for the first term on the RHS of the bound. It is seen from Lemma 2.8 that there exists an absolute constant $C > 0$ such that

$$\begin{aligned} P(Z_n'' > \epsilon) &\leq e^{C\sqrt{n} - n\cdot \inf I(\kappa)} + P(H_n^c) \\ &= e^{C\sqrt{n} - n\cdot \inf I(\kappa)} + o(1). \end{aligned}$$

where $I(\kappa)$ is as in Lemma 2.8 and the infimum is taken over all $\kappa \in H_n \cap \{Z_n'' > \epsilon\}$. We claim

$$n^{1/2} \cdot \inf_{\kappa\in H_n;\, Z_n''\geq\epsilon} I(\kappa) \to \infty \tag{2.72}$$

as $n \to \infty$. If this is true, we then obtain (2.69), and the proof is completed. Review

$$I(\kappa) = \|L_\kappa - \bar{\Omega}\|_\theta^2 - 4\int_{|s|>1} (L_\kappa(s) - \bar{\Omega}(s)) \cosh^{-1}|s|\, ds.$$

Lemma 2.9 says that the last term above is of order $O(n^{-2/3}(\log n)^2)$ as $\kappa \in H_n$ by taking $t_n = \log n$. To get (2.72), it suffices to show

$$n^{1/2} \cdot \inf_{\kappa\in H_n;\, Z_n''\geq\epsilon} \|L_\kappa - \bar{\Omega}\|_\theta^2 \to \infty \tag{2.73}$$

as $n \to \infty$. By the definitions of L_κ and $\bar{\Omega}$, we see from (2.54) that

$$\begin{aligned} \|L_\kappa - \bar{\Omega}\|_\theta^2 &\geq \frac{1}{4}\int_{-2}^2\int_{-2}^2 \Big(\frac{h_\kappa(s) - h_\kappa(t)}{s-t}\Big)^2 dsdt \\ &\geq \frac{1}{4^3}\int_{-2}^2\int_{-2}^2 (h_\kappa(s) - h_\kappa(t))^2\, dsdt \\ &= \frac{1}{4}E(h_\kappa(U) - h_\kappa(V))^2 \end{aligned}$$

where U and V are independent random variables with the uniform distribution on $[-2, 2]$. By the Jensen inequality, the last integral is bounded below by $E(h_\kappa(U) - Eh_\kappa(V))^2 = E[h_\kappa(U)^2] - [Eh_\kappa(V)]^2$. Consequently,

$$\begin{aligned} \|L_\kappa - \bar{\Omega}\|_\theta^2 &\geq \frac{1}{16}\int_{-2}^2 h_\kappa(u)^2\, du - \frac{1}{64}\Big(\int_{-2}^2 h_\kappa(u)\, du\Big)^2 \\ &\geq \frac{\epsilon}{16} n^{-1/2}\cdot a_n^2 - \frac{1}{64} n^{-2/3} \end{aligned}$$

for $\kappa \in H_n \cap \{Z_n'' \geq \epsilon\}$. This implies (2.73). □

With the above preparation we proceed to prove Theorem 5.

Proof of Theorem 5. By Lemma 2.3,

$$\lambda_\kappa = \big(m - \frac{\alpha}{2}\big)n + \sum_{i=1}^{m}\big(\frac{\alpha}{2}k_i - i\big)k_i.$$

Thus

$$\begin{aligned}&\frac{\lambda_\kappa - 2n^{3/2} - (\alpha-1)(\frac{128}{27}\pi^{-2})n^{3/2}}{n^{5/4}\cdot a_n}\\ = \quad & \frac{m-2\sqrt{n}}{n^{1/4}\cdot a_n} - \frac{\alpha}{2n^{1/4}\cdot a_n} + \frac{\sum_{i=1}^{m}(\frac{\alpha}{2}k_i - i)k_i - (\alpha-1)(\frac{128}{27}\pi^{-2})n^{3/2}}{n^{5/4}\cdot a_n}.\end{aligned}$$

We claim

$$\frac{\sum_{i=1}^{m}(\frac{\alpha}{2}k_i - i)k_i - (\alpha-1)(\frac{128}{27}\pi^{-2})n^{3/2}}{n^{5/4}\cdot a_n} \to 0 \tag{2.74}$$

in probability as $n \to \infty$. If this is true, by (2.52), we finish the proof. Now let us show (2.74).

We first claim

$$\frac{1}{n}\sum_{i=1}^{m}\big(\frac{1}{2}k_i - i\big)k_i \to N\big(-\frac{1}{2}, \sigma^2\big) \tag{2.75}$$

for some $\sigma^2 \in (0,\infty)$. To see why this is true, we get from (1.3) and Lemma 2.3 that

$$a(\kappa') - a(\kappa) = \frac{1}{2}n + \sum_{i=1}^{m}\big(\frac{1}{2}k_i - i\big)k_i.$$

By Theorem 1.2 from Fulman (2004), there is $\sigma^2 \in (0,\infty)$ such that

$$\frac{a(\kappa') - a(\kappa)}{n} \to N(0,\sigma^2)$$

weakly as $n \to \infty$. Then (2.75) follows.

Second, from (2.52), we know $\xi_n := (m - 2\sqrt{n})n^{-1/6}$ converges weakly to F_2 as $n \to \infty$. Write

$$m^3 = (2\sqrt{n} + n^{1/6}\xi_n)^3 = n^{1/2}\xi_n^3 + 6n^{5/6}\xi_n^2 + 12n^{7/6}\xi_n + 8n^{3/2}.$$

This implies that

$$\frac{m^3 - 8n^{3/2}}{n^{5/4}} \to 0$$

in probability as $n \to \infty$. Let Z_n be as in Lemma 2.11 and $\Omega(x)$ as in (1.12). It is seen from Lemmas 2.10 and 2.11 that

$$\begin{aligned}\sum_{i=1}^{m} ik_i &= \frac{1}{8}n^{3/2}\int_{-m/\sqrt{n}}^{k_1/\sqrt{n}}(g_\kappa(x)-x)^2\,dx - \frac{1}{6}m^3+\frac{1}{2}n\\ &= \frac{1}{8}n^{3/2}\left(Z_n+\int_{-2}^{2}(\Omega(x)-x)^2\,dx\right)-\frac{1}{6}m^3+\frac{1}{2}n\end{aligned}$$

with $\frac{n^{1/4}}{8a_n}Z_n \to 0$ in probability as $n\to\infty$. The last two assertions imply

$$\begin{aligned}&\frac{1}{n^{5/4}\cdot a_n}\Big[\sum_{i=1}^{m} ik_i - \frac{1}{8}n^{\frac{3}{2}}\int_{-2}^{2}(\Omega(x)-x)^2\,dx+\frac{4}{3}n^{\frac{3}{2}}\Big] \qquad (2.76)\\ =\ &\frac{n^{1/4}}{8a_n}Z_n - \frac{1}{6a_n}\cdot\frac{m^3-8n^{3/2}}{n^{5/4}}+\frac{1}{2a_n n^{1/4}} \to 0\end{aligned}$$

in probability as $n\to\infty$. It is trivial and yet a bit tedious to verify

$$\int_{-2}^{2}(\Omega(x)-x)^2\,dx = \frac{32}{3}+\frac{1024}{27\pi^2}. \qquad (2.77)$$

The calculation of (2.77) is included in Appendix 3.2. Plug this into (2.76) to see

$$\frac{\sum_{i=1}^{m} ik_i - \frac{128}{27\pi^2}n^{3/2}}{n^{5/4}\cdot a_n} \to 0 \qquad (2.78)$$

in probability as $n\to\infty$.

Third, observe

$$\sum_{i=1}^{m}(\frac{\alpha}{2}k_i-i)k_i = \alpha\sum_{i=1}^{m}\big(\frac{1}{2}k_i-i\big)k_i+(\alpha-1)\sum_{i=1}^{m} ik_i.$$

Therefore

$$\begin{aligned}&\frac{\sum_{i=1}^{m}(\frac{\alpha}{2}k_i-i)k_i-(\alpha-1)(\frac{128}{27}\pi^{-2})n^{3/2}}{n^{5/4}\cdot a_n}\\ =\ &\alpha\frac{\sum_{i=1}^{m}\big(\frac{1}{2}k_i-i\big)k_i}{n^{5/4}\cdot a_n}+(\alpha-1)\frac{\sum_{i=1}^{m} ik_i-(\frac{128}{27}\pi^{-2})n^{3/2}}{n^{5/4}\cdot a_n}\to 0\end{aligned}$$

in probability by (2.75) and (2.78). We finally arrive at (2.74). □

3 Appendix

In this section we will prove (1.6), verify (2.77) and derive the density functions of the random variable appearing in Theorem 1 for two cases. They are placed in three subsections.

3.1 Proof of (1.6)

Recall $(2s-1)!! = 1 \cdot 3 \cdots (2s-1)$ for integer $s \geq 1$. Set $(-1)!! = 1$. The following is Lemma 2.4 from Jiang (2009).

Lemma 3.1. *Suppose $p \geq 2$ and $Z_1, \cdots, Z_p$ are i.i.d. random variables with $Z_1 \sim N(0,1)$. Define $U_i = \frac{Z_i^2}{Z_1^2 + \cdots + Z_p^2}$ for $1 \leq i \leq p$. Let $a_1, \cdots, a_p$ be non-negative integers and $a = \sum_{i=1}^p a_i$. Then*

$$E\big(U_1^{a_1} \cdots U_p^{a_p}\big) = \frac{\prod_{i=1}^p (2a_i - 1)!!}{\prod_{i=1}^a (p + 2i - 2)}.$$

Proof of (1.6). Recall (1.5). Write $(r-1)s^2 = \sum_{i=1}^r x_i^2 - r\bar{x}^2$. In our case,

$$\bar{x} = \frac{1}{|\mathcal{P}_n(m)|} \sum_{\kappa \in \mathcal{P}_n(m)} \lambda_\kappa = E\lambda_\kappa;$$
$$s^2 = \frac{1}{|\mathcal{P}_n(m)| - 1} \sum_{\kappa \in \mathcal{P}_n(m)} (\lambda_\kappa - \bar{x})^2 \sim E(\lambda_\kappa^2) - (E\lambda_\kappa)^2$$

as $n \to \infty$, where E is the expectation about the uniform measure on $\mathcal{P}_n(m)'$. Therefore,

$$\frac{\bar{x}}{n^2} = \frac{E\lambda_\kappa}{n^2} \quad \text{and} \quad \frac{s^2}{n^4} \sim E\Big(\frac{\lambda_\kappa}{n^2}\Big)^2 - \Big(\frac{E\lambda_\kappa}{n^2}\Big)^2. \tag{3.1}$$

From Lemma 2.3, we see a trivial bound that $0 \leq \lambda_\kappa / n^2 \leq 1 + \frac{\alpha}{2}m$ for each partition $\kappa = (k_1, \cdots, k_m) \vdash n$ with $k_m \geq 1$. By Theorem 1, under $\mathcal{P}_n'(m)$,

$$\frac{\lambda_\kappa}{n^2} \to \frac{\alpha}{2} \cdot Y \quad \text{and} \quad Y := \frac{\xi_1^2 + \cdots + \xi_m^2}{(\xi_1 + \cdots + \xi_m)^2}$$

as $n \to \infty$, where $\{\xi_i;\, 1 \leq i \leq m\}$ are i.i.d. random variables with density $e^{-x} I(x \geq 0)$. By bounded convergence theorem and (3.1),

$$\frac{\bar{x}}{n^2} \to \frac{\alpha}{2} EY \quad \text{and} \quad \frac{s^2}{n^4} \to \frac{\alpha^2}{4}[E(Y^2) - (EY)^2] \tag{3.2}$$

as $n \to \infty$. Now we evaluate EY and $E(Y^2)$. Easily,

$$EY = m \cdot E\frac{\xi_1^2}{(\xi_1 + \cdots + \xi_m)^2};$$
$$E(Y^2) = m \cdot E\frac{\xi_1^4}{(\xi_1 + \cdots + \xi_m)^4} + m(m-1) \cdot E\frac{\xi_1^2 \xi_2^2}{(\xi_1 + \cdots + \xi_m)^4}. \tag{3.3}$$

Let $Z_1, \cdots, Z_{2m}$ be i.i.d. random variables with $N(0,1)$ and $U_i = \frac{Z_i^2}{Z_1^2 + \cdots + Z_{2m}^2}$ for $1 \leq i \leq 2m$. Evidently, $(Z_1^2 + Z_2^2)/2$ has density function $e^{-x} I(x \geq 0)$. Then,

$$\Big(\frac{\xi_i}{\xi_1 + \cdots + \xi_m}\Big)_{1 \leq i \leq m} \quad \text{and} \quad (U_{2i-1} + U_{2i})_{1 \leq i \leq m}$$

have the same distribution. Consequently, by taking $p = 2m$ in Lemma 3.1,

$$\begin{aligned} EY &= m \cdot E(U_1 + U_2)^2 \\ &= 2m[E(U_1^2) + E(U_1U_2)] \\ &= 2m\Big[\frac{3}{4m(m+1)} + \frac{1}{4m(m+1)}\Big] = \frac{2}{m+1}. \end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned} E\frac{\xi_1^4}{(\xi_1 + \cdots + \xi_m)^4} &= E[(U_1 + U_2)^4] \\ &= 2E(U_1^4) + 8E(U_1^3U_2) + 6E(U_1^2U_2^2) \\ &= \frac{105}{8}\frac{1}{m(m+1)(m+2)(m+3)} + \frac{15}{2}\frac{1}{m(m+1)(m+2)(m+3)} \\ &\qquad + \frac{27}{8}\frac{1}{m(m+1)(m+2)(m+3)} \\ &= \frac{24}{m(m+1)(m+2)(m+3)} \end{aligned}$$

and

$$\begin{aligned} E\frac{\xi_1^2\xi_2^2}{(\xi_1 + \cdots + \xi_m)^4} &= E[(U_1 + U_2)^2(U_3 + U_4)^2] \\ &= 4E(U_1^2U_2^2) + 8E(U_1^2U_2U_3) + 4E(U_1U_2U_3U_4) \\ &= \frac{9}{4}\frac{1}{m(m+1)(m+2)(m+3)} + \frac{3}{2}\frac{1}{m(m+1)(m+2)(m+3)} \\ &\qquad + \frac{1}{4}\frac{1}{m(m+1)(m+2)(m+3)} \\ &= \frac{4}{m(m+1)(m+2)(m+3)}. \end{aligned}$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} E(Y^2) &= \frac{4m+20}{(m+1)(m+2)(m+3)}; \\ E(Y^2) - (EY)^2 &= \frac{4m+20}{(m+1)(m+2)(m+3)} - \Big(\frac{2}{m+1}\Big)^2 \\ &= \frac{4m-4}{(m+1)^2(m+2)(m+3)}. \end{aligned}$$

This and (3.2) say that

$$\frac{\bar{x}}{n^2} \to \frac{\alpha}{m+1} \quad \text{and} \quad \frac{s^2}{n^4} \to \frac{(m-1)\alpha^2}{(m+1)^2(m+2)(m+3)}.$$

3.2 Verification of (2.77)

Verification of (2.77). Trivially, $\Omega(x)$ in (1.12) is an even function and $\Omega(x)' = \frac{2}{\pi}\arcsin\frac{x}{2}$ for $|x| < 2$. Then

$$\begin{aligned}
&\int_{-2}^{2}(\Omega(x)-x)^2\,dx = \int_{-2}^{2}\Omega(x)^2\,dx + \int_{-2}^{2}x^2\,dx \\
&= x\cdot\Omega(x)^2\Big|_{-2}^{2} - \int_{-2}^{2} x\cdot 2\Omega(x)\cdot\Omega(x)'\,dx + \frac{x^3}{3}\Big|_{-2}^{2} \\
&= \frac{64}{3} - \frac{16}{\pi^2}\int_0^2 x\arcsin\frac{x}{2}\cdot\Big(x\arcsin\frac{x}{2}+\sqrt{4-x^2}\Big)\,dx.
\end{aligned}$$

Now, set $x = 2\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$, the above integral becomes

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} 2\theta\sin\theta(2\theta\sin\theta + 2\cos\theta)2\cos\theta\,d\theta \\
= \quad & 2\int_0^{\frac{\pi}{2}}(\theta\sin\theta + \theta\sin(3\theta) + \theta^2\cos\theta - \theta^2\cos(3\theta))\,d\theta
\end{aligned} \tag{3.5}$$

by trigonometric identities. It is easy to verify that

$$\begin{aligned}
&\theta\sin\theta = (\sin\theta - \theta\cos\theta)'; \qquad \theta\sin(3\theta) = \frac{1}{9}(\sin(3\theta) - 3\theta\cos(3\theta))'; \\
&\theta^2\cos\theta = (\theta^2\sin\theta + 2\theta\cos\theta - 2\sin\theta)'; \\
&\theta^2\cos(3\theta) = \frac{1}{27}(9\theta^2\sin(3\theta) + 6\theta\cos(3\theta) - 2\sin(3\theta))'.
\end{aligned}$$

Thus, the term in (3.5) is equal to

$$2\Big(1 + (-\frac{1}{9}) + (\frac{\pi^2}{4} - 2) - \frac{1}{27}(-\frac{9\pi^2}{4} + 2)\Big) = \frac{2}{3}\pi^2 - \frac{64}{27}.$$

It follows that

$$\int_{-2}^{2}(\Omega(x)-x)^2\,dx = \frac{64}{3} - \frac{16}{\pi^2}\Big(\frac{2}{3}\pi^2 - \frac{64}{27}\Big) = \frac{32}{3} + \frac{1024}{27\pi^2}.$$

This completes the verification. □

3.3 Derivation of density functions in Theorem 1

In this section, we will derive explicit formulas for the limiting distribution in Theorem 1. For convenience, we rewrite the conclusion as

$$\frac{2}{\alpha}\cdot\frac{\lambda_\kappa}{n^2} \to \nu,$$

where ν is different from μ in Theorem 1 by a factor of $\frac{2}{\alpha}$. We will only evaluate the cases $m = 2, 3$. We first state the conclusions and prove them afterwards.

Case 1. For $m=2$, the support of ν is $[\frac{1}{2},1]$ and the cdf of ν is

$$F(t)=\sqrt{2t-1} \tag{3.6}$$

for $t\in[\frac{1}{2},1]$. Hence the density function is given by

$$f(t)=\frac{1}{\sqrt{2t-1}},\quad t\in[\frac{1}{2},1].$$

Case 2. For $m=3$, the support of ν is $[\frac{1}{3},1]$, and the cdf of ν is

$$F(t)=\begin{cases}\frac{2}{\sqrt{3}}\pi(t-\frac{1}{3}), & \text{if } \frac{1}{3}\le t<\frac{1}{2};\\ \frac{2}{\sqrt{3}}\Big((t-\frac{1}{3})(\pi-3\arccos\frac{1}{\sqrt{6t-2}})+\frac{\sqrt{6}}{2}\sqrt{t-\frac{1}{2}}\Big), & \text{if } \frac{1}{2}\le t<1.\end{cases} \tag{3.7}$$

By differentiation, we get the density function

$$f(t)=\begin{cases}\frac{2}{\sqrt{3}}\pi, & \text{if } \frac{1}{3}\le t<\frac{1}{2};\\ \frac{2}{\sqrt{3}}\big(\pi-3\arccos\frac{1}{\sqrt{6t-2}}\big), & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$

The above are the two density functions claimed below the statement of Theorem 1. Now we prove them.

From a comment below Theorem 1, the limiting law of $\frac{2}{\alpha}\cdot\frac{\lambda_\kappa}{n^2}$ is the same as the distribution of $\sum_{i=1}^m Y_i^2$, where $(Y_1,\dots,Y_m)$ has uniform distribution over the set

$$\mathcal{H}:=\Big\{(y_1,\dots,y_m)\in[0,1]^m;\ \sum_{i=1}^m y_i=1\Big\}.$$

By (2.2) the volume of $\mathcal{H}$ is $\frac{\sqrt{m}}{(m-1)!}$. Therefore, the cdf of $\sum_{i=1}^m Y_i^2$ is

$$F(t)=P\Big(\sum_{i=1}^m Y_i^2\le t\Big)=\frac{(m-1)!}{\sqrt{m}}\cdot\text{volume of }\Big\{\sum_{i=1}^m y_i^2\le t\Big\}\cap\mathcal{H},\quad t\ge 0. \tag{3.8}$$

Denote $B_m(t):=\{\sum_{i=1}^m y_i^2\le t\}\subset\mathbb{R}^m$. Let $V(t)$ be the volume of $B_m(t)\cap\mathcal{H}$. We start with some facts for any $m\ge 2$.

First, $V(t)=0$ for $t<\frac{1}{m}$. In fact, if $(y_1,\cdots,y_m)\in B_m(t)\cap\mathcal{H}$, then

$$\frac{1}{m}=\frac{(\sum_{i=1}^m y_i)^2}{m}\le\sum_{i=1}^m y_i^2\le t.$$

Further, for $t>1$, $\mathcal{H}$ is inscribed in $B_m(t)$ and thus $V(t)=\frac{\sqrt{m}}{(m-1)!}$. Now assume $1/m\le t\le 1$.

The proof of (3.6). Assume $m=2$. If $1/2\le t\le 1$, then $\{(y_1,y_2)\in[0,1]^2: y_1+y_2=1\}\cap\{y_1^2+y_2^2\le t\}$ is a line segment. Easily, the endpoints of the line segment are

$$\Big(\frac{1+\sqrt{2t-1}}{2},\frac{1-\sqrt{2t-1}}{2}\Big)\quad\text{and}\quad\Big(\frac{1-\sqrt{2t-1}}{2},\frac{1+\sqrt{2t-1}}{2}\Big),$$

respectively. Thus $V(t) = \sqrt{2(2t-1)}$. Therefore the conclusion follows directly from (3.8).

The proof of (3.7). We first observe that as t increases from $\frac{1}{3}$ to 1, the intersection $B_3(t) \cap \mathcal{H}$ expands and passes through $\mathcal{H}$ as t exceeds some critical value t_0; see Figure 6.

We claim that $t_0 = \frac{1}{2}$. Indeed, the center C of the intersection of $B_3(t)$ and the hyperplane $\mathcal{I} := \{(y_1, y_2, y_3) \in \mathbb{R}^3; y_1 + y_2 + y_3 = 1\} \supset \mathcal{H}$ is $C = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Thus, the distance from the origin to $\mathcal{I}$ is $d = ((\frac{1}{3})^2 + (\frac{1}{3})^2 + (\frac{1}{3})^2)^{1/2} = \frac{1}{\sqrt{3}}$. By Pythagorean's theorem, the radius of the intersection (disc) on $\mathcal{I}$ is

$$R(t) = \sqrt{t - d^2} = \sqrt{t - \frac{1}{3}}.$$

Let t_0 be the value such that the intersection $B_3(t) \cap \mathcal{H}$ exactly inscribes $\mathcal{H}$. By symmetry, the intersection point at the (x, y)-plane is $M = (\frac{1}{2}, \frac{1}{2}, 0)$; see Figure 6(b). Therefore $|CM| = \sqrt{\frac{1}{6}}$. Solving t_0 from $|CM| = R(t_0)$, we have $t_0 = \frac{1}{2}$.

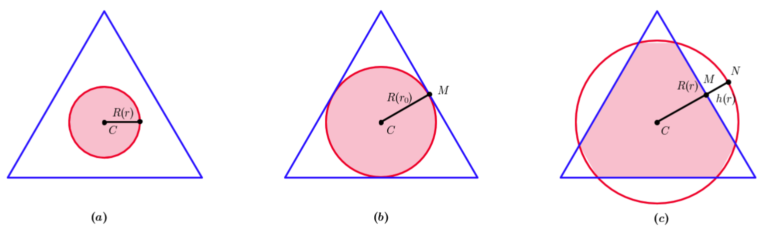

Figure 6: Shaded region indicates volume $V(t)$ of intersection as t changes from 1/3 to 1 as $m = 3$.

When $\frac{1}{3} \le t < \frac{1}{2}$, the intersection locates entirely in $\mathcal{H}$; see Figure 6(a). Then

$$V(t) = \pi R(t)^2 = \pi(t - \frac{1}{3}).$$

When $\frac{1}{2} \le t \le 1$, the volume of the intersection part [see Figure 6(c)] is given by

$$V(t) = \pi R(t)^2 - 3 \cdot V_{\text{cs}}(h(t), R(t)),$$

where $V_{\text{cs}}(h(t), R(t))$ is the area of circular segment with radius $R(t)$ and height

$$h(t) = R(t) - |CM| = \sqrt{t - \frac{1}{3}} - \sqrt{\frac{1}{6}}.$$

Therefore, it is easy to check

$$V(t) = \pi(t - \frac{1}{3}) - 3(t - \frac{1}{3}) \arccos \frac{1}{\sqrt{6t-2}} + 3\sqrt{\frac{1}{6}(t - \frac{1}{2})}\,.$$

This and (3.8) yield the desired conclusion.

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