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ABSTRACT
We study a non-Hermitian chiral random matrix of which the eigenvalues are complex random variables. The empirical distributions and the radius of the eigenvalues are investigated. The limit of the empirical distributions is a new probability distribution defined on the complex plane. The graphs of the density functions are plotted; the surfaces formed by the density functions are understood through their convexity and their Gaussian curvatures. The limit of the radius is a Gumbel distribution. The main observation is that the joint density function of the eigenvalues of the chiral ensemble, after a transformation, becomes a rotation-invariant determinantal point process on the complex plane. Then, the eigenvalues are studied by the tools developed by Jiang and Qi [J. Theor. Probab. 30, 326 (2017); 32, 353 (2019)]. Most efforts are devoted to deriving the central limit theorems for distributions defined by the Bessel functions via the method of steepest descent and the estimates of the zero of a non-trivial equation as the saddle point.

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I. INTRODUCTION AND MAIN RESULTS
Hermitian random matrices have been studied by experts from many disciplines such as Mathematics, Physics, Statistics, and Engineering. Many deep results are obtained. For example, for the limits of eigenvalues, we see the semi-circle law, the Marchenko–Pastur law, the Tracy–Widom law, and the connections to stochastic differential equations. See the book treatment and the references from, for instance, Anderson et al., Bai and Silverstein, Forrester, Akemann et al., Baik et al., and Erdős and Yau.

Non-Hermitian random matrices have many applications in, for example, the fractional quantum Hall effect (Di Francesco et al.) and quantum chromodynamics (Stephanov); see the work of Khoruzhenko and Sommers for introductions and further applications. The eigenvalues of this type of matrices are complex numbers rather than real numbers as mentioned in the previous paragraph. We understand many properties on the Ginibre ensembles, truncations of Haar-unitary matrices, the product of Ginibre ensembles, the product of truncations of Haar-unitary matrices, the elliptic ensemble, and the chiral non-Hermitian random matrix ensembles. Their limit properties will be further elaborated following our results later on.

In this paper, we will focus on a special case of the chiral non-Hermitian random matrix ensemble. For integers $n \geq 1$ and $v \geq 0$, let $P$ and $Q$ be two independent and identically distributed (i.i.d.) $(n + v) \times n$ matrices, where the entries from the two matrices are i.i.d. standard complex normals. For each $\tau \in [0, 1]$, define the $(2n + v) \times (2n + v)$ matrix,

$$D = \begin{pmatrix} 0 & \sqrt{1 + \tau P^* - \sqrt{1 - \tau Q}} & \sqrt{1 + \tau P + \sqrt{1 - \tau Q}} \\ \sqrt{1 + \tau P^* - \sqrt{1 - \tau Q}} & 0 & \sqrt{1 - \tau Q} \end{pmatrix}. \tag{1.1}$$

where $P^*$ stands for the complex conjugate of matrix $P$. This matrix is referred to as the chiral non-Hermitian random matrix ensemble. It has $v$-multiple zeros and $n$ pairs of eigenvalues, and each pair has the opposite sign. To understand these non-zero eigenvalues, we only need to
consider the $n$ eigenvalues, $z_1, \ldots, z_n$, with a positive $x$-coordinate. Akemann and Bender\cite{Akemann2009} derived that the joint probability density function (pdf) of these $n$ eigenvalues is equal to

$$
C \prod_{1 \leq j < k \leq n} |z_j^2 - z_k^2|^2 \cdot \prod_{j=1}^n |z_j|^{2(\nu+1)} \exp \left( \frac{2\tau \text{Re}(z_j^2)}{1 - \tau^2} \right) K_0 \left( \frac{2n|z_j|^2}{1 - \tau^2} \right)
$$

(1.2)

for all $z_1, \ldots, z_n \in \mathbb{C}$ with $\text{Re}(z_j) > 0$ for each $j$, where $C$ is a normalizing constant and $K_0$ is the modified Bessel function of the second kind (see more details in Lemma 2.1). Here and later, the density is relative to the Lebesgue measure on $\mathbb{C}^n$. This model includes special cases studied by Osborn\cite{Osborn2003} and Bender.\cite{Bender2003} The parameter $\tau$ reflects the strength of how the matrix is non-Hermitian. For example, at the extreme case of $\tau = 1$, the eigenvalues of the matrix $D$ in (1.1) are essentially those of a complex Wishart matrix and, hence, are non-negative.

Write $x_j = x_j + iy_j$ for each $j$. Given $\tau \in [0,1)$, Akemann and Bender\cite{Akemann2009} obtained that (1) $\max_{1 \leq j \leq n} x_j$ with a normalization converges to the Gumbel distribution; (2) renormalizing $x_j$ and $y_j$ with different constants, the new pairs $(x'_j, y'_j), 1 \leq j \leq n$, as a point process, converges to a Poisson process. Here, $\nu$ is fixed and the limit is taken as $n \to \infty$. As $\tau \to 1$, the aforementioned says that the eigenvalues of $D$ in (1.1) are just those of a complex Wishart matrix. The largest eigenvalue of the complex Wishart matrix has the asymptotic Tracy–Widom distribution (Johansson\cite{Johansson2002}); the empirical distribution of those eigenvalues converges to the Marchenko–Pastur law (Marchenko and Pastur;\cite{Marchenko1967} Bai and Silverstein\cite{Bai1994}). In this paper, we will study the same classical problems: the spectral radius $\max_{1 \leq j \leq n} x_j$ and the empirical distribution of $x_j$’s not counting zero eigenvalues for $\tau = 0$. In particular, our targets are different from those in the work of Akemann and Bender.\cite{Akemann2009} Assuming $\tau = 0$, the density in (1.2) then becomes

$$
f(x_1, \ldots, x_n) = C \prod_{1 \leq i < k \leq n} |x_i^2 - x_k^2|^2 \cdot \prod_{j=1}^n |x_j|^{2(\nu+1)} K_0 \left( 2n|x_j|^2 \right)
$$

(1.3)

for all $x_1, \ldots, x_n \in \mathbb{C}$ with $\text{Re}(x_j) > 0$ for each $j$. The parameter $\nu$ is allowed to be any non-negative real number and can change with $n$.

In this paper, we prove that, with suitable normalization, $\max_{1 \leq j \leq n} x_j$ converges to the Gumbel distribution and the empirical distribution goes to a new distribution defined on the complex plane $\mathbb{C}$. The density of the distribution is explicit. When considering the density function as a surface defined on $\mathbb{C}$, both the convexity and the Gaussian curvature are studied. Figures 1 and 2 show the change of the surface as the limit of $\nu/n$ changes.

**FIG. 1.** At $\alpha = 0$, the density surface is flat. At the next moment, there is a tiny hole at $z = 0$. With $\alpha$ increasing, the hole becomes larger and larger. The convex part (the Hessian is non-negative) in the density surface becomes large.
With $\alpha$ increasing, the convex part and the part with the positive Gaussian curvature becomes larger and larger. As $\alpha > 10 + 2\sqrt{30} \approx 20.95$, the whole density surface becomes convex and the Gaussian curvature is positive everywhere. No much change for the shape visually for $\alpha > 21$.

Now, we state our results. For $y > 3$, set
\[
a(y) = (\log y)^{1/2} - (\log y)^{-1/2} \log(\sqrt{2\pi} \log y) \quad \text{and} \quad b(y) = (\log y)^{-1/2}. \tag{1.4}
\]

Let $\Lambda$ be the cumulative distribution of the Gumbel distribution such that
\[
\Lambda(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}. \tag{1.5}
\]

**Theorem 1.** Let $v = v_n$ be a sequence of non-negative numbers. Let $z_1, \ldots, z_n$ have density $f_z$ as in (1.3). Then,
\[
\frac{1}{b(\frac{(n+v)}{2n+v})} \left[ \max_{1 \leq j \leq n} |z_j| - \left( \frac{2n}{n+v} \right)^{1/4} - d \left( \frac{n(n+v)}{2n+v} \right) \right] \xrightarrow{d} \Lambda. \tag{1.6}
\]

As aforementioned, $D$ from (1.1) has eigenvalues $\pm z_1, \ldots, \pm z_n$ with $z_1, \ldots, z_n$ having density $f_z(z_1, \ldots, z_n)$ from (1.3); the other eigenvalues are zero with $v$ multiples. Therefore, Theorem 1 also holds if “max$_{1 \leq j \leq n} |z_j|$” is replaced by the “spectral radius of $D$.”

If $v_n \equiv v$, Theorem 1 has the following consequence.

**Corollary 1.** Assume $v_n \equiv v \geq 0$. Let $z_1, \ldots, z_n$ have density $f_z$ as in (1.3). Then, $\beta_n \cdot \max_{1 \leq j \leq n} |z_j| - a_n \xrightarrow{d} \Lambda$, where
\[
a_n = \sqrt{8n \log \frac{n}{2} + \log \frac{n}{2} - \log(\sqrt{2\pi} \log \frac{n}{2})} \quad \text{and} \quad \beta_n = \sqrt{8n \log \frac{n}{2}}.
\]

The proof of this corollary is given after that of Theorem 1. On the other hand, (ii) of Theorem 3 from Akemann and Bender implies that
\[
\max_{1 \leq j \leq n} \Re(z_j) = 1 + \left(\frac{\log n}{16n}\right)^{1/2} (1 + o_P(1))
\]  \tag{1.7}

as \( n \to \infty \), where \( o_P(1) \) stands for a random variable converging to zero in probability as \( n \to \infty \). Our Corollary 1 says that

\[
\max_{1 \leq j \leq n} |z_j| = 1 + \left(\frac{\log n}{8n}\right)^{1/2} (1 + o_P(1)).
\]  \tag{1.8}

Note that \( \max_{1 \leq j \leq n} |z_j| \geq \max_{1 \leq j \leq n} \Re(z_j) \). (1.7) and (1.8) not only confirm this fact but also indicate the difference. Even so, it is still hard to quantify the size of \( \max_{1 \leq j \leq n} \Im(z_j) \). One can see the reason by a quick glimpse at the two complex numbers \( \epsilon + i\sqrt{1 - \epsilon^2} \) and \( \sqrt{1 - \epsilon^2} + \epsilon i \) for \( \epsilon \in (0,1) \) which have norms equal to 1 but have very different imaginary parts.

For a matrix \( M \) with eigenvalues \( z_1, \ldots, z_n \), the quantity \( \max_{1 \leq j \leq n} |z_j| \) is referred to as the spectral radius of \( M \). Rider \( 15,16 \) and Rider and Sinclair \( 17 \) showed that the spectral radii of the real, complex, and symplectic Ginibre ensembles, which are non-Hermitian, asymptotically follow the Gumbel distribution. This fact is very different from the Tracy–Widom distribution in the Hermitian case; see, for example, the work of Tracy and Widom. \( 18,19 \) Similar phenomena are observed for other non-Hermitian ensembles. Jiang and Qi \( 20 \) showed that, for the spherical ensembles, truncations of Haar-unitary matrices, and product of independent complex Ginibre ensembles, the limits of their spectral radii are a new distribution, the Gumbel distribution, and the normal distribution, respectively.

Now, we study the empirical measure of \( z_1, \ldots, z_n \) in (1.3). Reviewing the scaling \( [n/(n + v)]^{1/4} \) in (1.6), we define

\[
\rho_n = \frac{1}{n} \sum_{j=1}^{n} \delta_{n^{-1/4}z_j}.
\]

For \( \alpha \in [0, \infty) \), set \( a = \alpha^2/(1 + \alpha)^2 \) and \( b = 4/(1 + \alpha) \), and let \( \Phi_\alpha \) be a probability measure with the density function

\[
\phi_\alpha(z) = \frac{4}{\pi} \frac{|z|^2 \sqrt{\alpha + b - |z|^2}}{\sqrt{\alpha^2 - |z|^2}}
\]

for \( z \) with \( |z| \leq 1, \Re(z) > 0 \). Obviously, \( \Phi_0 \) is the uniform distribution on the “half moon” region \( \{z; |z| \leq 1, \Re(z) > 0\} \). Let \( \Phi_\infty \) denote the limit of \( \Phi_\alpha \) as \( \alpha \to \infty \), that is,

\[
\phi_\infty(z) = \frac{4}{\pi} \frac{|z|^2}{\sqrt{1 - |z|^2}}
\]

for \( z \) with \( |z| \leq 1, \Re(z) > 0 \).

**Theorem 2.** Let \( v = v_n \) be a sequence of non-negative numbers. Let \( z_1, \ldots, z_n \) have density \( f_z \) as in (1.3). Assume \( \lim_{n \to \infty} v_n / n = \alpha \in [0, \infty) \). Then, with probability one, \( \rho_n \) converges weakly to a probability distribution \( \rho \) with the density function \( \Phi_\alpha \) as \( n \to \infty \).

As explained below the statement of Theorem 1, we are able to draw a conclusion for \( \rho_n \), the empirical distribution of the eigenvalues of \( D \) from (1.1). If \( \lim_{n \to \infty} v_n / n = \alpha \in [0, \infty) \), then the limit of the fraction of the number of zero eigenvalues relative to the total number \( 2n + v_n \) is \( \frac{\delta_0}{2n} \). Second, the other eigenvalues of \( D \) are \( \alpha z_1, \ldots, \alpha z_n \), with \( z_1, \ldots, z_n \) having density \( f_z(z_1, \ldots, z_n) \) from (1.3), so we know the limit of \( \rho_n \) is \( \frac{\alpha}{2n} \delta_0 + \frac{2n}{2n} \rho \), where \( \rho \) has density \( \frac{1}{2} \cdot \frac{|z|^2}{\sqrt{\alpha^2 - |z|^2}} \) \( \Chi \{ |z| \leq 1 \} \). If \( \alpha = \infty \), of course, the limit is degenerated to \( \delta_0 \).

Observe \( \lim_{n \to \infty} [n/(n + v)]^{1/4} \equiv 1 \) for fixed \( v \). A quick consequence of Theorem 2 is the following.

**Corollary 2.** Assume \( v \geq 0 \) is fixed. Let \( z_1, \ldots, z_n \) have density \( f_z \) as in (1.3). Then, with probability one, \( \frac{1}{n} \sum_{j=1}^{n} \delta_{n^{-1/4}z_j} \) converges weakly to the uniform distribution on \( \{z; |z| \leq 1, \Re(z) > 0\} \) as \( n \to \infty \).

We draw Figs. 1 and 2 to see the change of the density function \( \Phi_\alpha \) as \( \alpha \) changes. If we think of \( \alpha \) as time and think of the surface \( \{(z, \phi_\alpha(z)); |z| \leq 1, \Re(z) > 0\} \) as the roof of a house, then the roof is flat at time 0, and suddenly the roof starts leaking at the center \( z = 0 \). With time going by, the leaking area becomes larger and larger, and eventually the whole roof collapses with the final shape \( \{(z, \phi_\alpha(z)); |z| \leq 1, \Re(z) > 0\} \), which is similar to the picture with \( \alpha = 21 \) in Fig. 2 (the shape of \( \phi_\alpha(z) \) visually does not change as \( \alpha \geq 21 \)). The roof does not completely land on the ground because the volume under the roof has to be one \( \{|\phi_\alpha(z)| \text{ is a probability density function for all } \alpha \geq 0\} \).
Realizing the interesting phenomenon above, we further look at the geometry of the density surface. It is shown that the surface

\[ \{(z, \phi_z(z)) : |z| < 1, \text{Re}(z) > 0\} \] is convex and has a positive Gaussian curvature

\[ \text{if } \alpha > 10 + 2\sqrt{30}, \text{the surface is not convex, and it has a negative Gaussian curvature for some } z \text{ if } 0 < \alpha < 10 + 2\sqrt{30}. \] (1.9)

When the Gaussian curvature at a point is positive, the surface will be like a dome, locally lying on one side of its tangent plane. When the curvature at a point is negative, the point is hyperbolic. From (1.9), we see that the whole surface, indeed, looks like a dome as \( \alpha > 10 + 2\sqrt{30} \). The fact (1.9) will be checked at the end of this paper.

Now, we make some remarks on the literature related to Theorem 2. For complex Ginibre ensembles, products of Ginibre ensembles, and product of truncated Haar-invariant unitary matrices, their eigenvalues form determinantal point processes. Their limiting laws of the empirical distributions of the eigenvalues are derived by using the determinantal point processes. See, for example, the work of Burda et al., Götzte and Tikhomirov, Bordenave, O’Rourke and Soshnikov, Burda, O’Rourke et al., and Jiang and Qi.

Review Corollary 2. In the literature, the uniform distribution on \( |z| \leq 1 \) is referred to as the circular law, which is the limit of the empirical distribution of the eigenvalues of a square matrix with entries being independent and identically distributed random variables; see, for example, the work of Girko, Bai, Tao and Vu, or Bordenave and Chafaï.

For the proofs of the two main theorems, we utilize the tools developed by Jiang and Qi for rotation-invariant and non-Hermitian random matrices. The tools provide sufficient conditions for the convergence of the spectral radius and the empirical distribution of the eigenvalues. However, the chiral non-Hermitian random matrix is not rotation-invariant. In our proofs, we first make a transform such that the eigenvalues of the chiral non-Hermitian random matrix form a rotation-variant ensemble. Under this setting, both the limit of the spectral radius and that of the empirical spectral distribution depend on a set of \( n \) independent random variables \( Y_1, \ldots, Y_n \) (Lemma 2.5). The distribution of each \( Y_j \) is determined by the modified Bessel function of the second kind. To apply the tools by Jiang and Qi, we spend many efforts to derive a “uniform” central limit theorem (CLT) for \( \{Y_j : m \leq j \leq n\} \) (Lemma 2.8), where \( m \) depends on \( n \). Since the distribution of \( Y_j \) depends on the modified Bessel function and it has not been understood to our knowledge, we employ the method of steepest descent to derive the CLT for \( \{Y_j : m \leq j \leq n\} \). In particular, the saddle point is the solution of a non-trivial equation, and hence a great energy is spent to estimate the solution. Noting both theorems allow that \( v_n \geq 0 \) is arbitrary, we use another trick of subsequence argument such that \( \lim_{n \to \infty} v_n/\sqrt{n} \in [0, \infty) \) and \( \lim_{n \to \infty} \sqrt{n}/\sqrt{n+1} \) in Theorem 1 and Theorem 2, respectively.

We study the spectral properties of eigenvalues with joint density in (1.2) for \( \tau = 0 \) in this paper. A generalization of our methods to the general case (1.2) with \( \tau \in (0, 1) \) remains unknown and may be challenging. We leave it as a future work.

The rest of the paper is organized as follows. The proofs of Theorems 1 and 2, Corollary 1, and the check of (1.9) are presented in Sec. II.

II. PROOFS

This section is divided into four parts. In Sec. II A, we make some preparations; in Sec. II B, we prove Theorems 1 and 2 as well as Corollary 1; in Sec. II C, we prove the technical results of Lemmas 2.9 and 2.10 used in Sec. II B; the verification of (1.9) is given in Sec. II D.

A. Some technical tools

We first list some properties of \( K_\nu(x) \), the modified Bessel function of the second kind. To avoid confusion, for all lemmas in this section, the parameter \( v \in [0, \infty) \) is reserved for the subscript in the modified Bessel function of the second kind \( K_v \). In the case that the parameter \( v \) also depends on \( n \), we will write \( v = v_n \).

**Lemma 2.1 (Properties of \( K_v \)).** The following statements hold.

(a) \( \text{[Formula 9.6.24 in the work of Abramowitz and Stegun]} \). For any \( v \geq 0 \),

\[ K_v(x) = \int_0^\infty e^{-x} \cos(t) \cosh(vt) dt, \quad x > 0, \]

where \( \cosh(t) = (e^t + e^{-t})/2, \quad t \in \mathbb{R} \).

(b) \( \text{[Formulas 9.6.8 and 9.6.9 in the work of Abramowitz and Stegun]} \). If \( v = 0 \), then

\[ K_0(x) \sim -\log x \quad \text{as } x \downarrow 0, \]

and if \( v > 0 \) is fixed, then

\[ K_v(x) \sim 2^{v-1} \Gamma(v)x^{-v} \quad \text{as } x \downarrow 0. \]
Thus, the joint probability density function (pdf) of \( z_1, \ldots, z_n \) as in (1.3), then the density function of \( (z_1, \ldots, z_n) =: u = (u_1, \ldots, u_n) \) is given by

\[
f(u) = C \cdot \prod_{1 \leq j < k \leq n} |u_j - u_k|^2 \cdot \prod_{j=1}^n |u_j|^n K_v(2n|u_j|)
\]

for all \( u \in \mathbb{C}^n \). Since the Lebesgue measure of \( \mathbb{C}^n \setminus C'' \) is zero, we will simply regard \( (u_1, \ldots, u_n) \) has density \( f(u) \) defined for all \( u \in \mathbb{C}^n \).

**Proof.** Identify \( \mathbb{C} \) with \( \mathbb{R}^2 \). Then, \( C_1 = \{(x, y) : x > 0, y \in \mathbb{R}\} \) and \( C_2 = \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\} \). First, we show the transform \( u = z^2 : C_1 \to C_2 \) is a one-to-one and onto map. Write \( z = (x, y) \). Define a transform \( u = z^2 \), which is the same as \( u(x, y) = (x, y)^2 \) in \( C_1 \to C_2 \) such that \( u(x, y) = (s, t) = (x^2 - y^2, 2xy) \). By solving the last equation, we obtain the inverse function \( z = z(s, t) : C_2 \to C_1 \) given by

\[
z(s, t) = \left(\frac{1}{\sqrt{2}} \sqrt{s^2 + t^2 + s}, \frac{1}{\sqrt{2}} \text{sgn}(t) \sqrt{s^2 + t^2 - s}\right),
\]

where "sgn" is the sign function.

Second, let us compute the Jacobian of the transform,

\[
\frac{\partial (s, t)}{\partial (x, y)} = \text{det} \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 4(x^2 + y^2) = 4|z|^2 = 4|u|.
\]

Thus, the joint probability density function (pdf) of \( u_i = z^2_i, i = 1, \ldots, n, \) is given by

\[
f(u_1, \ldots, u_n) = C \cdot \prod_{1 \leq j < k \leq n} |u_j - u_k|^2 \cdot \prod_{j=1}^n |u_j|^n K_v(2n|u_j|)
\]

for all \( u \in \mathbb{C}^n \).

**Lemma 2.3.** Define \( \xi(x) = x - \log(1 + x) \) for \( x \geq 0 \). Then,

\[
\xi(x) \geq \frac{1}{4} ax
\]

for all \( a \in (0, 1] \) and \( x \geq a \).

**Proof.** Fix \( a \in (0, 1] \). Note that

\[
\xi(x) = x^2 \int_0^1 \frac{t}{1 + tx} dt \geq x^2 \int_0^1 \frac{1}{1 + x} dt = \frac{x}{1 + x} \frac{x}{2}.
\]

Since \( \frac{x}{1 + x} \) is increasing in \( x \geq 0 \), we get \( \frac{x}{1 + x} \geq \frac{x}{1 + x} \geq \frac{x}{2} \) if \( x \geq a \). The lemma follows.

**Lemma 2.4.** Let \( f \) and \( g \) be density functions of real random variables \( X \) and \( Y \), respectively. Assume \( f \) and \( g \) have a common support \( D \subset \mathbb{R} \) and \( q(x) := f(x)/g(x) \) is non-decreasing in \( x \in D \). Then, \( P(X > x) \geq P(Y > x) \) for all \( x \in \mathbb{R} \).
Proof. Let \( \omega_i = \inf \{x : x \in D\} \) and \( \omega_u = \sup \{x : x \in D\} \). It suffices to show that \( P(X > x) \geq P(Y > x) \) for \( \omega_i < x < \omega_u \). Now, we extend function \( q \) to the range \( \omega_i < x < \omega_u \) by defining \( q(x) = \inf \{q(y) : y \geq x, y \in D\} \) if \( \omega_i < x < \omega_u \) but \( x \not\in D \). Then, \( q(x) \) is non-decreasing for \( \omega_i < x < \omega_u \) and \( f(x) = q(x)g(x) \) for \( \omega_i < x < \omega_u \). Furthermore, for \( \omega_i < x < \omega_u \),

\[
P(X > x) = \int_{\omega_i}^{x} q(t)g(t)dt \geq q(x)\int_{\omega_i}^{x} g(t)dt = q(x)P(Y > x)
\]

and

\[
P(X \leq x) = \int_{\omega_i}^{x} q(t)g(t)dt \leq q(x)\int_{\omega_i}^{x} g(t)dt = q(x)P(Y \leq x),
\]

that is,

\[
P(X > x) \geq q(x)P(Y > x) \quad \text{and} \quad P(X \leq x) \leq q(x)P(Y \leq x)
\]

for all \( \omega_i < x < \omega_u \). By multiplying \( P(Y \leq x) \) on both sides of the first inequality, multiplying \( P(Y > x) \) on both sides of the second one and comparing, we see that

\[
P(X > x)P(Y \leq x) \geq P(X \leq x)P(Y > x)
\]

for \( \omega_i < x < \omega_u \). The desired result is obtained by adding \( P(X > x)P(Y > x) \) to the both sides of the above inequality. \( \square \)

Lemma 2.5. Let \( u_1, \ldots, u_n \) have the joint density function \( f(u_1, \ldots, u_n) \) as in (2.3). Let \( Y_j, 1 \leq j \leq n, \) be independent random variables, and the density of \( Y_j \) is proportional to \( y^{3v_{u1}}K_v(2ny)(y > 0) \). Then, \( g \) \( (u_1, \ldots, [u_n]) \) and \( g \) \( (Y_1, \ldots, Y_n) \) have the same distribution for any symmetric function \( g \).

Proof. Define \( f(u_1, \ldots, u_n) = 0 \) for any \( u = (u_1, \ldots, u_n) \) with \( u_j \in (-\infty, 0] \) for some \( 1 \leq j \leq n \). Then, the conclusion follows from Lemma 1.1 in the work of Jiang and Qi. \( \square \)

Lemma 2.6. Let \( Y_j, 1 \leq j \leq n, \) be independent random variables defined in Lemma 2.5. Then, for each \( y, P(Y_j > y) \) is non-decreasing in \( j \) for \( 1 \leq j \leq n \).

Proof. Since the pdf of \( Y_j \) is proportional to \( y^{3v_{u1}}K_v(2ny)(y > 0) \), the ratio of pdfs of \( Y_j \) and \( Y_{j-1} \) is proportional to \( y/(y > 0) \) which is increasing in \( y > 0 \). It follows from Lemma 2.4 that \( P(Y_j > y) \geq P(Y_{j-1} > y) \) for \( 2 \leq j \leq n \). This completes the proof. \( \square \)

Now, we study the Central Limit Theorem (CLT) for \( Y_j \) as \( v = v_n \to \infty \). We use \( \Phi(x) \) to denote the cumulative distribution function of the standard normal \( N(0, 1) \), that is,

\[
\Phi(x) = \int_{-\infty}^{x} \phi(t)dt
\]

with \( \phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2} \) for \( t \in \mathbb{R} \).

Lemma 2.7. Let \( \Phi(x) \) be as in (2.5). Then,

\[
1 - \Phi(ct + d) = [1 - \Phi(t)](1 + o(1)) + O\left(\frac{1}{n^2}\right).
\]

uniformly over \( t \in \mathbb{R}, |c - 1| \leq n^{-3/8}, \) and \( |d| \leq n^{-3/8} \) as \( n \to \infty \).

Proof. First, by (2.26),

\[
1 - \Phi(ct + d) = P(N(0, 1) \geq ct + d) \leq e^{-(ct+d)^2/2}
\]

as \( t \geq 2 \) and \( n \) is sufficiently large. Thus, as \( n \to \infty \),

\[
1 - \Phi(ct + d) \leq \frac{1}{n^2},
\]

uniformly over \( |c - 1| \leq n^{-3/8}, |d| \leq n^{-3/8} \), and \( t \geq \log n \). In particular, it holds with \( c = 1 \) and \( d = 0 \). Thus, (2.6) is true uniformly over \( |c - 1| \leq n^{-3/8}, |d| \leq n^{-3/8}, \) and \( t \geq \log n \) as \( n \to \infty \). If \( t \leq -\log n \), then
uniformly over $|c-1| \leq n^{-3/8}$ and $|d| \leq n^{-3/8}$ by the same argument as obtaining (2.7). Hence, (2.6) is true uniformly over $|c-1| \leq n^{-3/8}$, $|d| \leq n^{-3/8}$, and $t \leq \log n$ as $n \to \infty$.

Now, assume $|t| < \log n$. Review $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ for $t \in \mathbb{R}$. Then,

\[
1 - \Phi(ct + d) = c \int_t^\infty \phi(cs + d) \, ds.
\]

Note that

\[
\frac{\phi(cs + d)}{\phi(s)} = \exp\left\{-\frac{1}{2} \left(c^2 - 1\right)s^2 - cs + \frac{1}{2} c^2 \right\} = 1 + O\left(\frac{1}{n^{1/8}}\right),
\]

uniformly over $|s| \leq \log n$ as $n \to \infty$. Thus,

\[
1 - \Phi(ct + d) = 1 - \Phi(c \log n + d) + c \int_t^{\log n} \phi(cs + d) \, ds
\]

\[
= O\left(\frac{1}{n^2}\right) + \left[1 + O\left(\frac{1}{n^{1/8}}\right)\right] \cdot c \int_t^{\log n} \phi(s) \, ds,
\]

(2.8)

uniformly over $|c-1| \leq n^{-3/8}$ and $|d| \leq n^{-3/8}$ by (2.7). In particular,

\[
1 - \Phi(t) = O\left(\frac{1}{n^2}\right) + \left[1 + O\left(\frac{1}{n^{1/8}}\right)\right] \cdot \int_t^{\log n} \phi(s) \, ds.
\]

Solve for $\int_t^{\log n} \phi(s) \, ds$ and then plug it in (2.8) to see that

\[
1 - \Phi(ct + d) = O\left(\frac{1}{n^2}\right) + \left[1 + O\left(\frac{1}{n^{1/8}}\right)\right] \cdot [1 - \Phi(t)],
\]

uniformly over $|c-1| \leq n^{-3/8}$, $|d| \leq n^{-3/8}$, and $|t| < \log n$ as $n \to \infty$. This together with the earlier conclusions yields the desired conclusion.

\[\square\]

Lemma 2.8. Let $Y_j$ be as in Lemma 2.5. Let $v = v_n \to \infty$ as $n \to \infty$. Fix $\delta \in (0, 1)$. Then, as $n \to \infty$,

\[
P\left(\frac{2nY_j - 2\sqrt{j(v + j)}}{\sqrt{2j + v}} > t\right) = (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^2}\right),
\]

(2.9)

uniformly over $t \in \mathbb{R}$ and $\delta n \leq j \leq n$.

Proof. Define

\[
\tau(x) = \eta(x) + \frac{1}{4v} \log(1 + x^2) - \frac{2j + v - 1}{v} \log x
\]

\[
= \sqrt{1 + x^2} \log x - \log(1 + \sqrt{1 + x^2}) + \frac{1}{4v} \log(1 + x^2) - \frac{2j + v - 1}{v} \log x.
\]

(2.10)

It is easy to check that

\[
\tau'(x) = \frac{\sqrt{1 + x^2}}{x} + \frac{x}{2v(1 + x^2)} - \frac{2j + v - 1}{v} \frac{x}{v x},
\]

\[
\tau''(x) = \frac{1}{2v(1 + x^2)} - \frac{2j - 1.5 + v}{v},
\]

(2.11)

and
\[ r''(x) = -\frac{1}{x^2 \sqrt{1 + x^2}} + \frac{1 - x^2}{2v(1 + x^2)^2} + \frac{2j + v - 1}{vx^2}, \]  
\[ j > \frac{2j - 1.5}{vx^2}, \]  

(2.12)

(2.13)

by the trivial facts that
\[ -\frac{1}{x^2 \sqrt{1 + x^2}} > -\frac{1}{x^2}, \quad -\frac{1}{2v(1 + x^2)^2} > -\frac{1}{2vx^2}, \quad \text{and} \quad \frac{1}{v(1 + x^2)^2} > 0. \]

Note that \( \tau'(0+) = -\infty \) and \( \tau'(\infty) = 1 \). Since \( r''(x) > 0 \) for all \( x > 0 \), \( r'(x) \) is strictly increasing, and thus a unique root to the equation \( r'(x) = 0 \) exists in \((0, \infty)\). Denote this root by \( \mu_{\nu,j} \). Then, \( \mu_{\nu,j} > 0 \) and \( \tau'(\mu_{\nu,j}) = 0 \).

Define
\[ a_{\nu,j}^2 = \frac{2j + v}{v^2}, \quad \beta_{\nu,j}^2 = \frac{\sigma_{\nu,j}}{\mu_{\nu,j}}, \quad \text{and} \quad \gamma_{\nu,j}(t) = \mu_{\nu,j}(1 + \beta_{\nu,j}t), \]

for \( t \in \mathbb{R} \). Note that the density function of \( \frac{2nY}{v} \) as a function of \( y \) is proportional to \( y^{2jv-1} K_v(y)I(y > 0) \). Set
\[ V_j = \frac{1}{a_{\nu,j}^2} \frac{2nY_j}{v - \mu_{\nu,j}}, \]

(2.16)

for \( 1 \leq j \leq n \). Then, the density function of \( V_j \) as a function of \( t \) is proportional to \( \xi_{\nu,j}(t)(1 + tf_{\nu,j} > 0) \), where
\[ \xi_{\nu,j}(t) = (\gamma_{\nu,j}(t))^2 v^{2jv-1} K_v(\gamma_{\nu,j}(t)). \]

Hence, the density function of \( V_j \) is equal to
\[ f_{\nu,j}(t) = \frac{1}{C_{\nu,j} \xi_{\nu,j}(0)} I(1 + tf_{\nu,j} > 0), \]

where
\[ C_{\nu,j} = \int_{-\infty}^{\infty} \xi_{\nu,j}(t)(1 + tf_{\nu,j} > 0) dt \]

and \( \xi_{\nu,j}(0) = (\mu_{\nu,j})^{2jv-1} K_v(\mu_{\nu,j}) > 0 \) by Lemma 2.1(a) and (2.14). Review (2.17). Take \( x = \gamma_{\nu,j}(t) \). Then, from (2.2),
\[ \xi_{\nu,j}(t) = x^{2jv-1} K_v(vx) \]
\[ = \sqrt{\frac{\pi v}{2v^x}} x^{2jv-1} \exp \left[ -\frac{1}{4} \log(1 + x^2) - v\eta(x) \right] \left( 1 + O \left( \frac{1}{v} \right) \right) \]
\[ = \sqrt{\frac{\pi}{2v^x}} \exp \left[ -v(\tau(\gamma_{\nu,j}(t)) - \tau(\mu_{\nu,j})) \right] \left( 1 + O \left( \frac{1}{v} \right) \right). \]

With \( y_{\nu,j}(0) = \mu_{\nu,j} \), we then have
\[ f_{\nu,j}(t) = \frac{1 + O \left( \frac{1}{v} \right)}{C_{\nu,j}} \exp \left[ -v(\tau(\gamma_{\nu,j}(t)) - \tau(\mu_{\nu,j})) \right] I(1 + tf_{\nu,j} > 0) \]

(2.18)

and
\[ C_{\nu,j} = \left( 1 + O \left( \frac{1}{v} \right) \right) \int_{t \beta_{\nu,j} > 0} \exp \left[ -v(\tau(\gamma_{\nu,j}(t)) - \tau(\mu_{\nu,j})) \right] dt. \]

(2.19)

Some properties of \( \mu_{\nu,j} \) will be provided in Lemmas 2.9 and 2.10. Since the proof of each lemma costs a considerable length, we postpone them until Sec. II C.
Lemma 2.9. [Estimate of $\mu_{v,j}$ from (2.14)]. Let $\delta \in (0, 1)$ be fixed. Let $v = v_n \to \infty$ as $n \to \infty$. Define

$$d_{v,j} = \frac{1}{\sigma_{v,j}} \left( \frac{2\sqrt{j(v+j)}}{\nu} - \mu_{v,j} \right),$$

$$\bar{\mu}_{v,j} = \frac{(2j + v - 1.5)}{v} \left( \frac{v}{2(2j + v - 1.5)^2} \right) - 1$$

for $\delta n \leq j \leq n$. Then,

$$\mu_{v,j} = \bar{\mu}_{v,j} \left[ 1 + O \left( \frac{v^4}{n(n + v)^2} \right) \right]$$

(2.20)

holds uniformly over $\delta n \leq j \leq n$ as $n \to \infty$. Furthermore, $\max_{\delta n \leq j \leq n} |d_{v,j}| \to O(n^{-1/2})$.

The next result presents estimates of $\tau(y_{v,j}(t)) - \tau(\mu_{v,j})$.

Lemma 2.10. Review the notations $\tau(x), \mu_{v,j}, y_{v,j}$ and $\sigma_{v,j}$, and $\beta_{v,j}$ in (2.10), (2.14), and (2.15), respectively. Let $\delta \in (0, 1)$ be fixed. Then, the following statements hold uniformly over $|t| \leq n^{1/8}$ and $\delta n \leq j \leq n$ as $n$ is sufficiently large.

(i) $\sigma_{v,j}^2 \tau''(\mu_{v,j}) = \frac{1}{t} \left[ 1 + O \left( \frac{t}{n} \right) \right]$.

(ii) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \frac{1}{2} \sigma_{v,j}^2 \tau''(\mu_{v,j}) t (1 + O(\beta_{v,j}|t|))$.

(iii) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) > \frac{1}{2} \beta_{v,j}^2 t - \log(1 + \beta_{v,j}|t|)$ for $t \geq 0$.

(iv) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) \geq \frac{1}{2} \beta_{v,j}^2 t - \log(1 + \beta_{v,j}|t|)$ for $-1/\beta_{v,j} < t < 0$.

Let $V_j$ be as in (2.16). Let $\delta \in (0, 1)$ be fixed. With Lemmas 2.9 and 2.10 at hand, we claim that $V_j$’s satisfy a “uniform” CLT, that is,

$$P(V_j > t) = \left[ 1 - \Phi(t) \right] (1 + o(1)) + O \left( \frac{1}{n^2} \right),$$

(2.21)

uniformly over $t \in \mathbb{R}$ and $\delta n \leq j \leq n$ as $n \to \infty$. If this is true, by (2.15) and (2.16),

$$\frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j + v}} = V_j - d_{v,j},$$

where $d_{v,j} = \left( \frac{2\sqrt{(v+j)}}{\nu} - v \right) / \sigma_{v,j}$. By Lemma 2.9, $\max_{\delta n \leq j \leq n} |d_{v,j}| = O(n^{-1/2})$. Therefore, we have from (2.21) and Lemma 2.7 that

$$P \left( \frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j + v}} > t \right) = P(V_j \geq t + d_{v,j})$$

$$= (1 + o(1)) \left( 1 - \Phi(t + d_{v,j}) \right) + O \left( \frac{1}{n^2} \right)$$

$$= (1 + o(1)) \left( 1 - \Phi(t) \right) + O \left( \frac{1}{n^2} \right),$$

uniformly over $t \in \mathbb{R}$ and $\delta n \leq j \leq n$ as $n \to \infty$. Hence, (2.9) is obtained. So, to complete the whole proof, it remains to show (2.21). We will prove this next via the method of steepest descent.

Recall $\beta_{v,j} = \sigma_{v,j}^2 / \mu_{v,j}$ from (2.15), where $\sigma_{v,j}^2 = \frac{2\sqrt{v+1}}{\nu}$ and $\mu_{v,j}$ is as in (2.14). Easily, from (2.20),

$$\frac{c_1}{\sqrt{n}} \leq \min_{\delta n \leq j \leq n} \beta_{v,j} \leq \max_{\delta n \leq j \leq n} \beta_{v,j} \leq \frac{c_2}{\sqrt{n}},$$

(2.22)

where $c_1 > 0$ and $c_2 > 0$ are two constants depending on $\delta$ only. It follows from (i) and (ii) of Lemma 2.10 that

$$v(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})) = \frac{\beta_{v,j}^2}{2} \left[ 1 + O \left( \frac{\beta_{v,j}|t|}{n + v} \right) \right]$$

$$= \frac{\beta_{v,j}^2}{2} \left[ 1 + O \left( n^{-1/2} \right) \right],$$

uniformly over $|t| \leq n^{1/8}$ and $\delta n \leq j \leq n$, which implies

$$\exp \left[ - v(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})) \right] = e^{-\beta_{v,j}^2 t / 2} \left[ 1 + O \left( n^{-1/8} \right) \right],$$

(2.23)
uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq j \leq n\). From (2.22),
\[
\max_{\delta n \leq j \leq n} \beta_{v,j} n^{1/8} \leq c_2 n^{-3/8} < 1,
\]
for all large \(n\). Thus, from Lemma 2.10(iii) and Lemma 2.3 (take \(x = a\)), we get
\[
v(t(y_{v,j}(t)) - \tau(\mu_{v,j})) \geq \frac{1}{4} n^{1/8} \beta_{v,j} t \geq \frac{c_1^2 \delta}{4} n^{1/8} t,
\]
for any \(t > n^{1/8}\). Therefore, for all large \(n\),
\[
\int_{\mu_{v,j}}^{\infty} \exp\left[-v(t(y_{v,j}(t)) - \tau(\mu_{v,j}))\right] dt \\
\leq \int_{\mu_{v,j}}^{\infty} \exp\left(-\frac{c_1^2 \delta}{4} n^{1/8} t\right) dt \\
= \frac{4}{c_1^2 \delta n^{1/8}} \exp\left(-\frac{c_1^2 \delta}{4} n^{1/8}\right) \\
= O\left(\frac{1}{n^{1/8}}\right).
\]
(2.24)
It follows from (i) and (iv) of Lemma 2.10 that, uniformly over \(\delta n \leq j \leq n\) and \(-1/\beta_{v,j} < t < 0\), we have
\[
v(t(y_{v,j}(t)) - \tau(\mu_{v,j})) \geq \frac{t^2}{4},
\]
for all large \(n\), which yields
\[
\int_{-1/\beta_{v,j}}^{-n^{1/8}} \exp\left[-v(t(y_{v,j}(t)) - \tau(\mu_{v,j}))\right] dt \\
\leq \int_{-n^{1/8}}^{-1/\beta_{v,j}} \exp\left(-\frac{t^2}{4}\right) dt \\
\leq \frac{2}{n^{1/8}} \exp\left(-\frac{1}{4} n^{1/4}\right) \\
= O\left(\frac{1}{n^{1/8}}\right).
\]
(2.25)
uniformly over \(\delta n \leq j \leq n\), where the second integral is equal to \(\sqrt{4\pi} \cdot P(N(0,1) \geq n^{1/8}/\sqrt{2})\), and the second inequality is obtained by the inequality
\[
P(N(0,1) \geq x) \leq \frac{1}{\sqrt{2\pi} x} e^{-x^2/2},
\]
(2.26)
for all \(x > 0\). Similar to (2.25), we have
\[
\int_{-n^{1/8}}^{n^{1/8}} e^{-t^2/2} dt = \int_{-\infty}^{\infty} e^{-t^2/2} dt - 2 \int_{n^{1/8}}^{\infty} e^{-t^2/2} dt \\
= \sqrt{2\pi} + O\left(\frac{1}{n}\right).
\]
(2.27)
Consequently, from (2.23),
\[
\int_{-n^{1/8}}^{n^{1/8}} \exp\left[-v(t(y_{v,j}(t)) - \tau(\mu_{v,j}))\right] dt = \int_{-n^{1/8}}^{n^{1/8}} e^{-t^2/2} dt \cdot (1 + O(n^{-1/8})) \\
= \sqrt{2\pi} + O(n^{-1/8}),
\]
uniformly over \(\delta n \leq j \leq n\). By taking into account the above estimates, we have from (2.19) that
\[
C_{v,j} = \left(1 + O\left(\frac{1}{n}\right)\right) \sqrt{2\pi} + O\left(\frac{1}{n^{1/2}}\right) = \sqrt{2\pi} + o(1),
\]
uniformly over $\delta n \leq j \leq n$ as $n \to \infty$. Furthermore, remember $V_j$ has the density function $f_{v_j}(t)$ as in (2.28). Easily, $V_j \geq -1/\beta_{v_j}$ from (2.16) for each $j$. By the expression of $f_{v_j}(t)$ from (2.18),

$$P(V_j > t) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{h(t)}^{\infty} \exp\left[ -v(\tau(y_{v_j}(s)) - \tau(\mu_{v_j})) \right] ds,$$

(2.28)

uniformly over $\delta n \leq j \leq n$ as $n \to \infty$, where $h(t) := \max\{-1/\beta_{v_j}, t\}$ for $t \in \mathbb{R}$. If $t < -n^{-1/8}$, then

$$0 \leq \left( \int_{h(t)}^{\infty} - \int_{-n^{-1/8}}^{\infty} \right) \exp\left[ -v(\tau(y_{v_j}(s)) - \tau(\mu_{v_j})) \right] ds$$

$$\leq \int_{-n^{-1/8}}^{\infty} \exp\left[ -v(\tau(y_{v_j}(s)) - \tau(\mu_{v_j})) \right] ds$$

$$\leq \frac{1}{n^2},$$

by (2.25). Hence,

$$P(V_j > t) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{-n^{-1/8}}^{\infty} \exp\left[ -v(\tau(y_{v_j}(s)) - \tau(\mu_{v_j})) \right] ds + O\left(\frac{1}{n^2}\right)$$

$$= \frac{1 + o(1)}{\sqrt{2\pi}} \left[ \sqrt{2\pi + O\left(\frac{1}{n^2}\right)} + O\left(\frac{1}{n^2}\right) \right]$$

$$= 1 + o(1)$$

$$= (1 + o(1))(1 - \Phi(t)) + (1 + o(1))\Phi(t)$$

$$= (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^2}\right).$$

uniformly over $\delta n \leq j \leq n$ as $n \to \infty$ by (2.24) and (2.27). In the last step, we have used the inequality $\Phi(t) \leq P(N(0,1) \geq n^{1/8}) \leq \frac{1}{\sqrt{n}}$ for all $t < -n^{-1/8}$. Therefore, (2.21) holds uniformly over $t < -n^{-1/8}$ and $\delta n \leq j \leq n$ as $n \to \infty$. If $t \geq -n^{-1/8}$, we see from (2.28) that

$$P(V_j > t) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{t}^{\infty} \exp\left[ -v(\tau(y_{v_j}(s)) - \tau(\mu_{v_j})) \right] ds.$$

Therefore, by the same arguments as in (2.24) and (2.27), we conclude that the above integral is equal to $\int_{t}^{\infty} e^{-t^2/2} + O(n^{-2})$ uniformly over $t \geq -n^{-1/8}$ and $\delta n \leq j \leq n$ as $n \to \infty$. This combining with the earlier conclusion for $t < -n^{-1/8}$ completes the proof of (2.21).

Lemma 2.8 focuses on the CLT for $Y_j$ as $v = v_n \to \infty$. Now, we work on the same problem under the assumption that $\{v_n; n \geq 1\}$ is a bounded sequence. A lemma is needed first.

Lemma 2.11. Let $g_1$ and $g_2$ be non-negative functions defined over $(0, \infty)$ and $b_i(\theta) := \int_{0}^{\infty} t^i g_i(t) dt < \infty$ for all $\theta \geq \theta_0$, $i = 1, 2$, where $\theta_0 > 0$ is a constant. Assume that $g_1(x) \sim g_2(x)$ as $x \to \infty$. The following holds.

(a) Uniformly over $\theta \in [\theta_0, \infty)$,

$$\int_{x}^{\infty} t^i g_1(t) dt \sim \int_{x}^{\infty} t^i g_2(t) dt \quad \text{as} \quad x \to \infty.$$

(b) If $\lim_{\theta \to \infty} \frac{\log b_1(\theta)}{\theta} = \infty$, then $b_1(\theta) \sim b_2(\theta)$ as $\theta \to \infty$.

Proof. (a). Write

$$\int_{x}^{\infty} t^i g_1(t) dt = \int_{x}^{\infty} t^i g_2(t) dt + \int_{x}^{\infty} t^i g_2(t) \left( \frac{g_1(t)}{g_2(t)} - 1 \right) I(g_2(t) = 0) \, dt.$$

Denote by $e(x)$ the last integral. Then,

$$|e(x)| \leq \max_{t \geq x} \left[ \left| \frac{g_1(t)}{g_2(t)} - 1 \right| I(g_2(t) = 0) \right] \cdot \int_{x}^{\infty} t^i g_2(t) dt.$$

The conclusion then follows from the fact that $g_1(x) \sim g_2(x)$ as $x \to \infty$.

(b) Define $x_0 = \frac{1}{2}(b_1(\theta))^{1/\theta}$ for large $\theta$. Then, $x_0 = \frac{1}{2} \exp\left(\frac{\log b_1(\theta)}{\theta}\right) \to \infty$ as $\theta \to \infty$, and

$$\lim_{\theta \to \infty} \frac{(x_0)^{\theta}}{b_1(\theta)} = 0.$$
which implies that
\[ \int_{0}^{\infty} t^\theta \xi(t) dt \leq (x_0)^\theta \int_{0}^{\infty} \xi(t) dt = O((x_0)^\theta) = o(b_1(\theta)), \quad (2.29) \]
for \( i = 1, 2 \). It follows from part (a) that, as \( \theta \to \infty \),
\[ \int_{x_0}^{\infty} t^\theta \xi(t) dt \sim \int_{x_0}^{\infty} t^\theta \xi_1(t) dt = b_1(\theta) - \int_{x_0}^{\infty} t^\theta \xi(t) dt = b_1(\theta)(1 + o(1)). \]
Therefore,
\[ \int_{x_0}^{\infty} t^\theta \xi(t) dt = b_1(\theta)(1 + o(1)). \]
This and (2.29) conclude that
\[ \int_{0}^{\infty} t^\theta \xi(t) dt = b_1(\theta)(1 + o(1)) \]
as \( \theta \to \infty \).

\[ \square \]

**Lemma 2.12.** Let \( v_0 > 0 \) be a fixed number. Let \( Y_j \) be as in Lemma 2.5. Fix \( \delta \in (0, 1) \). Then, as \( n \to \infty \),
\[ P \left( \frac{2nY_j - 2\sqrt{j(v + j)}}{\sqrt{2j + v}} > t \right) = (1 + o(1))(1 - \Phi(t)) + O \left( \frac{1}{n^2} \right). \]
uniformly over \( t \in \mathbb{R}, \delta n \leq j \leq n, \) and \( 0 \leq v \leq v_0 \).

**Proof.** The proof is divided into a few steps.

**Step 1. Reduction of \( Y_j \) to a Gamma distribution.** Let \( \text{Gamma}(\alpha, \beta) \) denote a Gamma distribution with the density function given by
\[ y_{\alpha, \beta}(t) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} t^{\alpha-1} e^{-t/\beta}, \quad t > 0, \]
where \( \alpha > 0 \) and \( \beta > 0 \) are parameters. It follows from part (b) and part (c) in Lemma 2.1 that
\[ \int_{0}^{\infty} t^\theta \xi(t) dt < \infty, \quad (2.30) \]
for any \( \theta \geq v \). Since we consider \( \nu \in [0, v_0] \) here, the above integral is well defined for \( \theta \geq v_0 =: \theta_0 \). For each \( \nu \in [0, v_0] \), define
\[ \xi(t) = t^{-1/2} e^{-t} \quad \text{and} \quad \xi_{\nu}(t) = \sqrt{\frac{2}{\pi}} K_\nu(t), \quad t > 0. \]
Then, for fixed \( v \in [0, v_0] \), we have from (2.1) that \( \xi(t) \sim \xi_{\nu}(t) \) as \( t \to \infty \). Furthermore, we have for \( \theta \geq \theta_0 \) that
\[ b_1(\theta) := \int_{0}^{\infty} t^\theta \xi(t) dt = \int_{0}^{\infty} t^{\theta-1/2} e^{-t} dt = \Gamma \left( \theta + \frac{1}{2} \right) \]
and
\[ b_{\nu}(\theta) := \int_{0}^{\infty} t^\theta \xi_{\nu}(t) dt = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} t^\theta K_\nu(t) dt < \infty \]
from (2.30). From Chap. 6 in the work of Abramowitz and Stegun,\(^3\) we also have that
\[ \Gamma \left( \theta + \frac{1}{2} \right) \sim e^{-\theta - \frac{1}{2}} \left( \theta + \frac{1}{2} \right)^\theta (2\pi)^{1/2} \]
as \( \theta \to \infty \), which implies that \( \lim_{\theta \to \infty} \frac{\ln \Gamma(\theta)}{\theta} = \infty \).
In virtue of Lemma 2.11(b), for fixed \( \nu \in [0, v_0] \),
\[ b_{\nu}(\theta) \sim b_1(\theta) \quad (2.31) \]
as \( \theta \to \infty \); by Lemma 2.11(a), as \( x \to \infty \),
\[
\int_x^\infty t^\theta g_{v2}(t) dt \sim b_1(\theta) \int_x^\infty g_{\theta^v+1}(t) dt,
\]
uniformly over all \( \theta \geq \theta_0 \). Note that for each \( t > 0 \), \( \cosh(\tau(t)) \) is increasing in \( \nu \in [0, \infty) \). From Lemma 2.1(a), we know \( K_\nu(x) \) is increasing in \( \nu \in [0, \infty) \) for fixed \( x > 0 \), so is \( g_{v2}(x) \). This implies that
\[
\frac{b_{v2}(\theta)}{b_1(\theta)} \leq \frac{b_{v2}(\theta)}{b_1(\theta)} \leq \frac{b_{v2}(\theta)}{b_1(\theta)},
\]
\[
\frac{1}{I(x, \theta)} \int_x^\infty t^\theta g_{v2}(t) dt \leq \frac{1}{I(x, \theta)} \int_x^\infty t^\theta g_{v2}(t) dt \leq \frac{1}{I(x, \theta)} \int_x^\infty t^\theta g_{v2}(t) dt,
\]
for any \( 0 \leq v \leq v_0 \), where \( I(x, \theta) := b_1(\theta) \int_x^\infty g_{\theta^v+1}(t) dt \). Then, (2.31) holds uniformly over \( 0 \leq \nu \leq v_0 \). Second, the left and the right expressions of (2.32) do not depend on \( v \), and both converge uniformly in \( \theta \in [\theta_0, \infty) \). This guarantees the uniform convergence of the middle term in (2.32) to one over \( 0 \leq v \leq v_0 \) and \( \theta \geq \theta_0 \) as \( x \to \infty \). Thus, we conclude that
\[
\int_x^\infty \frac{t^\theta g_{v2}(t)}{b_{v2}(\theta)} dt \sim \frac{b_1(\theta)}{b_2(\theta)} \int_x^\infty g_{\theta^v+1}(t) dt
\]
uniformly over \( 0 \leq v \leq v_0 \) as \( x \to \infty \) and \( \theta \to \infty \), that is,
\[
\int_x^\infty \frac{t^\theta g_{v2}(t)}{b_{v2}(\theta)} dt = (1 + o(1)) \int_x^\infty g_{\theta^v+1}(t) dt,
\]
uniformly over \( 0 \leq v \leq v_0 \) as \( x \to \infty \) and \( \theta \to \infty \). By Lemma 2.5,
\[
\frac{t^{2jv-1}g_{v2}(t)}{b_{v2}(2j + v - 1)} = \sqrt{\frac{2}{\pi}} \frac{t^{2jv-1}K_{v}(t)}{b_{v2}(2j + v - 1)}
\]
is the density of \( 2nY_j \) for each \( 1 \leq j \leq n \). Immediately, for fixed \( \delta \in (0, 1) \) and any divergent sequence \( x_n \) with \( \lim_{n \to \infty} x_n = \infty \), we have that
\[
P(2nY_j > x) = \int_x^\infty \sqrt{\frac{2}{\pi}} \frac{t^{2jv-1}K_{v}(t)}{b_{v2}(2j + v - 1)} dt = (1 + o(1)) \int_x^\infty g_{2jv+1}(t) dt,
\]
uniformly over \( \delta \leq j \leq n \), \( 0 \leq v \leq v_0 \), and \( x \geq x_n \) as \( n \to \infty \).

Step 2. Estimation of the probability on the right-hand side of (2.33). Let \( S_m \) denote a random variable with density \( g_{m^\delta}(t) \). Note that \( S_m \) can be written as the sum of \( m \) i.i.d. random variables having a Gamma(1, 1) distribution and Gamma(1, 1) has mean 1 and variance 1. Then, it follows from Theorem 1 on page 217 of the book by Petrov\(^7\) that, for any sequence of positive numbers \( \tau_m \) such that \( \tau_m = o(m^{1/8}) \),
\[
P(S_m > m + \sqrt{mx}) = (1 + o(1))(1 - \Phi(x)), \text{ uniformly over } |x| \leq \tau_m
\]
and
\[
P(S_m < m - \sqrt{mx}) = (1 + o(1))(1 - \Phi(x)), \text{ uniformly over } |x| \leq \tau_m
\]
as \( m \to \infty \). Now set \( \tau_m = m^{1/7} \). Since
\[
P(S_m > m + \sqrt{mx}) \leq P(S_m > m + \sqrt{mm^{1/7}})
\]
\[
= (1 + o(1))(1 - \Phi(m^{1/7}))
\]
\[
= O\left(\frac{1}{m}\right),
\]
uniformly for \( x \geq m^{1/7} \). We have from (2.35) the following equation:
\[ P(S_m \leq m + \sqrt{mx}) \leq P(S_m \leq m - \sqrt{mn^{1/7}}) \]
\[ = (1 + o(1))(1 - \Phi(m^{1/7})) \]
\[ = O\left(\frac{1}{m^2}\right), \]
for \( x \leq -m^{1/7} \), and hence from (2.34),
\[ P(S_m > m + \sqrt{mx}) = 1 - P(S_m \leq m + \sqrt{mx}) \]
\[ = 1 + O\left(\frac{1}{m^2}\right) \]
\[ = 1 - \Phi(x) + O\left(\frac{1}{m^2}\right) \]
\[ = 1 - \Phi(x) + O\left(\frac{1}{m^2}\right). \]

uniformly for \( x \leq -m^{1/7} \) as \( m \to \infty \), where in the last step, we use the fact that \( \Phi(x) \leq \exp\{-m^{2/7}/2\} \) for all \( x \leq -m^{1/7} \) and \( m \geq 1 \) based on (2.26). Then, we conclude
\[ P(S_m > m + \sqrt{mx}) = (1 + o(1))(1 - \Phi(x)) + O\left(\frac{1}{m^2}\right). \] (2.36)
uniformly over \( x \in \mathbb{R} \) as \( m \to \infty \).

Let \( m_j \) denote the integer such that \( m_j - 1 < 2j + v - \frac{1}{4} \leq m_j \), and set \( m_j' = m_j - 1 \). Then, we have from (2.36) that
\[ P(S_m > m_j + \sqrt{m_j}x) = (1 + o(1))(1 - \Phi(x)) + O\left(\frac{1}{m^2}\right) \]
(2.37)
and
\[ P(S_m > m'_j + \sqrt{m'_j}x) = (1 + o(1))(1 - \Phi(x)) + O\left(\frac{1}{m^2}\right). \]
uniformly over \( x \in \mathbb{R} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). Note that
\[ P(S_m > 2\sqrt{f(j + v)} + t\sqrt{2j + v}) = P(S_m > m_j + \sqrt{m_j}x(t)), \]
where
\[ x(t) = t\sqrt{\frac{2j + v}{m_j}} + \frac{2\sqrt{f(j + v)} - m_j}{\sqrt{m_j}} \]
\[ = t\left[1 + O\left(\frac{1}{n}\right)\right] + O\left(\frac{1}{\sqrt{n}}\right). \]
uniformly over \( \delta n \leq j \leq n \) as \( n \to \infty \). Then, combining Lemma 2.7 and (2.37), we obtain
\[ P(S_m > 2\sqrt{f(j + v)} + t\sqrt{2j + v}) \]
\[ = (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^2}\right). \] (2.38)
uniformly over \( t \in \mathbb{R} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). Similarly, we have
\[ P(S_{m_j} > 2\sqrt{f(j + v)} + t\sqrt{2j + v}) \]
\[ = (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^2}\right). \] (2.39)
uniformly over \( t \in \mathbb{R} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \).

In view of Lemma 2.4, we have
\[ \int_{n_1}^{\infty} y_{n,1}(t)dt \geq \int_{n_2}^{\infty} y_{n,2}(t)dt, \]
whenever \( \alpha_1 > \alpha_2 > 0 \) and, thus,
\[
P(S_n) > 2 \sqrt{j(j + v)} + t \sqrt{2j + v} \leq \int_{2 \sqrt{j(j + v)} + t \sqrt{2j + v}}^{\infty} \frac{Y_{j/2}^{(1)}(s)}{s} ds \leq P(S_n) > 2 \sqrt{j(j + v)} + t \sqrt{2j + v},
\]
which coupled with (2.38) and (2.39) yields
\[
\int_{2 \sqrt{j(j + v)} + t \sqrt{2j + v}}^{\infty} \frac{Y_{j/2}^{(1)}(s)}{s} ds = (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^T}\right)
\]
uniformly over \( t \in \mathbb{R} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). From (2.33), we conclude
\[
P(2nY_j > 2 \sqrt{j(j + v)} + t \sqrt{2j + v}) = (1 + o(1))(1 - \Phi(t)) + O\left(\frac{1}{n^T}\right),
\]
uniformly over \( t \geq -n^{1/3} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \), where the choice of \( -n^{1/3} \) is such that
\[
\inf_{n \leq j \leq n^{rac{1}{3}}} \left\{ 2 \sqrt{j(j + v)} + t \sqrt{2j + v} \right\} \to \infty
\]
as \( n \to \infty \) required by (2.33). To complete the proof, we need to verify that (2.40) also holds uniformly over \( t < -n^{1/3} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \).

In fact, by taking \( t = -n^{1/3} \) in (2.41) and by the same argument as obtaining (2.40), we have
\[
P(2nY_j > 2 \sqrt{j(j + v)} + t \sqrt{2j + v}) \geq P(2nY_j > 2 \sqrt{j(j + v)} - n^{1/3} \sqrt{2j + v})
\]
\[
= (1 + o(1))(1 - \Phi(-n^{1/3})) + O\left(\frac{1}{n^T}\right)
\]
\[
= 1 + o(1).
\]
Therefore,
\[
P(2nY_j > 2 \sqrt{j(j + v)} + t \sqrt{2j + v - 1}) = 1 + o(1)
\]
\[
= (1 + o(1))(1 - \Phi(t)) + (1 + o(1))\Phi(t)
\]
\[
= 1 + o(1).
\]
uniformly over \( t < -n^{1/3} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). This completes the proof. \( \square \)

**Lemma 2.13.** (Lemma 2.2 from Jiang and Qi [12]). Let \( \{j_n, n \geq 1\} \) and \( \{x_n, n \geq 1\} \) be positive numbers with \( \lim_{n \to \infty} x_n = \infty \) and \( \lim_{n \to \infty} x_n^{-1/2} \log x_n \to \infty \). Let \( a(y) \) and \( b(y) \) be as in (1.4). For fixed \( y \in \mathbb{R} \), if \( \{c_{nj}, 1 \leq j \leq j_n, n \geq 1\} \) are real numbers such that \( \lim_{n \to \infty} \max_{1 \leq j \leq j_n} |c_{nj} x_n^{1/2} - 1| = 0 \), then
\[
\lim_{n \to \infty} \frac{1}{j_n} \sum_{j=1}^{j_n} \left[ 1 - \Phi((j - 1)c_{nj} + a(x_n) + b(x_n)y) \right] = c^{-y}.
\]

**Lemma 2.14.** Let \( v = v_n \) be a sequence of non-negative numbers. Let \( a(y) \) and \( b(y) \) be as in (1.4). Recall \( \Lambda \) in (1.5). If \( u_1, \ldots, u_n \) have density \( f(u_1, \ldots, u_n) \) as in (2.3), then
\[
\frac{1}{b(2n^{1/2})} \left[ 2n \max_{1 \leq j \leq n} [u_j] - 2 \sqrt{n(n + v)} \right] - a(2n^{1/2}) \to \Lambda.
\]

**Proof.** Set \( x_n = \frac{n(n + v)}{2n^{1/2}}, a_n = a(x_n), \) and \( b_n = b(x_n) \). Let \( Y_1, \ldots, Y_n \) be as in Lemma 2.5. Then, from the lemma, it suffices to show that
\[
\frac{1}{b_n} \left[ 2n \max_{1 \leq j \leq n} Y_j - 2 \sqrt{n(n + v)} \right] - a_n \to \Lambda
\]
or equivalently
\[
\prod_{j=1}^{n} P\left(2nY_{n-j+1} \leq 2\sqrt{n(n + v) + \sqrt{2n + v}(a_n + b_ny)}\right) \to \exp(-\epsilon^j),
\]
(2.42)

for any \(y \in \mathbb{R}\) as \(n \to \infty\).

To prove (2.42), it suffices to verify that for every subsequence \(\{n'\}\) of \(\{n\}\), there exists its further subsequence, say \(\{n''\}\), such that (2.42) is true along the subsequence \(\{n''\}\). Subsequences \(\{n'\}\) are selected in such a way that \(v_{n''}\) has a limit, say \(v_1\), where \(v_1 \in [0, \infty]\). Verification of (2.42) along a subsequence \(\{n''\}\) with \(\lim_{n'' \to \infty} v_{n''} = v_1 \in [0, \infty]\) is similar to proving (2.42) along the entire sequence with \(\lim_{n \to \infty} v_n \in [0, \infty]\). Therefore, for brevity, we will show (2.42) under each of the following conditions: (i) \(\lim_{n \to \infty} v_n \in [0, \infty]\); (ii) \(\lim_{n \to \infty} v_n = \infty\).

Fix \(y \in \mathbb{R}\). Set \(a_{nj} = P(2nY_{n-j+1} > 2\sqrt{n(n + v) + \sqrt{2n + v}(a_n + b_ny)}), 1 \leq j \leq n\). According to Lemma 2.6, \(a_{n1} \geq a_{n2} \geq \cdots \geq a_{nn}\). We need to prove

\[
\lim_{n \to \infty} \prod_{j=1}^{n} (1 - a_{nj}) = \exp(-\epsilon^j),
\]
which is equivalent to

\[
\lim_{n \to \infty} \sum_{j=1}^{n} a_{nj} = e^\epsilon
\]
(2.43)

if we can show

\[
a_{n1} = \max_{1 \leq j \leq n} a_{nj} \to 0 \text{ as } n \to \infty.
\]
(2.44)

Note that \(\frac{2}{n} \leq x_n \leq n\). It is seen that \(a_{n} + b_{n}y \to \infty\) and is of order \(\sqrt{\log n}\). Under condition (i), there exists a \(v_0 > 0\) such that \(0 \leq v_n \leq v_0\) for all \(n \geq 1\). Then, from Lemma 2.12, \(a_{n1} = (1 + o(1))(1 - \Phi(a_n + b_ny)) + O(n^{-2}) \to 0\) as \(n \to \infty\). The same is true by applying Lemma 2.8 under condition (ii) where \(v_n \to \infty\). This proves (2.44).

To prove (2.43), define \(j_n = \lceil n^{2/3} \rceil + 2\), where \(\lceil \cdot \rceil\) denotes the integer part of \(\cdot\). Set

\[
f_j = \frac{2nY_j - 2\sqrt{j(j + v)}}{\sqrt{2j + v}}
\]
for \(1 \leq j \leq n\). Then,

\[
a_{nj} = P(2nY_{n-j+1} > 2\sqrt{n(n + v) + \sqrt{2n + v}(a_n + b_ny)})
= P(Y_{n-j+1} > d_{nj} + a_n + b_ny),
\]

where

\[
d_{nj} = \frac{2\sqrt{n(n + v)}}{2n + v} - 2\sqrt{(n - (j - 1))(n + v - (j - 1))}
+ \left[\sqrt{2n + v - 2(j - 1)} - 1\right](a_n + b_ny)
= 2\sqrt{x_n} \left[1 - \frac{1}{n} - \frac{1}{n + v} - \frac{1}{n + v} - \left(1 - \frac{2(j - 1)}{2n + v}\right)^{1/2}\right]
= \left(1 - \frac{2(j - 1)}{2n + v}\right)^{1/2} - 1\right](a_n + b_ny).
\]

Write \((1 + x)^a = 1 + ax + \epsilon_1(x)\). Then, there exist constants \(C_a > 0\) and \(x_0 > 0\) such that \(|\epsilon_1(x)| \leq C_a x^2\) as \(|x| \leq x_0\). Taking \(a = 1/2\) and \(a = -1/2\), respectively, and using the trivial formula \((1 + a)(1 + b) = 1 + a + b + ab\), we obtain

\[
d_{nj} = \sqrt{x_n} \left[\frac{1}{n} + \frac{1}{n + v} + \epsilon_1^2(a_n + b_ny),
\right]
\]
where \(|\epsilon_1| \leq C_a x_n\) uniformly over \(1 \leq j \leq j_n\) and \(n\) is sufficiently large, and \(C_a\) is a constant not depending on \(n\) or \(j\). Note that \(a_n = O(\sqrt{\log n})\) and \(b_n \to 0\) as \(n \to \infty\). Therefore,
uniformly over $1 \leq j \leq j_n$ as $n \to \infty$. Define $c_{n1} = \sqrt{x_n}$ and $c_{nj} = \frac{d_n}{j^{\alpha}}$ for $2 \leq j \leq j_n$. Then, we have

$$a_{nj} = \Phi(Y_n - j_n),$$

and $\lim_{n \to \infty} \max_{1 \leq j \leq n} |c_{nj}/x_n - 1| = 0$. By applying Lemmas 2.8 and 2.12, respectively, we have

$$a_{nj} = (1 + o(1)) \left[ 1 - \Phi((j_n - 1)c_{nj} + a_n + b_ny) \right] + O\left(\frac{1}{n}\right).$$

uniformly over $1 \leq j \leq j_n$ as $n \to \infty$. Consequently, it follows from Lemma 2.13 that

$$\lim_{n \to \infty} \sum_{j = 1}^{j_n} a_{nj} = e^{-\gamma}.$$  

(2.47)

Moreover, by (2.45) and (2.46),

$$\sum_{j = j_n + 1}^{n} a_{nj} \leq (n - j_n)a_{nj},$$

$$= (1 + o(1)) \cdot n \left[ 1 - \Phi((j_n - 1)c_{nj} + a_n + b_ny) \right] + O\left(\frac{1}{n}\right),$$

$$\leq (1 + o(1)) \cdot n \left[ 1 - \Phi\left(\frac{j_n - 1}{2\sqrt{x_n}}\right) \right] + O\left(\frac{1}{n}\right),$$

where the facts $(j_n - 1)c_{nj} = d_{nj}, a_n = O(\sqrt{\log n})$, and $b_n \to 0$ are used. Noting that $(j_n - 1)/\sqrt{x_n} > n^{1/6}/2$ as $n$ is large enough, the sum above goes to zero by (2.26). This together with (2.47) yields (2.43).

\[\square\]

**B. Proofs of Theorems 1 and 2**

**Proof of Theorem 1.** Let $\mathbf{u_1}, \ldots, \mathbf{u_n}$ have density $f(u_1, \ldots, u_n)$ as in (2.3). Review $a(y)$ and $b(y)$ from (1.4). As in Lemma 2.14, set $x_n = \frac{n(x_n + v)}{2n + v}, a_n = a(x_n)$, and $b_n = b(x_n)$. Then,

$$\Lambda_n := \frac{1}{b_n} \left[ \frac{2\max_{1 \leq j \leq n} |\mathbf{u}_j| - 2\sqrt{n(n + v)}}{\sqrt{2n + v}} - a_n \right] \Rightarrow \Lambda$$

(2.48)

as $n \to \infty$, where $\Lambda$ is as in (1.5). Solve for $\max_{1 \leq j \leq n} |\mathbf{u}_j|$ to get

$$\max_{1 \leq j \leq n} |\mathbf{u}_j| = \sqrt{\frac{n + v}{n}} + \sqrt{\frac{2n + v}{2n}} (a_n + b_n\Lambda_n)$$

$$= \sqrt{\frac{n + v}{n}} \left[ 1 + \frac{1}{2} \sqrt{\frac{2n + v}{n(n + v)}} (a_n + b_n\Lambda_n) \right].$$

By the formula $(1 + x)^{1/2} = 1 + \frac{1}{2} x + O(x^2)$ as $x \to 0$, we have
The proof of the above equation is trivial since
where we have used the following facts in the second equality:

\[
\begin{align*}
\text{as } n \to \infty. \quad \square
\end{align*}
\]

Proof of Corollary 1. Observe that

\[
\begin{align*}
\left(\frac{n + v}{n}\right)^{1/4} &= 1 + O\left(\frac{n}{n^{1/4}(n + v)^{1/4}}\right) - 2\sqrt{2n + O\left(\frac{n}{n^{1/2}}\right)}, \\
\frac{1}{b(\frac{n(n + v)}{2n + v})} &= O\left(\frac{1}{n}\right),
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{b(\frac{n(n + v)}{2n + v})} &= \log y - \log\left(\sqrt{2\pi} \log y\right) - \log \frac{n}{2} - \log\left(\sqrt{2\pi} \log \frac{n}{2}\right) + o(1)
\end{align*}
\]

since \(\log y = \log \frac{n}{2} + O\left(\frac{1}{n}\right)\) at \(y = \frac{n(n + v)}{2n + v}\). Take these quantities into (1.6) to see that

\[
\begin{align*}
\left[\left(8n \log \frac{n}{2}\right)^{1/2} + O\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right)\right] \cdot \left(\max_{\mathcal{L} \le n} |z| - 1 + O\left(\frac{1}{n}\right)\right) \\
- \log \frac{n}{2} + \log\left(\sqrt{2\pi} \log \frac{n}{2}\right)
\end{align*}
\]

converges weakly to \(\Lambda\). This implies that

\[
\begin{align*}
\left[\left(8n \log \frac{n}{2}\right)^{1/2} + O\left(\frac{(\log n)^{1/2}}{n^{1/2}}\right)\right] \cdot \left(\max_{\mathcal{L} \le n} |z| - 1\right) \\
- \log \frac{n}{2} + \log\left(\sqrt{2\pi} \log \frac{n}{2}\right)
\end{align*}
\]

converges weakly to \(\Lambda\). By the Slutsky lemma, we know \(\max_{\mathcal{L} \le n} |z| - 1 \to 0\) in probability. Therefore,

\[
\begin{align*}
\left(8n \log \frac{n}{2}\right)^{1/2} \cdot \max_{\mathcal{L} \le n} |z| - \left(8n \log \frac{n}{2}\right)^{1/2} - \log \frac{n}{2} + \log\left(\sqrt{2\pi} \log \frac{n}{2}\right)
\end{align*}
\]

converges weakly to \(\Lambda\). \(\square\)
Recall Lemma 2.2. The joint density of \( u_j = \mathbf{z}_j^2 \), \( 1 \leq j \leq n \), is given by

\[
    f(u_1, \ldots, u_n) = C \cdot \prod_{1 \leq j < k \leq n} |u_j - u_k|^2 \left[ \prod_{j=1}^{n} \varphi_n(|u_j|) \right] du_1 \cdots du_n,
\]

for all \( u_j \in \mathbb{C} \), \( 1 \leq j \leq n \), where \( \varphi_n(y) = y^n K_0(2ny) \) for \( y > 0 \) and \( du_j = dx dy \) for \( u_j = x + iy \). Then, \( \lambda_n(du) := \varphi_n(|u|)du \) is a finite measure, that is,

\[
    \lambda_n(C) = \int_C \varphi_n(|u|)du = \int_0^\infty \int_0^{2\pi} r \varphi_n(r)d\theta dr = 2\pi \int_0^\infty r \varphi_n(r)dr < \infty
\]

from (b) and (c) in Lemma 2.1.

**Lemma 2.15.** (Proposition 2 from Jiang and Qi\(^{27}\)) Let \( u_1, \ldots, u_n \) have the density function \( f(u_1, \ldots, u_n) \) as in (2.3). For any measurable function \( h : \mathbb{C} \to \mathbb{R} \) with \( \sup_{u \in \mathbb{C}} |h(u)| \leq 1 \), we have

\[
    E\left[ \sum_{j=1}^{n} \left( h(u_j) - Eh(u_j) \right)^4 \right] \leq Kn^2
\]

for \( n \geq 1 \), where \( K \) is a constant not depending on \( n \), \( \varphi(u) \), or \( h(u) \).

**Lemma 2.16.** Let \( v = v_n \) be a sequence of non-negative numbers. Let \( u_1, \ldots, u_n \) have the density function \( f(u_1, \ldots, u_n) \) as in (2.3). Let \( \tau_n \) be the empirical probability measure of \( \lfloor n/(n + v_n) \rfloor^{1/2} u_j \), \( 1 \leq j \leq n \). Assume \( \lim_{n \to \infty} v_n/n = \alpha \in [0, \infty] \). Then, with probability one, \( \tau_n \to \tau \) as \( n \to \infty \), where \( \tau \) is a probability measure on \( \mathbb{C} \) with the density function

\[
    \varphi_a(u) = \frac{1}{\pi} \frac{1}{\sqrt{a + b|u|^2}}, \quad |u| \leq 1,
\]

where \( a = a^2/(1 + a)^2 \) and \( b = 4/(1 + a) \).

**Proof.** Let \( \tau'_n \) be the empirical probability measure of \( \lfloor n/(n + v_n) \rfloor^{1/2} u_j \), \( 1 \leq j \leq n \). We first show that

\[
    \tau'_n \to \tau', \quad (2.49)
\]

where \( \tau' \) has the density function \( l(r) := \frac{2r}{\sqrt{a + br^2}} \), \( 0 \leq r < 1 \). If this is true, by Theorem 1 from Jiang and Qi\(^{27}\) \( \tau_n \) converges weakly to the distribution of \( \text{Re}^{\Theta} \), where the random vector \( (R, \Theta)' \) has the product law of \( \tau' \) and the uniform distribution on \( [0, 2\pi] \). Therefore, for any bounded and continuous function \( g(z) \) defined on \( \mathbb{C} \), with probability one,

\[
    \lim_{n \to \infty} \int_{\mathbb{C}} g(u) \tau_n(du) = Eg(\text{Re}^{\Theta})
\]

\[
    = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 g(re^{i\theta}) \frac{2r}{\sqrt{a + br^2}} dr d\theta
    = \int_0^1 g(re^{i\theta}) \frac{2r}{\sqrt{a + br^2}} dr d\theta.
\]

On the other hand, by the polar transformation \( u = re^{i\theta} \),

\[
    \int_{\mathbb{C}} g(u) \tau(du) = \frac{1}{\pi} \int_{|u| \leq 1} g(u) \frac{1}{\sqrt{a + b|u|^2}} du
    = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} g(re^{i\theta}) \frac{r}{\sqrt{a + br^2}} dr d\theta
    = \int_0^1 g(re^{i\theta}) \frac{2r}{\sqrt{a + br^2}} dr d\theta
\]

The two assertions assure that \( \tau_n \to \tau \) as \( n \to \infty \). Now, we prove (2.49). To simplify notation, set

\[
    \zeta_n := \left( \frac{n}{n + v_n} \right)^{1/2}. \quad (2.50)
\]
It is enough to show

\[
\frac{1}{n} \sum_{i=1}^{n} I(c_i | u_i | \leq y) \to v'(0, y]
\]

for each \( y \geq 0 \). Regarding \( I(c_i | u_i | \leq y) \) as a function of \( u_i \), which takes values in \([0, 1]\). Recall the Markov inequality \( P(|V| \geq t) \leq t^{-d}E(V^d) \) for any random variable \( V \) and constant \( t > 0 \). By Lemma 2.15 and the Borel–Cantelli lemma, with probability one,

\[
\frac{1}{n} \sum_{i=1}^{n} [I(c_i | u_i | \leq y) - P(c_i | u_i | \leq y)] \to 0
\]

as \( n \to \infty \). Therefore, to complete the proof of (2.49), by Lemma 2.5, it remains to check that

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} P(c_j Y_j \leq y) = \begin{cases} 
1, & \text{if } y \geq 1 \\
\int_{0}^{y} \frac{2r}{\sqrt{a + br^2}} dr, & \text{if } 0 < y < 1,
\end{cases}
\]

(2.51)

where \( Y_j, 1 \leq j \leq n \), are random variables appeared in Lemma 2.5. Now, we use Lemmas 2.8 and 2.12 to estimate the above probabilities.

By using the subsequence argument similar to the proof of (2.42) in the Proof of Lemma 2.14, it suffices to show that (2.51) holds under condition (I): \( \lim_{n \to \infty} v_n/n = a \in [0, \infty) \) with \( v_n \to \infty \) or condition (II): \( \lim_{n \to \infty} v_n = 0 \). We will only prove (I) by using Lemma 2.8. The item (II) can be proved by using the same argument with "Lemma 2.8" replaced by "Lemma 2.12" and, hence, is omitted.

Now, we assume condition (I): \( v_n \to \infty \) and \( a_n := \frac{v_n}{n} \to a \in [0, \infty] \) as \( n \to \infty \). We prove this via a few steps.

**Step 1.** \( y \geq 1 \). Obviously,

\[
\frac{1}{n} \sum_{j=1}^{n} P(c_j Y_j > y) \leq \delta + \frac{1}{n} \sum_{n \delta \leq j \leq n} P(c_j Y_j > y)
\]

for any \( \delta \in (0, 1) \). Set

\[
\omega_{n \delta} = \frac{2nc_a^{-1}y - 2\sqrt{j+v}}{\sqrt{j+v}} = 2\sqrt{n(n+v)}y - \sqrt{j+v}, \quad j = 1, 2, \ldots, n.
\]

From Lemma 2.8,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} P(c_j Y_j > y) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left[1 - \Phi(\omega_{n \delta})\right].
\]

Use the facts \( \sqrt{j+v} \leq \sqrt{2(n+v)} \) and \( \sqrt{j+n+v} \leq \sqrt{n(n+v)} \) to see that

\[
\omega_{n \delta} \geq 2\sqrt{n(n+v)} - \sqrt{j+n+v} \geq 2\sqrt{n\delta'}
\]

for each \( n \delta \leq j \leq n(1 - \delta') \) if \( y \geq 1 \). It follows that

\[
\frac{1}{n} \sum_{j=1}^{n} \left[1 - \Phi(\omega_{n \delta})\right] \leq 1 - \Phi\left(\frac{1}{2} \sqrt{n\delta'}\right) + \frac{1}{n} \sum_{j=n(1-\delta')}^{n} 1 \leq 1 - \Phi\left(\frac{1}{2} \sqrt{n\delta'}\right) + \delta' + \frac{1}{n}
\]

for any \( \delta' \in (0, 1/2) \). By sending \( n \to \infty \) and then \( \delta' \downarrow 0 \), we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=n \delta}^{n} P(c_j Y_j > y) = 0
\]
for \( y \geq 1 \). This shows the first identity in (2.51).

**Step 2.** \( y \in (0, 1) \). It is easy to check that

\[
\int_0^y \frac{2r}{\sqrt{a + br^2}} \, dr = \frac{2}{b} \left[ \frac{\sqrt{a + by^2} - \sqrt{a}}{y} \right]_0^y = \frac{1}{2} \left( \sqrt{a^2 + 4(1 + a)y^2} - a \right)
\]

for \( \alpha \in [0, \infty) \) and

\[
\int_0^y \frac{2r}{\sqrt{a + br^2}} \, dr = \int_0^y 2 \, dr = y^2
\]

for \( \alpha = \infty \). For \( y \in [0, 1] \), define

\[
F_n(y) = \begin{cases} 
\frac{1}{2} \left( \sqrt{\alpha^2 + 4(1 + \alpha)y^2} - \alpha \right), & \text{if } 0 \leq \alpha < \infty \\
\frac{1}{y^2}, & \text{if } \alpha = \infty
\end{cases}
\]

and

\[
G_n(y) := \frac{1}{n} \sum_{j=1}^{n} P(\text{if } \alpha = \infty
\]

for \( y \in \mathbb{R} \). To complete the proof of (2.51), it is enough to show that

\[
\lim_{n \to \infty} G_n(y) = F_n(y), \quad y \in (0, 1). \tag{2.53}
\]

Review \( \alpha_i = \frac{\alpha}{n} \to \alpha \in [0, \infty] \) as \( n \to \infty \). Evidently,

\[
F_n(y)F_n(y) + \alpha_n = y^2(1 + \alpha_n) \tag{2.54}
\]

and \( \lim_{n \to \infty} F_n(y) = F_n(y) \) for any \( y \in (0, 1) \). Since \( F_n(0) < F_n(1) \) for \( y \in (0, 1) \), we have \( F_n(y) \in (0, 1) \) for \( y \in (0, 1) \).

Now, fix an \( y \in (0, 1) \) and let \( \epsilon \in (0, 1) \) be any small number such that \( y(1 + \epsilon) \in (0, 1) \). Set

\[
k_n^+ = \left[ nF_n(y(1 + \epsilon)) \right] + 1 \quad \text{and} \quad k_n^- = \left[ nF_n(y(1 - \epsilon)) \right],
\]

where \([x]\) denotes the integer part of \( x \) as before. Review \( c_n \) from (2.50). We will prove that

\[
\lim_{n \to \infty} P(Y_{k_n} \leq yc_n^{-1}) = 1 \quad \text{and} \quad \lim_{n \to \infty} P(Y_{k_n} \leq yc_n^{-1}) = 0. \tag{2.55}
\]

Assuming this is true, we have from (2.52) and Lemma 2.6 that

\[
\limsup_{n \to \infty} G_n(y) = \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P(Y_k \leq yc_n^{-1}) \leq \limsup_{n \to \infty} \frac{1}{n} \left[ \sum_{k=1}^{k_n^+} P(Y_k \leq yc_n^{-1}) + \sum_{k=k_n^++1}^{n} P(Y_k \leq yc_n^{-1}) \right] \leq \limsup_{n \to \infty} \frac{1}{n} [ k_n^+ + (n - k_n^+) P(Y_{k_n^-} \leq yc_n^{-1}) ] \leq \limsup_{n \to \infty} \frac{1}{n} [ k_n^+ + P(Y_{k_n^-} \leq yc_n^{-1}) ] = F_n(y(1 + \epsilon)).
\]

This implies

\[
\limsup_{n \to \infty} G_n(y) \leq F_n(y)
\]

by letting \( \epsilon \downarrow 0 \). Similarly, we have
\begin{align*}
\liminf_{n \to \infty} G_n(y) & \geq \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{k_n} P(Y_k \leq y \varsigma_n^{-1}) \\
& \geq \liminf_{n \to \infty} \left[ \frac{k_n}{n} P(Y_{k_n} \leq y \varsigma_n^{-1}) \right] \\
& = \liminf_{n \to \infty} \frac{k_n}{n} \\
& = \mathcal{F}_n(y(1-\epsilon)).
\end{align*}

By letting \( \epsilon \downarrow 0 \), we conclude

\[ \liminf_{n \to \infty} G_n(y) \geq F_n(y). \]

Therefore, we get (2.53).

Now, we proceed to prove (2.55). We will prove the first limit only. The second one can be proved in the same manner. In fact, from (2.54),

\[ k_n^2 (k_n + v_n) \leq n \mathcal{F}_n(y(1-\epsilon) [n \mathcal{F}_n(y(1-\epsilon) + v_n] \\
= n^2 \mathcal{F}_n(y(1-\epsilon)) [\mathcal{F}_n(y(1-\epsilon) + \alpha_n] \\
= n^2 \nu^2 (1-\epsilon)^2 (1+\alpha_n) \quad (2.56) \]

and

\[ 2k_n^2 + v_n \leq 2n(1+\alpha_n). \quad (2.57) \]

Set

\[ L_n = \frac{2nY_{k_n} - 2\sqrt{k_n (k_n + v_n)}}{\sqrt{2k_n + v_n}} \quad \text{and} \quad l_n = \frac{2y(n + v_n)^{1/2} - 2\sqrt{k_n (k_n + v_n)}}{\sqrt{2k_n + v_n}}. \]

By (2.56) and (2.57),

\[ l_n \geq \frac{2ny(1+\alpha_n)^{1/2} - 2ny(1-\epsilon) \sqrt{1+\alpha_n}}{\sqrt{2n(1+\alpha_n)}} = \nu \sqrt{2n}. \]

Hence,

\[ P(Y_{k_n} \leq y \varsigma_n^{-1}) = P(L_n \leq l_n) \geq P(l_n \leq \nu \sqrt{2n}) \\
= 1 - (1+o(1))(1 - \Phi(\nu \sqrt{2n})) + O\left(\frac{1}{n} \right) \\
= 1 + o(1) \]

by Lemma 2.8. This completes the proof of the first limit in (2.55). \( \square \)

**Proof of Theorem 2.** Let \( v = v_n \) be any sequence of non-negative numbers. Our assumption is that \( \lim_{n \to \infty} \frac{v_n}{n} = \alpha \in [0, \infty] \). Recall (2.50) that

\[ \varsigma_n = \left( \frac{n}{n+v_n} \right)^{1/2}. \]

Let \( \rho \) be the probability distribution with the density function \( \Phi_\alpha \) appeared in the statement of Theorem 2. To prove that \( \rho_n \xrightarrow{d} \rho \), it suffices to verify that, with probability one,

\[ \frac{1}{n} \sum_{k=1}^{n} h(\varsigma_n^{1/2} z_k) \to \int_{\mathbb{C}} h(z) \rho(dz) \quad (2.58) \]

for any continuous function \( h \) defined on \( \mathbb{C}_1 = \{ z \in \mathbb{C}; \ \text{Re}(z) > 0 \} \) with \( 0 \leq h(z) \leq 1 \) for all \( z \in \mathbb{C}_1 \). Here and later, “\( dz \)” stands for “\( dx \)dy” when \( z = x + iy \).
Let \( u_1, \ldots, u_n \) have the density function \( f(u_1, \ldots, u_n) \) as in \((2.3)\). Assume \( \rho_1 \) is a probability measure with the density function

\[
\Psi_a(z) = \frac{1}{n} \cdot \frac{1}{\sqrt{a + b|z|^2}}, \quad |z| \leq 1,
\]

where \( a = a^2/(1 + a)^2 \) and \( b = 4/(1 + a) \). By Lemma 2.16, with probability one,

\[
\frac{1}{n} \sum_{j=1}^{n} \delta_{u_j, n} \to \rho_1 \tag{2.59}
\]

as \( n \to \infty \). Recall \( C_2 = \mathbb{C}\{z \in \mathbb{R}; |z| \leq 0\} \). Obviously the Lebesgue measure of \( \{z \in \mathbb{R}; |z| \leq 0\} \) is zero. This implies that \((2.59)\) also holds if we restrict \( u_1, \ldots, u_n \) and \( \rho_1 \) on \( C_2 \) (one way to show this is the approximation of \( C_2 \) via \( C_{2, \epsilon} := \mathbb{C}\{s + it; s \leq \epsilon, |t| \leq \epsilon\} \) as \( \epsilon \downarrow 0 \)). Therefore, with probability one,

\[
\frac{1}{n} \sum_{j=1}^{n} h_1(\cdot, u_j) \to \int_{C_2} h_1(z) \rho_1(dz) \tag{2.60}
\]

for any continuous function \( h_1(z) \) defined on \( C_2 \) with \( 0 \leq h_1(z) \leq 1 \) for all \( z \in C_2 \). Review Lemma 2.2. Let \( u = z^2 \) be the one-to-one and onto transform from \( C_1 \) to \( C_2 \). Denote by \( u^{-1} \) the inverse transform from \( C_2 \) to \( C_1 \). As seen from \((2.4)\), \( u^{-1} \) is also continuous. Take \( h_1 = h \circ u^{-1/2} : C_2 \to [0, 1] \). Then, \( h_1 : C_2 \to [0, 1] \) is a continuous function. By \((2.60)\),

\[
\frac{1}{n} \sum_{j=1}^{n} h_1(\cdot, z_j) \to \int_{C_2} h(z^{1/2}) \rho_1(dz).
\]

For \( z = re^{i\theta} \in C_2 \), it is easy to check that \( z^{1/2} = \sqrt{r}e^{i\theta/2} \) with \( 0 < r \leq 1 \) and \( \theta \in (-\pi, \pi) \). Then,

\[
\int_{C_2} h(z^{1/2}) \rho_1(dz) = \frac{1}{\pi} \int_{-\pi}^{\pi} \int_{0}^{1} h(\sqrt{r}e^{i\theta/2}) \frac{1}{\sqrt{a + br^2}} \cdot r \, dr \, d\theta = \frac{4}{\pi} \int_{-\pi/2}^{\pi/2} \int_{0}^{1} h(se^{i\eta}) \frac{s^2}{\sqrt{a + bs^2}} \cdot s \, ds \, d\eta = \frac{4}{\pi} \int_{C_1} h(z) \frac{|z|^2}{\sqrt{a + b|z|^2}} \, dz,
\]

where we make a transform that \( s = \sqrt{r} \) and \( \eta = \theta/2 \) in the second equality and last step follows from making a similar polar transformation. The last two assertions yield \((2.58)\).

\[\square\]

**C. Proofs of Lemmas 2.9 and 2.10**

**Proof of Lemma 2.9.** First,

\[
\sqrt{1 + \beta_{v,j}^2} = \frac{2j + v - 1.5}{v} + \frac{v}{2(2j + v - 1.5)^2}, \tag{2.61}
\]

\[
\frac{1}{C_6} \left( \frac{v + n}{v} \right)^2 \leq 1 + \beta_{v,j}^2 \leq C_6 \cdot \left( \frac{v + n}{v} \right)^2, \tag{2.62}
\]

uniformly over \( \delta n \leq j \leq n \) as \( n \to \infty \), where \( C_6 > 0 \) is a constant depending on \( \delta \) but not depending on \( n \). Second, write \( \frac{2v + 1.5}{v} = \frac{2v + 1.5}{v} + 1 \). Then,

\[
\left( \frac{2j + v - 1.5}{v} \right)^2 = \left( \frac{2j - 1.5}{v} \right)^2 + 1. \tag{2.63}
\]

Consequently, from the definition of \( \beta_{v,j} \),
\[
\bar{\mu}_{v,j}^2 = \frac{(2j - 1.5)(2j + 2v - 1.5)}{v^2} + \frac{1}{2j + v - 1.5} + O\left(\frac{v^2}{(n + v)^2}\right)
\]

(2.64)

\[
\bar{\mu}_{v,j}^2 = \frac{(2j - 1.5)(2v + 2j - 1.5)}{v^3} + \frac{1}{(2j - 1.5)(2j + v - 1.5)(2j + 2v - 1.5)} + O\left(\frac{v^2}{(n + v)^3}\right).
\]

(2.65)

By (2.11) and (2.61), this again implies

\[
\bar{\mu}_{v,j} = \frac{v}{2} - \frac{1}{2}(1 + \bar{\mu}_{v,j}^2) - \frac{1}{2v(1 + \bar{\mu}_{v,j}^2)}
\]

(2.66)

\[
\frac{\delta^2}{v^2} n(n + v) \leq \bar{\mu}_{v,j} \leq \frac{4}{v^2} n(n + v),
\]

where \( \kappa := 2j + v - 1.5 \), and in the last step, we use the fact \( \frac{\delta^2}{v^2} \to 0 \) and the fact that \( (1 + a)^{-2} = 1 + O(a) \) as \( a \to 0 \). From the definition of \( \bar{\mu}_{v,j} \)

(2.67)

\[
\frac{\delta'(\bar{\mu}_{v,j})}{\bar{\mu}_{v,j}} = O\left(\frac{v^5}{n(n + v)^6}\right)
\]

and (2.63), we see that

\[
\frac{\delta^2}{v^2} n(n + v) \leq \bar{\mu}_{v,j} \leq \frac{4}{v^2} n(n + v),
\]

uniformly for \( \delta n \leq j \leq n \) as \( n \) is sufficiently large. It follows from (2.65) that

Now, define \( x_{v,j}(t) = \bar{\mu}_{v,j}(1 + \bar{\mu}_{v,j}^2) \), where

(2.68)

\[
\bar{\mu}_{v,j} := \sqrt{\frac{2j + v - 1.5}{(2j - 1.5)(2v + 2j - 1.5)}}
\]

is of order \( n^{-1/2} \) uniformly over \( \delta n \leq j \leq n \) as \( n \to \infty \). Then,

(2.69)

\[
x_{v,j}(t) = \bar{\mu}_{v,j}(1 + \bar{\mu}_{v,j}^2)^2,
\]

(2.70)

\[
1 + x_{v,j}^2(t) = 1 + \bar{\mu}_{v,j} + \bar{\mu}_{v,j}(2\bar{\mu}_{v,j} + \bar{\mu}_{v,j}^2)
\]

\[
= (1 + \bar{\mu}_{v,j}^2)^2\left[1 + \frac{\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2}(2\bar{\mu}_{v,j} + \bar{\mu}_{v,j}^2)^2\right].
\]

By the formula \( \sqrt{1 + x} = 1 + \frac{x}{2} + O(x^2) \) as \( x \to 0 \), we have

\[
\sqrt{1 + x_{v,j}^2(t)} = \sqrt{1 + \bar{\mu}_{v,j}^2 + \frac{\bar{\mu}_{v,j}^2\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2}(1 + O(n^{-3/2}))}.
\]

Which holds uniformly over \( |t| \leq n^{-1/8} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). Similarly, from the fact that \( \frac{1}{1 + x} = 1 - x(1 + o(1)) \) as \( x \to 0 \), we obtain
\[
\frac{1}{1 + x_{v,j}^2}(t) = \frac{1}{1 + \bar{\mu}_{v,j}^2} \left[ 1 + \frac{\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2} (2\bar{\mu}_{v,j}t + \bar{\mu}_{v,j}^2 t^2) \right]^{-1}
\]

\[
= \frac{1}{1 + \bar{\mu}_{v,j}^2} \left[ 1 - \frac{\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2} (2\bar{\mu}_{v,j}t + \bar{\mu}_{v,j}^2 t^2)(1 + o(1)) \right]
\]

\[
= \frac{1}{1 + \bar{\mu}_{v,j}^2} - \frac{\bar{\mu}_{v,j}^2}{(1 + \bar{\mu}_{v,j}^2)^2} (2\bar{\mu}_{v,j}t + \bar{\mu}_{v,j}^2 t^2)(1 + o(1)).
\]

Hence,

\[
- \frac{1}{2v} \frac{1}{1 + x_{v,j}^2}(t) = - \frac{1}{2v} \frac{1}{1 + \bar{\mu}_{v,j}^2} + \frac{\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2} \left( 1 + \frac{1}{2} \bar{\mu}_{v,j} t (1 + o(1)) \right) \frac{(1 + O(n^{-3/8}))}{\sqrt{(1 + \bar{\mu}_{v,j}^2)^{3/2}}}
\]

\[
= - \frac{1}{2v} \frac{1}{1 + \bar{\mu}_{v,j}^2} + \frac{\bar{\mu}_{v,j}^2}{1 + \bar{\mu}_{v,j}^2} \cdot O\left( \frac{v^2}{(v + n)^2} \right),
\]

uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq t \leq n\) as \(n \to \infty\), where the fact (2.62) is applied in the last step. Then, we conclude from (2.11) that

\[
x_{v,j}(t)r'(x_{v,j}(t)) = \sqrt{1 + x_{v,j}^2} - \frac{1}{2v(1 + x_{v,j}^2)} \frac{2j + v - 1.5}{v} \frac{\bar{\mu}_{v,j}^2}{(1 + \bar{\mu}_{v,j}^2)} r'(\bar{\mu}_{v,j})
\]

\[
= \bar{\mu}_{v,j} r'(\bar{\mu}_{v,j}) + \frac{1}{v(1 + \bar{\mu}_{v,j}^2)} \left( 1 + O(n^{-3/8}) \right),
\]

uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq t \leq n\) as \(n \to \infty\). The assertion from (2.61) implies \(\sqrt{1 + \bar{\mu}_{v,j}^2} = \frac{2j + v - 1.5}{v} + O\left( \frac{v^2}{(v + n)^2} \right)\). Furthermore, \(x_{v,j}(t) = \bar{\mu}_{v,j}(1 + O(n^{-3/8}))\) from (2.69). With these at hand, we conclude from (2.67) and (2.68) that

\[
\frac{r'(x_{v,j}(t))}{x_{v,j}(t)} = \frac{\bar{\mu}_{v,j}^2}{(1 + O(n^{-3/8})) + r'(\bar{\mu}_{v,j})} \frac{r'(\bar{\mu}_{v,j})}{\bar{\mu}_{v,j}} (1 + O(n^{-3/8}))}
\]

\[
= \sqrt{(2j - 1.5)(2j + v - 1.5)(2j + v - 2.5)} (1 + O(n^{-3/8})) + O\left( \frac{v^5}{n(n + v)^2} \right),
\]

uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq t \leq n\) as \(n \to \infty\).

Now, set \(t = t_0 = \frac{\alpha v^4}{n(n + v)^2}\) in (2.71). If \(\alpha\) is large enough, the first term in (2.71) dominates the last one. Then, \(r'(x_{v,j}(t_0)) > 0\) and \(r'(x_{v,j}(-t_0)) < 0\) uniformly over all \(\delta n \leq t \leq n\) and large \(n\). We obtain (2.20) by the continuity of function \(r(x)\) and the definitions of \(\mu_{v,j}\) from (2.14) and \(x_{v,j}(t)\).

Finally, by (2.64),

\[
\bar{\mu}_{v,j}^2 = \frac{4j(v + f) + O(n + v)}{v^2} \left[ 1 + O\left( \frac{v^2}{n(n + v)^2} \right) \right],
\]

which combining with (2.20) implies

\[
\mu_{v,j}^2 = \frac{4j(v + f) + O(n + v)}{v^2} \left[ 1 + O\left( \frac{v^2}{n(n + v)^2} \right) \right].
\]

Then, use (2.15) and the fact \(|a - b| \leq |a^2 - b^2| \cdot a^{-1}\) for any \(a > 0\), \(b > 0\) to see that
uniformly over all \( \delta n \leq j \leq n \) as \( n \to \infty \). The proof is completed.

**Proof of Lemma 2.10.** It follows from (2.15) that

\[
1 + y_{\nu,ij}^2(t) = 1 + \mu_{\nu,ij}^2 \cdot \left[ 1 + \frac{\mu_{\nu,ij}^2}{1 + \mu_{\nu,ij}^2} \right] + \mu_{\nu,ij}^2 (1 + O(n^{-3/8}))
\]

From (2.20) and (2.66),

\[
\frac{\delta^2}{2v^2} n(n + v) \leq \mu_{\nu,ij}^2 \leq \frac{\delta}{v^2} n(n + v),
\]

uniformly for \( \delta n \leq j \leq n \) as \( n \) is sufficiently large. Similar to the argument between (2.70) and (2.71), we see

\[
\sqrt{1 + y_{\nu,ij}^2(t)} = \sqrt{1 + \mu_{\nu,ij}^2 + \frac{\mu_{\nu,ij}^2}{1 + \mu_{\nu,ij}^2} \beta_{\nu,ij} t (1 + O(n^{-3/8}))};
\]

\[
- \frac{1}{2v} \frac{1}{1 + y_{\nu,ij}^2(t)} = \frac{1}{2v} - \frac{1}{1 + \mu_{\nu,ij}^2 + \frac{\mu_{\nu,ij}^2}{1 + \mu_{\nu,ij}^2} \beta_{\nu,ij} t O\left(\frac{1}{(v + n)^2}\right)}
\]

hold uniformly over \( |t| \leq n^{1/8} \) and \( \delta n \leq j \leq n \) as \( n \to \infty \). Therefore, we get from (2.11) and the fact \( \tau'(\mu_{\nu,ij}) = 0 \) that

\[
y_{\nu,ij} \tau'(y_{\nu,ij}(t)) = \sqrt{1 + y_{\nu,ij}^2} - \frac{1}{2v(1 + y_{\nu,ij}^2)} \cdot \frac{2j + v - 1 - 0.5}{v}
\]

\[
= \frac{\mu_{\nu,ij}^2 \beta_{\nu,ij} t (1 + O(n^{-3/8}))) + \mu_{\nu,ij}^2 \tau'(\mu_{\nu,ij})}{\sqrt{1 + \mu_{\nu,ij}^2}}
\]

\[
= \frac{\mu_{\nu,ij}^2 \beta_{\nu,ij} t (1 + O(n^{-3/8})))}{\sqrt{1 + \mu_{\nu,ij}^2}}
\]

and

\[
\frac{\tau'(y_{\nu,ij}(t))}{y_{\nu,ij}(t)} = \frac{\beta_{\nu,ij} t}{\sqrt{1 + \mu_{\nu,ij}^2}} (1 + O(n^{-3/8})).
\]

Note that \( 1 + \mu_{\nu,ij}^2 \) and \( \left(\frac{n + v}{v}\right)^2 \) are of the same order from (2.61). From (2.20) and the fact \( \sqrt{1 + x} = 1 + O(x) \) as \( x \to 0 \), we have

\[
\sqrt{1 + \mu_{\nu,ij}^2} = \frac{2j + v - 1.5}{v} + \frac{v}{2(2j + v - 1.5)^2} + O\left(\frac{v^3}{n(n + v)^t}\right)
\]

\[
= \frac{2j + v - 1.5}{v} \left[ 1 + O\left(\frac{v^2}{(n + v)^t}\right)\right].
\]

By the identity between (2.12) and (2.13),

\[
\tau''(x) = \frac{1}{1 + x^2} \frac{1}{v(1 + x^2)^2} - \frac{\tau'(x)}{x}.
\]
It follows that

\[
\tau''(y_{v,j}(0)) = \tau''(\mu_{v,j}) = \frac{1}{\sqrt{1 + \mu_{v,j}^2}} + \frac{1}{v(1 + \mu_{v,j}^2)}
\]

\[
= \frac{v}{2} + \frac{1}{2} \left[ 1 + O\left(\frac{v^2}{n + v}\right) \right]
\]

\[
= \frac{v}{2} + \frac{1}{2} \left[ 1 + O\left(\frac{1}{n + v}\right) \right]
\]  

(2.76)

and

\[
\mu_{v,j}^2 \beta_{v,j} \tau''(\mu_{v,j}) = \sigma_{v,j} \tau''(\mu_{v,j}) = \frac{1}{v} \left[ 1 + O\left(\frac{1}{n + v}\right) \right]
\]  

(2.77)

from (2.15). We obtain (i). Furthermore, by (2.72)–(2.74),

\[
\tau''(y_{v,j}(t)) = \frac{1}{\sqrt{1 + y_{v,j}(t)}^2} + \frac{1}{v(1 + y_{v,j}(t))^2} - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)},
\]

(2.78)

\[
= \frac{1}{\sqrt{1 + \mu_{v,j}^2}} \left[ 1 + \frac{\mu_{v,j}^2 \beta_{v,j}}{1 + \mu_{v,j}^2} O(1) \right]
\]

\[
+ \frac{1}{v(1 + \mu_{v,j}^2)} \left[ 1 + \frac{\mu_{v,j}^2 \beta_{v,j}}{1 + \mu_{v,j}^2} O(1) \right] - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)}
\]

\[
= \tau''(y_{v,j}(0)) + \frac{\mu_{v,j}^2 \beta_{v,j}}{(1 + \mu_{v,j}^2)^2}O(1) - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)}.
\]  

(2.79)

uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq j \leq n\) as \(n \to \infty\), where (2.75) is used in the first equality; (2.72) is used in the second equality; (2.76) is used in the third equality. From (2.79), we claim that

\[
\tau''(y_{v,j}(t)) = \tau''(y_{v,j}(0)) \left[ 1 + O(\beta_{v,j}|t|) \right],
\]  

(2.80)

uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq j \leq n\) as \(n \to \infty\). In fact, we know that \(1 + \mu_{v,j}^2\) has the same order as \(\frac{(\mu_{v,j}^2)^2}{\mu_{v,j}^2}\) from (2.73). Therefore,

\[
\tau''(y_{v,j}(0)) = \frac{1}{\sqrt{1 + \mu_{v,j}^2}} \left[ 1 + O\left(\frac{v}{n + v}\right) \right]
\]

by (2.76). This is equivalent to

\[
\frac{1}{\sqrt{1 + \mu_{v,j}^2}} = \tau''(y_{v,j}(0)) \left[ 1 + O\left(\frac{v}{n + v}\right) \right].
\]

This yields that

\[
\frac{\mu_{v,j}^2 \beta_{v,j}}{(1 + \mu_{v,j}^2)^2} = \frac{1}{\sqrt{1 + \mu_{v,j}^2}} \frac{\mu_{v,j}^2 \beta_{v,j}}{1 + \mu_{v,j}^2} \beta_{v,j} = \tau''(y_{v,j}(0)) \cdot O(\beta_{v,j}|t|),
\]

\[
\tau'(y_{v,j}(t)) \left[ 1 + O(\beta_{v,j}|t|) \right] = \tau''(y_{v,j}(0)) \cdot O(\beta_{v,j}|t|)
\]

by (2.74). These two facts conclude (2.80).

Review \(y_{v,j}(t) = \mu_{v,j}(1 + \beta_{v,j}|t|)\) and \(y_{v,j}(0) = \mu_{v,j}\). We have

\[
\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \mu_{v,j} \beta_{v,j} \int_0^t \tau'(y_{v,j}(s))ds = \mu_{v,j} \beta_{v,j} \int_0^t \int_0^s \tau''(y_{v,j}(w))dwds.
\]
From (2.80), we have
\[ \tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \frac{\mu_v^2 \beta_{v,j} r''(\mu_{v,j})}{2} t^2 [1 + O(\beta_{v,j}|t|)], \]
uniformly over \(|t| \leq n^{1/8}\) and \(\delta n \leq j \leq n\) as \(n \to \infty\). We get (ii) by the notation \(\beta_{v,j} = \sigma_{v,j}^2 / \mu_{v,j}^2\) in (2.15). It is seen from (2.13) that
\[ \tau(y_{v,j}(t)) - \tau(\mu_{v,j}) > \beta_{v,j}^2 \frac{2j - 1.5}{v} \cdot \int_0^1 \int_0^t \frac{1}{(1 + \beta_{v,j}u)^2} du ds \]
\[ = \frac{2j - 1.5}{v} \left[ \beta_{v,j} t - \log(1 + \beta_{v,j} t) \right] \]
for any \(t > -1/\beta_{v,j}\). This implies (iii) since \(2j - 1.5 > j\) for all \(j \geq 1\).

Finally, from (2.15), \(y_{v,j}(t) = \mu_{v,j}(1 + \beta_{v,j} t)\) with \(\mu_{v,j} > 0\) and \(\beta_{v,j} > 0\). Thus, \(y_{v,j}(t)\) is an increasing function in \(t > -1/\beta_{v,j}\). Keep in mind that we will interchange \("y_{v,j}(0)"\) and \("\mu_{v,j}"\) next. By (2.13), \(\tau'(x)\) is increasing in \(x > 0\) and \(\tau'(\mu_{v,j}) = \tau'(y_{v,j}(0)) = 0\). This says that \(\tau'(y_{v,j}(t)) \leq \tau'(y_{v,j}(0)) = 0\) for \(-1/\beta_{v,j} < t < 0\). Observe that the first two terms on the right-hand side of (2.78) are decreasing in \(-1/\beta_{v,j} < t \leq 0\). The two facts show that \(\tau'(y_{v,j}(t)) \geq \tau'(y_{v,j}(0)) = \tau'(\mu_{v,j})\) for \(-1/\beta_{v,j} < t \leq 0\). Note that \(\frac{d}{dt} \tau(y_{v,j}(t)) = \mu_{v,j} \beta_{v,j} \tau'(y_{v,j}(t))\) and \(\frac{d^2}{dt^2} \tau(y_{v,j}(t)) = \mu_{v,j}^2 \beta_{v,j}^2 \tau''(y_{v,j}(t))\) by the chain rule. For any \(t\) with \(-1/\beta_{v,j} < t < 0\), by using Taylor's theorem, there exists \(w \in (t,0)\) such that
\[ \tau(y_{v,j}(t)) - \tau(y_{v,j}(0)) = \mu_v \beta_v \tau'(y_{v,j}(0)) \frac{1}{2} \mu_v^2 \beta_v^2 \tau''(y_{v,j}(w)) \]
\[ \geq \frac{t^2}{2} \mu_v^2 \beta_v^2 \tau''(\mu_{v,j}). \]
This leads to (iv).

\[ \square \]

**D. Proof of (1.9)**

From p. 163 from Do Carmo,\(^{36}\) for a surface \((x, y, h(x, y))\) on an open set \((x, y) \in U \subset \mathbb{R}^2\), the Gaussian curvature \(K\) is given by
\[ K = \frac{h_{xx} h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}, \quad (x, y) \in U. \]
In our case,
\[ h(x, y) = \frac{\sqrt{a(x^2 + y^2)}}{\sqrt{a + b(x^2 + y^2)}}, \quad x^2 + y^2 < 1, x > 0, \]
where \(a = \alpha^2 / (1 + \alpha)^2\), \(b = 4 / (1 + \alpha)\), and \(c = 4 / \pi\). To simplify notation, set \(A = a + b(x^2 + y^2)^2\). Then, we have from the formula \((f/g)' = (f'g - fg')/g^2\) that
\[ h_x = \frac{2cx \sqrt{A} - c(x^2 + y^2) 2bx(x^2 + y^2)}{2 \sqrt{A}} \]
\[ = \frac{2cx A - 2 b c x (x^2 + y^2)^2}{A^{3/2}} \]
\[ = \frac{2ac}{A^{3/2}}. \quad (2.81) \]
So
\[ h_{xx} = (2ac) A^{3/2} - x - \frac{\sqrt{A}}{2} 4bx(x^2 + y^2) \]
\[ = (2ac) \left( a + b(x^2 + y^2)^2 \right) - 6b x^2 (x^2 + y^2) \]
\[ = (2ac) \left( a + b(x^2 + y^2)(y^2 - 5x^2) \right) A^{3/2} \]
\[ = (2ac) \left( a + b(x^2 + y^2)(y^2 - 5x^2) \right) \]
\[ \quad \text{and} \]
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\[ h_{xy} = 2acx \left( -\frac{3}{2} \right) A^{3/2} \cdot 4by(x^2 + y^2) \]
\[ = -12abcxy(x^2 + y^2)A^{-3/2}. \]

By symmetry, we get the expressions for \( h_x \) and \( h_{yy} \) by replacing "\( x \)" with "\( y \)" in (2.81) and (2.82). In particular,
\[
 (1 + h_x^2 + h_y^2)^2 = \left[ 1 + \frac{4a^2c^2(x^2 + y^2)}{A^3} \right]
\]

and
\[
 h_{xx}h_{yy} - h_{xy}^2 = 4a^2c^2A^{-3}\left[ a + b(x^2 + y^2)(y^2 - 5x^2) \right] \\
= 4a^2c^2A^{-3}\left[ a + b(x^2 + y^2)(x^2 - 5y^2) \right] \\
= -144a^2b^2c^2x^2y^2(x^2 + y^2)^2A^{-3} \\
= 4a^2c^2A^{-3} \left[ a + b(x^2 + y^2)(y^2 - 5x^2) \right] \\
= 36b^2x^2y^2(x^2 + y^2)^2. 
\]

It is a bit tedious but easy to check that
\[
 [a + b(x^2 + y^2)(y^2 - 5x^2)] [a + b(x^2 + y^2)(x^2 - 5y^2) - 36b^2x^2y^2(x^2 + y^2)^2] \\
= a^2 - 4ab(x^2 + y^2)^2 - 5b^2(x^2 + y^2)^4 \\
= [a + b(x^2 + y^2)^2]^2 - 5b^2(x^2 + y^2)^4. 
\]

This says that
\[
 h_{xx}h_{yy} - h_{xy}^2 = 4a^2c^2A^{-3} [a + b(x^2 + y^2)^2]^2 - 5b(x^2 + y^2)^4. 
\]

Therefore, by the notation \( A = a + b(x^2 + y^2)^2 \),
\[
 K = \frac{A^6}{[A^3 + 4a^2c^2(x^2 + y^2)]^2} \cdot \frac{4a^2c^2}{A^3} \cdot A[a - 5b(x^2 + y^2)^2] \\
= (4c^2) \cdot \frac{a^2(a + bR^4)(a - 5bR^4)}{[a + bR^4]^2 + 4a^2c^2R^2]^2}, 
\]

where \( R := \sqrt{x^2 + y^2} \in (0, 1) \). So \( K > 0 \) for all \((x, y)\) with \( x^2 + y^2 < 1, |x| \leq 1 \) if \( a > 5b \) and \( K < 0 \) for all \((x, y)\) with \( x^2 + y^2 > \sqrt{a/5b} \) if \( a < 5b \). Now, \( a > 5b \) if and only if
\[
 \frac{a^2}{1 + a} > 20, 
\]

which again holds if and only if \( a > 10 + \sqrt{120} \). Similarly, \( a < 5b \) if and only if \( a < 10 + \sqrt{120} \).

To make \( h(x, y) \) be a convex function, it suffices to check that the Hessian of \( h \) is non-negative definite. That is, that matrix
\[
 H = \begin{pmatrix}
 h_{xx} & h_{xy} \\
 h_{xy} & h_{yy}
 \end{pmatrix}
\]

has to be non-negative definite. This holds if \( h_{xx} \geq 0, h_{yy} \geq 0, \) and \( \det(H) \geq 0 \). It is easy to see that \( \inf_{x^2 + y^2 < 1}\) \((x^2 + y^2)\)(\(y^2 - 5x^2\)) = \(-5\) by taking \( x \uparrow 1 \) and \( y \downarrow 0 \). So \( h_{xx} \geq 0 \) and \( h_{yy} \geq 0 \) if \( a \geq 5b \).

From (2.83), we know \( K \geq 0 \) for all \((x, y)\) with \( x^2 + y^2 < 1, |x| \leq 1 \) if and only if \( a > 10 + \sqrt{120} \).

\[ \square \]

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REFERENCES