Eigenvalues of Large Chiral Non-Hermitian Random Matrices

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(Dated: 20 December 2019)

We study a non-Hermitian chiral random matrix of which the eigenvalues are complex random variables. The empirical distributions and the radius of the eigenvalues are investigated. The limit of the empirical distributions is a new probability distribution defined on the complex plane. The graphs of the density functions are plotted; the surfaces formed by the density functions are understood through their convexity and their Gaussian curvatures. The limit of the radius is a Gumbel distribution. The main observation is that the joint density function of the eigenvalues of the chiral ensemble, after a transformation, becomes a rotation-invariant determinantal point process on the complex plane. Then the eigenvalues are studied by the tools developed by Jiang and Qi [J. Theor. Probab **30**, 326 (2017)] and Jiang and Qi [J. Theor. Probab. **32**, 353 (2019)]. Most efforts are devoted to deriving the central limit theorems for distributions defined by the Bessel functions via the method of steepest descent and the estimates of the zero of a non-trivial equation as the saddle point.

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1. INTRODUCTION AND MAIN RESULTS

Hermitian random matrices haven been studied by experts from many disciplines such as Mathematics, Physics, Statistics and Engineering. Many deep results are obtained. For example, for the limits of eigenvalues, we see the semi-circle law, the Marchenko-Pastur law, the Tracy-Widom law, the local semi-circle law, the connections to stochastic differential equations. See the book treatment and the references from, for instance, Anderson *et al.*[1], Bai and Silverstein[2], Forrester[3], Akemann *et al.*[4], Baik *et al.*[5] and Erdős and Yau[6].

Non-Hermitian random matrices have many applications in, for example, the fractional quantum Hall effect (Di Francesco *et al.*[7]) and quantum chromodynamics (Stephanov[8]); see Khoruzhenko and Sommers[9] for introductions and further applications. The eigenvalues of this type of matrices are complex numbers rather than real numbers as mentioned in the previous paragraph. We understand many properties on the Ginibre ensembles, truncations of Haar-unitary matrices, the product of Ginibre ensembles, the product of truncations of Haar-unitary matrices, the elliptic ensemble and the chiral non-Hermitian random matrix ensembles. Their limit properties will be further elaborated following our results later on.

In this paper, we will focus on a special case of the chiral non-Hermitian random matrix ensemble. For integers $n \geq 1$ and $v \geq 0$, let P and Q be two i.i.d. $(n+v)\times n$ matrices, where the entries from the two matrices are i.i.d. standard complex normals. For each $\tau \in [0,1]$, define the $(2n+v)\times(2n+v)$ matrix

$$\mathcal{D} = \begin{pmatrix} 0 & \sqrt{1+\tau}P + \sqrt{1-\tau}Q \\ \sqrt{1+\tau}P^* - \sqrt{1-\tau}Q^* & 0 \end{pmatrix}, \tag{1.1}$$

where P^* stands for the complex conjugate of matrix P. This matrix is refereed to as the chiral non-Hermitian random matrix ensemble. It has v-multiple zeros, n pairs of eigenvalues and each pair has the opposite sign. To understand these non-zero eigenvalues we only need to consider the n eigenvalues, $\mathbf{z}_1, \cdots, \mathbf{z}_n$, with positive x-coordinate. Akemann and Bender[10] derive that the joint probability density function of these n eigenvalues is equal to

$$C \prod_{1\leq j<k\leq n} |z_j^2 - z_k^2|^2 \cdot \prod_{j=1}^{n} |z_j|^{2(v+1)} \exp\Big(\frac{2\tau n \mathrm{Re}(z_i^2)}{1-\tau^2}\Big) K_v\Big(\frac{2n|z_j|^2}{1-\tau^2}\Big) \tag{1.2}$$

for all $z_1, \cdots, z_n \in \mathbb{C}$ with $\mathrm{Re}(z_j) > 0$ for each j, where C is a normalizing constant and K_v is the modified Bessel function of the second kind (see more details in Lemma 2.1). Here and later the density is relative to the Lebesgue measure on $\mathbb{C}^n$. This model includes special cases studied by Osborn[11] and Bender[12]. The parameter τ reflects the strength of how the matrix is non-Hermitian. For example, at the extreme case of $\tau = 1$, the eigenvalues of the matrix $\mathcal{D}$ in (1.1) are essentially those of a complex Wishart matrix, and hence are non-negative.

Write $\mathbf{z}_j = x_j + iy_j$ for each j. Given $\tau \in [0, 1)$, Akemann and Bender[10] obtain that (1) $\max_{1\le j\le n} x_j$ with a normalization converges to the Gumbel distribution; (2) renormalizing x_j and y_j with different constants, the new pairs (x'_j, y'_j), $1 \le j \le n$, as a point process, converges to a Poisson process. Here v is fixed and the limit is taken as $n \to \infty$. As $\tau = 1$, the aforementioned says that the eigenvalues of $\mathcal{D}$ in (1.1) are just those of a complex Wishart matrix. The largest eigenvalue of the complex Wishart matrix has the asymptotic Tracy-Widom distribution (Johansson[13]); the empirical distribution of those eigenvalues converges to the Marchenko-Pastur law (Marchenko and Pastur[14]; Bai and Silverstein[2]). In this paper we will study the same classical problems: the spectral radius $\max_{1\le j\le n} |\mathbf{z}_j|$ and the empirical distribution of $\mathbf{z}_j$'s not counting zero eigenvalues for $\tau = 0$. In particular, our targets are different from those in Akemann and Bender[10]. Assuming $\tau = 0$, the density in (1.2) then becomes that

$$f_z(z_1, \cdots, z_n) = C \prod_{1\le j<k\le n} |z_j^2 - z_k^2|^2 \cdot \prod_{j=1}^{n} |z_j|^{2(v+1)} K_v(2n|z_j|^2) \tag{1.3}$$

for all $z_1, \cdots, z_n \in \mathbb{C}$ with $\mathrm{Re}(z_j) > 0$ for each j. The parameter v is allowed to be any nonnegative real number and can change with n.

In this paper we prove that, with suitable normalization, $\max_{1\le j\le n} |\mathbf{z}_j|$ converges to the Gumbel distribution and the empirical distribution goes to a new distribution defined on the complex plane $\mathbb{C}$. The density of the distribution is explicit. When considering the density function as a surface defined on $\mathbb{C}$, both the convexity and the Gaussian curvature are studied. Figures 1 and 2 show the change of the surface as the limit of v/n changes.

Now we state our results. For $y > 3$, set

$$a(y) = (\log y)^{1/2} - (\log y)^{-1/2} \log(\sqrt{2\pi} \log y) \quad \text{and} \quad b(y) = (\log y)^{-1/2}. \tag{1.4}$$

Let Λ be the cumulative distribution of the Gumbel distribution such that

$$\Lambda(x) = \exp(-e^{-x}), \qquad x \in \mathbb{R}. \tag{1.5}$$

Theorem 1. *Let $v = v_n$ be a sequence of nonnegative numbers. Let $\mathbf{z}_1, \cdots, \mathbf{z}_n$ have density f_z as in (1.3). Then*

$$\frac{1}{b\big(\frac{n(n+v)}{2n+v}\big)}\Big[\frac{\max_{1\le j\le n}|\mathbf{z}_j| - (\frac{n+v}{n})^{1/4}}{\frac{1}{4}\frac{(2n+v)^{1/2}}{n^{3/4}(n+v)^{1/4}}} - a\Big(\frac{n(n+v)}{2n+v}\Big)\Big] \xrightarrow{d} \Lambda. \tag{1.6}$$

As aforementioned, $\mathcal{D}$ from (1.1) has eigenvalues $\pm\mathbf{z}_1, \cdots, \pm\mathbf{z}_n$ with $\mathbf{z}_1, \cdots, \mathbf{z}_n$ having density $f_z(z_1, \cdots, z_n)$ from (1.3); the other eigenvalues are zero with v multiples. Therefore, Theorem 1 also holds if "$\max_{1\le j\le n}|\mathbf{z}_j|$" is replaced by the "spectral radius of $\mathcal{D}$".

If $v_n \equiv v$, Theorem 1 has the following consequence.

Corollary 1. *Assume $v_n \equiv v \ge 0$. Let $\mathbf{z}_1, \cdots, \mathbf{z}_n$ have density f_z as in (1.3). Then $\beta_n \cdot \max_{1\le j\le n}|\mathbf{z}_j| - \alpha_n \xrightarrow{d} \Lambda$, where*

$$\alpha_n = \sqrt{8n\log\frac{n}{2}} + \log\frac{n}{2} - \log\big(\sqrt{2\pi}\log\frac{n}{2}\big) \quad and \quad \beta_n = \sqrt{8n\log\frac{n}{2}}.$$

The proof of this corollary is given after that of Theorem 1. On the other hand, (ii) of Theorem 3 from Akemann and Bender[10] implies that

$$\max_{1\le j\le n}\mathrm{Re}(\mathbf{z}_j) = 1 + \Big(\frac{\log n}{16n}\Big)^{1/2}(1 + o_P(1)) \tag{1.7}$$

as $n \to \infty$, where $o_P(1)$ stands for a random variable converging to zero in probability as $n \to \infty$. Our Corollary 1 says that

$$\max_{1\le j\le n}|\mathbf{z}_j| = 1 + \Big(\frac{\log n}{8n}\Big)^{1/2}(1 + o_P(1)). \tag{1.8}$$

Note that $\max_{1\le j\le n}|\mathbf{z}_j| \ge \max_{1\le j\le n}\mathrm{Re}(\mathbf{z}_j)$. (1.7) and (1.8) not only confirm this fact but also indicate the difference. Even so, it is still hard to quantify the size of $\max_{1\le j\le n}\mathrm{Im}(\mathbf{z}_j)$. One can see the reason by a quick glimpse at the two complex numbers $\varepsilon + i\sqrt{1-\varepsilon^2}$ and $\sqrt{1-\varepsilon^2} + \varepsilon i$ for $\varepsilon \in (0, 1)$ which have norms equal to 1 but have very different imaginary parts.

For a matrix $\mathbf{M}$ with eigenvalues $\mathbf{z}_1, \cdots, \mathbf{z}_n$, the quantity $\max_{1\le j\le n}|\mathbf{z}_j|$ is refereed to as the spectral radius of $\mathbf{M}$. Rider[15,16] and Rider and Sinclair[17] show that the spectral radii of the real, complex and symplectic Ginibre ensembles, which are non-Hermitian, asymptotically follow the Gumbel distribution. This fact is very different from the Tracy-Widom distribution in the Hermitian case; see, for example, Tracy and Widom[18,19]. Similar phenomena are observed for other non-Hermitian ensembles. Jiang and Qi[20] show that, for the spherical ensembles, truncations of Haar-unitary matrices and product of independent complex Ginibre ensembles, the limits of their spectral radii are a new distribution, the Gumbel distribution and the normal distribution, respectively.

Now we study the empirical measure of $\mathbf{z}_1, \cdots, \mathbf{z}_n$ in (1.3). Reviewing the scaling $[n/(n+v)]^{1/4}$ in (1.6), we define

$$\rho_n = \frac{1}{n}\sum_{j=1}^{n}\delta_{(\frac{n}{n+v})^{1/4}\mathbf{z}_j}.$$

For $\alpha \in [0, \infty)$, set $a = \alpha^2/(1+\alpha)^2$ and $b = 4/(1+\alpha)$, and let Φ_α be a probability measure with density function

$$\phi_\alpha(z) = \frac{4}{\pi}\frac{|z|^2}{\sqrt{a + b|z|^4}}$$

for z with $|z| \leq 1, \mathrm{Re}(z) > 0$. Obviously, Φ_0 is the uniform distribution on the "half moon" region $\{z;\, |z| \leq 1, \mathrm{Re}(z) > 0\}$. Let Φ_∞ denote the limit of Φ_α as $\alpha \to \infty$, that is,

$$\phi_\infty(z) = \frac{4}{\pi}|z|^2$$

for z with $|z| \leq 1$, $\mathrm{Re}(z) > 0$.

Theorem 2. *Let $v = v_n$ be a sequence of nonnegative numbers. Let $\mathbf{z}_1, \cdots, \mathbf{z}_n$ have density f_z as in (1.3). Assume $\lim_{n\to\infty} v_n/n = \alpha \in [0, \infty]$. Then, with probability one, ρ_n converges weakly to a probability distribution ρ with density function Φ_α as $n \to \infty$.*

As explained below the statement of Theorem 1, we are able to draw a conclusion for $\tilde{\rho}_n$, the empirical distribution of the eigenvalues of $\mathcal{D}$ from (1.1). If $\lim_{n\to\infty} v_n/n = \alpha \in [0, \infty)$, then the limit of the fraction of the number of zero eigenvalues relative to the total number $2n + v_n$ is $\frac{\alpha}{2+\alpha}$. Second, the other eigenvalues of $\mathcal{D}$ are $\pm\mathbf{z}_1, \cdots, \pm\mathbf{z}_n$ with $\mathbf{z}_1, \cdots, \mathbf{z}_n$ having density $f_z(z_1, \cdots, z_n)$ from (1.3), so we know the limit of $\tilde{\rho}_n$ is $\frac{\alpha}{2+\alpha}\delta_0 + \frac{2}{2+\alpha}\tilde{\rho}$, where $\tilde{\rho}$ has density $\frac{2}{\pi}\frac{|z|^2}{\sqrt{a+b|z|^4}}I(|z| \leq 1)$. If $\alpha = \infty$, of course, the limit is degenerate to δ_0.

Observe $\lim_{n\to\infty}[n/(n + v)]^{1/4} = 1$ for fixed v. A quick consequence of Theorem 2 is the following.

Corollary 2. *Assume $v \geq 0$ is fixed. Let $\mathbf{z}_1, \cdots, \mathbf{z}_n$ have density f_z as in (1.3). Then, with probability one, $\frac{1}{n}\sum_{j=1}^{n}\delta_{\mathbf{z}_j}$ converges weakly to the uniform distribution on $\{|z| \leq 1,\ Re(z) > 0\}$ as $n \to \infty$.*

We draw Figures 1 and 2 to see the change of the density function Φ_α as α changes. If we think of α as time and think of the surface $\{(z, \phi_\alpha(z));\, |z| \leq 1,\ \mathrm{Re}(z) > 0\}$ as the roof of a house, then the roof is flat at time 0, and suddenly the roof starts leaking at the center $z = 0$. With time going by, the leaking area becomes larger and larger, and eventually the whole roof collapses with the final shape $\{(z, \phi_\infty(z));\ |z| \le 1,\ \mathrm{Re}(z) > 0\}$, which is similar to the picture with $\alpha = 21$ in Figure 2 (the shape of $\phi_\alpha(z)$ visually does not change as $\alpha \ge 21$). The roof does not completely land on the ground because the volume under the roof has to be one ($\phi_\alpha(z)$ is a probability density function for all $\alpha \ge 0$).

Realizing the interesting phenomenon above, we further look at the geometry of the density surface. It is shown that the surface

$$\begin{aligned}&\{(z, \phi_\alpha(z));\ |z| < 1,\ \mathrm{Re}(z) > 0\} \text{ is convex and has positive Gaussian curvature}\\ &\text{if } \alpha > 10 + 2\sqrt{30};\ \text{the surface is not convex and it has negative Gaussian}\\ &\text{curvature for some } z \text{ if } 0 < \alpha < 10 + 2\sqrt{30}.\end{aligned} \tag{1.9}$$

When the Gaussian curvature at a point is positive, the surface will be like a dome, locally lying on one side of its tangent plane. When the curvature at a point is negative, the point is hyperbolic. From (1.9) we see that the whole surface indeed looks like a dome as $\alpha > 10 + 2\sqrt{30}$. The fact (1.9) will be checked at the end of this paper.

Now we make some remarks on literature related to Theorem 2. For complex Ginibre ensembles, products of Ginibre ensembles and product of truncated Haar-invariant unitary matrices, their eigenvalues form determinantal point processes. Their limiting laws of the empirical distributions of the eigenvalues are derived by using the determinantal point processes. See, for example, Burda *et al.*[21], Götze and Tikhomirov[22], Bordenave[23], O'Rourke and Soshnikov[24], Burda[25], O'Rourke *et al.*[26] and Jiang and Qi[27].

Review Corollary 2. In literature, the uniform distribution on $|z| \le 1$ is referred to as the circular law, which is the limit of the empirical distribution of the eigenvalues of a square matrix with entries being independent and identically distributed random variables; see, for example, Girko[28,29], Bai[30], Tao and Vu[31] or Bordenave and Chafaï[32]

For the proofs of the two main theorems, we utilize the tools developed by Jiang and Qi[20,27] for rotation-invariant and non-Hermitian random matrices. The tools provide sufficient conditions for the convergence of the spectral radius and the empirical distribution of the eigenvalues. However, the chiral non-Hermitian random matrix is not rotation-variant. In our proofs, we first make a transform such that the eigenvalues of the chiral non-Hermitian random matrix form a rotation-variant ensemble. Under this setting, both the limit of the

Plot of surface Φ_α

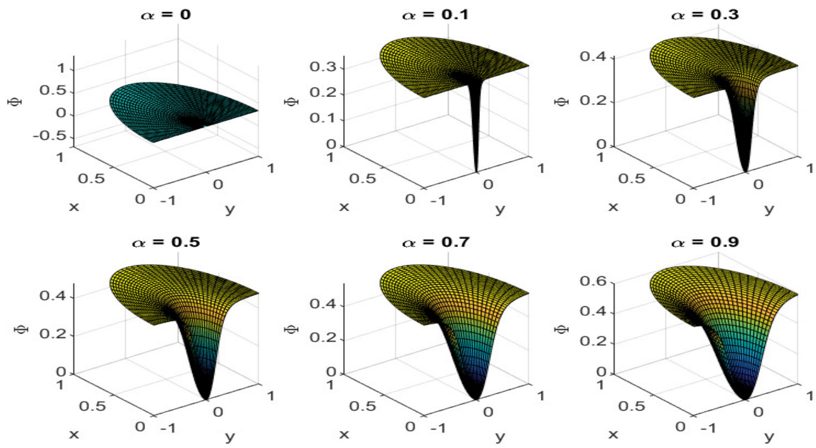

FIG. 1. *At $\alpha = 0$, the density surface is flat. At the next moment, there is a tiny hole at $z = 0$. With α increasing, the hole becomes larger and larger. The convex part (the Hessian is nonnegative) in the density surface becomes large.*

Plot of surface Φ_α

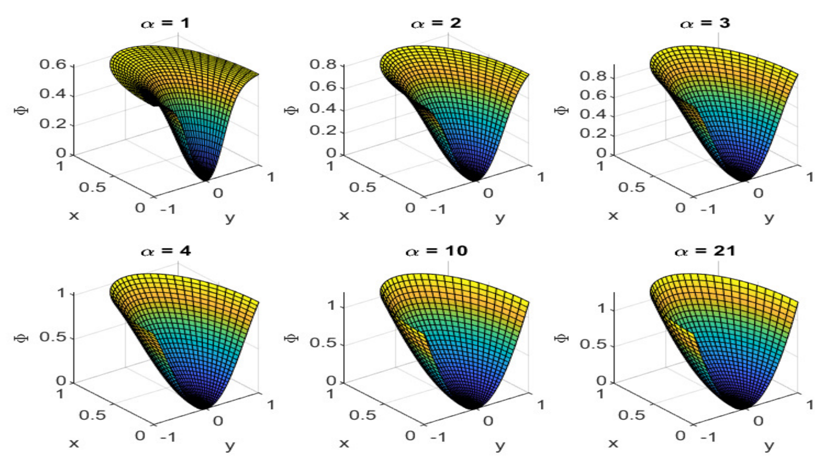

FIG. 2. *With α increasing, the convex part and the part with positive Gaussian curvature becomes larger and larger. As $\alpha > 10 + 2\sqrt{30} \simeq 20.95$, the whole density surface becomes convex and the Gaussian curvature is positive everywhere. No much change for the shape visually for $\alpha > 21$.*

spectral radius and that of the empirical spectral distribution depend on a set of n independent random variables $Y_1, \cdots, Y_n$ (Lemma 2.5). The distribution of each Y_j is determined by the modified Bessel function of the second kind. To apply the tools by Jiang and Qi[20,27], we spend many efforts to derive a "uniform" central limit theorem (CLT) for $\{Y_j;\, m \le j \le n\}$ (Lemma 2.8), where m depends on n. Since the distribution of Y_j depends on the Bessel function and it has not been understood to our knowledge, we employ the method of steepest descent to derive the CLT for $\{Y_j;\, m \le j \le n\}$. In particular, the saddle point is the solution of a non-trivial equation, and hence a great energy is spent to estimate the solution. Noting both theorems allow that $v_n \ge 0$ is arbitrary, we use another trick of subsequence argument such that $\lim_{n\to\infty} v_n \in [0, \infty]$ and $\lim_{n\to\infty} v_n/n \in [0, \infty]$ in Theorem 1 and Theorem 2, respectively.

We study the spectral properties of eigenvalues with joint density in (1.2) for $\tau = 0$ in this paper. A generalization of our methods to the general case (1.2) with $\tau \in (0, 1)$ remains unknown and may be challenging. We leave it as a future work.

The rest of the paper is organized as follows. The proofs of Theorems 1 and 2, Corollary 1 and the check of (1.9) are presented in Section 2.

2. PROOFS

This section is divided into four parts. In Section 2.1 we make some preparations; in Section 2.2 we prove Theorems 1 and 2 as well as Corollary 1; in Section 2.3, we prove the technical results Lemmas 2.9 and 2.10 used in Section 2.2; the verification of (1.9) is given in Section 2.4.

2.1. Some technical tools

We first list some properties of $K_v(x)$, the modified Bessel function of the second kind. To avoid confusion, for all lemmas in this section, the parameter $v \in [0, \infty)$ is reserved for the subscript in the modified Bessel function of the second kind K_v. In the case that the parameter v also depends on n, we will write $v = v_n$.

Lemma 2.1. *(Properties of K_v). The following statements hold.*

(a) (Formula 9.6.24 in Abramowitz and Stegun[33]). For any $v \geq 0$,

$$K_v(x) = \int_0^\infty e^{-x\cosh(t)} \cosh(vt)dt, \quad x > 0,$$

where $\cosh(t) = (e^t + e^{-t})/2,\ t \in \mathbb{R}$.

(b) (Formulas 9.6.8 and 9.6.9 in Abramowitz and Stegun[33]). If $v = 0$ *then*

$$K_0(x) \sim -\log x \qquad \text{as} \ \ x \downarrow 0,$$

and if $v > 0$ *is fixed then*

$$K_v(x) \sim 2^{v-1}\Gamma(v)x^{-v} \qquad \text{as} \ \ x \downarrow 0.$$

(c) (Formula 9.7.2 in Abramowitz and Stegun[33]). If $v \geq 0$ *is fixed, then*

$$K_v(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\Big[1 + O\Big(\frac{1}{x}\Big)\Big] \qquad \text{as } x \to \infty. \tag{2.1}$$

(d) (Equations (4.4) and (4.6) in Olver[34]). As $v \to \infty$

$$K_v(vx) = \sqrt{\frac{\pi}{2v}}(1+x^2)^{-1/4}\exp\big(-v\eta(x)\big)\Big[1 + O\Big(\frac{1}{v}\Big)\Big] \tag{2.2}$$

uniformly in $x \in (0, \infty)$*, where* $\eta(x) = \sqrt{1+x^2} + \log x - \log(1 + \sqrt{1+x^2}\,)$.

Set $\mathbb{C}_1 = \{z \in \mathbb{C};\, \mathrm{Re}(z) > 0\}$ and $\mathbb{C}_2 = \mathbb{C}\backslash\{z \in \mathbb{R};\, z \leq 0\}$. Review that K_v is the modified Bessel function of the second kind. Below we use the convention $0^v K_v(0) = 0$ for all $v \geq 0$.

Lemma 2.2. *For* $(z_1, \cdots, z_n) \in \mathbb{C}_1^n$*, define* $u_j = z_j^2$, $1 \leq j \leq n$*. Then, the map* $(z_1, \cdots, z_n) \to (u_1, \cdots, u_n) : \mathbb{C}_1^n \to \mathbb{C}_2^n$ *is a one-to-one and onto map. Let* $(\mathbf{z}_1, \cdots, \mathbf{z}_n) \in \mathbb{C}_1^n$ *have density function* $f_z(z_1, \cdots, z_n)$ *as in (1.3), then the density function of* $(\mathbf{z}_1^2, \cdots, \mathbf{z}_n^2) =:$ $\mathbf{u} = (\mathbf{u}_1, \cdots, \mathbf{u}_n)$ *is given by*

$$f(u) = C \cdot \prod_{1\leq j<k\leq n} |u_j - u_k|^2 \cdot \prod_{j=1}^n |u_j|^v K_v\big(2n|u_j|\big) \tag{2.3}$$

for all $u \in \mathbb{C}_2^n$*. Since the Lebesgue measure of* $\mathbb{C}^n\backslash\mathbb{C}_2^n$ *is zero, we will simply regard* $(\mathbf{u}_1, \cdots, \mathbf{u}_n)$ *has density* $f(u)$ *defined for all* $u \in \mathbb{C}^n$.

Proof. Identify $\mathbb{C}$ with $\mathbb{R}^2$. Then $\mathbb{C}_1 = \{(x, y) : x > 0, y \in \mathbb{R}\}$ and $\mathbb{C}_2 = \mathbb{R}^2\backslash\{(x, 0) : x \leq 0\}$. First, we show the transform $u = z^2$: $\mathbb{C}_1 \to \mathbb{C}_2$ is a one-to-one and onto map. Write

$z = (x, y)'$. Define a transform $u = z^2$, which is the same as $u(x, y) : (x, y)' \in \mathbb{C}_1 \to \mathbb{C}_2$ such that $u(x, y) = (s, t)' = (x^2 - y^2, 2xy)'$. By solving the last equation, we obtain the inverse function $z = z(s, t)$: $\mathbb{C}_2 \to \mathbb{C}_1$ given by

$$z(s,t) = \Big(\frac{1}{\sqrt{2}}\sqrt{\sqrt{s^2+t^2}+s},\ \frac{1}{\sqrt{2}}\mathrm{sgn}(t)\sqrt{\sqrt{s^2+t^2}-s}\ \Big)' \tag{2.4}$$

where "sgn" is the sign function.

Second, let us compute the Jacobian of the transform.

$$\frac{\partial(s,t)}{\partial(x,y)} = \det\begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 4(x^2+y^2) = 4|z|^2 = 4|u|.$$

Thus, the joint probability density function (pdf) of $\mathbf{u}_i = \mathbf{z}_i^2$, $i = 1, \cdots, n$ is given by

$$f(u_1, \cdots, u_n) = C \cdot \prod_{1 \le j < k \le n} |u_j - u_k|^2 \cdot \prod_{j=1}^{n} |u_j|^v K_v\big(2n|u_j|\big)$$

for all $u \in \mathbb{C}_2^n$. □

Lemma 2.3. *Define $\xi(x) = x - \log(1+x)$ for $x \ge 0$. Then*

$$\xi(x) \ge \frac{1}{4}ax$$

for all $a \in (0, 1]$ and $x \ge a$.

Proof. Fix $a \in (0, 1]$. Note that

$$\xi(x) = x^2 \int_0^1 \frac{t}{1+tx}dt \ge x^2 \int_0^1 \frac{t}{1+x}dt = \frac{x}{1+x}\frac{x}{2}.$$

Since $\frac{x}{1+x}$ is increasing in $x \ge 0$, we get $\frac{x}{1+x} \ge \frac{a}{1+a} \ge \frac{a}{2}$ if $x \ge a$. The lemma follows. □

Lemma 2.4. *Let f and g be density functions of real random variables X and Y, respectively. Assume f and g have a common support $D \subset \mathbb{R}$ and $q(x) := f(x)/g(x)$ is non-decreasing in $x \in D$. Then $P(X > x) \ge P(Y > x)$ for all $x \in \mathbb{R}$.*

Proof. Let $\omega_l = \inf\{x : x \in D\}$ and $\omega_u = \sup\{x : x \in D\}$. It suffices to show that $P(X > x) \ge P(Y > x)$ for $\omega_l < x < \omega_u$. Now we extend function q to the range $\omega_l < x < \omega_u$ by defining $q(x) = \inf\{q(y) : y \ge x, y \in D\}$ if $\omega_l < x < \omega_u$ but $x \notin D$. Then $q(x)$ is non-decreasing for $\omega_l < x < \omega_u$ and $f(x) = q(x)g(x)$ for $\omega_l < x < \omega_u$. Furthermore, for $\omega_l < x < \omega_u$,

$$P(X > x) = \int_x^{\omega_u} q(t)g(t)dt \ge q(x)\int_x^{\omega_u} g(t)dt = q(x)P(Y > x)$$

and

$$P(X \le x) = \int_{\omega_l}^{x} q(t)g(t)dt \le q(x)\int_{\omega_l}^{x} g(t)dt = q(x)P(Y \le x),$$

that is,

$$P(X > x) \ge q(x)P(Y > x) \ \text{ and } \ P(X \le x) \le q(x)P(Y \le x)$$

for all $\omega_l < x < \omega_u$. By multiplying $P(Y \le x)$ on both sides of the first inequality, multiplying $P(Y > x)$ on both sides of the second one and comparing, we see that

$$P(X > x)P(Y \le x) \ge P(X \le x)P(Y > x)$$

for $\omega_l < x < \omega_u$. The desired result is obtained by adding $P(X > x)P(Y > x)$ to the both sides of the above inequality. □

Lemma 2.5. *Let* $\mathbf{u}_1, \cdots, \mathbf{u}_n$ *have the joint density function* $f(u_1, \cdots, u_n)$ *as in (2.3). Let* Y_j*,* $1 \le j \le n$*, be independent random variables and the density of* Y_j *is proportional to* $y^{2j+v-1}K_v(2ny)I(y > 0)$*. Then* $g(|\mathbf{u}_1|, \cdots, |\mathbf{u}_n|)$ *and* $g(Y_1, \cdots, Y_n)$ *have the same distribution for any symmetric function* g*.*

Proof. Define $f(u_1, \cdots, u_n) = 0$ for any $u = (u_1, \cdots, u_n)$ with $u_j \in (-\infty, 0]$ for some $1 \le j \le n$. Then the conclusion follows from Lemma 1.1 in Jiang and Qi[20]. □

Lemma 2.6. *Let* Y_j*,* $1 \le j \le n$ *be independent random variables defined in Lemma 2.5. Then, for each* y*,* $P(Y_j > y)$ *is non-decreasing in* j *for* $1 \le j \le n$*.*

Proof. Since the pdf of Y_j is proportional to $y^{2j+v-1}K_v(2ny)I(y > 0)$, the ratio of pdfs of Y_j and Y_{j-1} is proportional to $yI(y > 0)$ which is increasing in $y > 0$. It follows from Lemma 2.4 that $P(Y_j > y) \ge P(Y_{j-1} > y)$ for $2 \le j \le n$. This completes the proof. □

Now we study the Central Limit Theorem (CLT) for Y_j as $v = v_n \to \infty$. We use $\Phi(x)$ to denote the cumulative distribution function of the standard normal $N(0, 1)$, that is,

$$\Phi(x) = \int_{-\infty}^{x} \phi(t)dt \tag{2.5}$$

with $\phi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ for $t \in \mathbb{R}$.

Lemma 2.7. *Let* $\Phi(x)$ *be as in (2.5). Then*

$$1 - \Phi(ct + d) = [1 - \Phi(t)](1 + o(1)) + O\Big(\frac{1}{n^2}\Big) \tag{2.6}$$

uniformly over $t \in \mathbb{R}$*,* $|c - 1| \le n^{-3/8}$ *and* $|d| \le n^{-3/8}$ *as* $n \to \infty$*.*

Proof. First, by (2.26),

$$1-\Phi(ct+d)=P(N(0,1)\geq ct+d)\leq e^{-(ct+d)^2/2}$$

as $t\geq 2$ and n is sufficiently large. Thus, as $n\to\infty$,

$$1-\Phi(ct+d)\leq \frac{1}{n^2} \tag{2.7}$$

uniformly over $|c-1|\leq n^{-3/8}$, $|d|\leq n^{-3/8}$ and $t\geq \log n$. In particular, it holds with $c=1$ and $d=0$. Thus, (2.6) is true uniformly over $|c-1|\leq n^{-3/8}$, $|d|\leq n^{-3/8}$ and $t>\log n$ as $n\to\infty$. If $t\leq -\log n$, then

$$\begin{aligned}\big|[1-\Phi(ct+d)]-[1-\Phi(t)]\big| &\leq \Phi(ct+d)+\Phi(t)\\ &= P(N(0,1)\geq c(-t)-d)+P(N(0,1)\geq(-t))\\ &\leq \frac{2}{n^2}\end{aligned}$$

uniformly over $|c-1|\leq n^{-3/8}$ and $|d|\leq n^{-3/8}$ by the same argument as obtaining (2.7). Hence, (2.6) is true uniformly over $|c-1|\leq n^{-3/8}$, $|d|\leq n^{-3/8}$ and $t\leq -\log n$ as $n\to\infty$.

Now, assume $|t|<\log n$. Review $\phi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ for $t\in\mathbb{R}$. Then

$$1-\Phi(ct+d)=c\int_t^{\infty}\phi(cs+d)\,ds.$$

Note that

$$\frac{\phi(cs+d)}{\phi(s)}=\exp\Big\{-\frac{1}{2}(c^2-1)s^2-cds-\frac{1}{2}d^2\Big\}=1+O\Big(\frac{1}{n^{1/8}}\Big)$$

uniformly over $|s|\leq \log n$ as $n\to\infty$. Thus,

$$\begin{aligned}1-\Phi(ct+d) &= 1-\Phi(c\log n+d)+c\int_t^{\log n}\phi(cs+d)\,ds\\ &= O\Big(\frac{1}{n^2}\Big)+\Big[1+O\big(n^{-1/8}\big)\Big]\cdot c\int_t^{\log n}\phi(s)\,ds\end{aligned} \tag{2.8}$$

uniformly over $|c-1|\leq n^{-3/8}$ and $|d|\leq n^{-3/8}$ by (2.7). In particular,

$$1-\Phi(t)=O\Big(\frac{1}{n^2}\Big)+\Big[1+O\big(n^{-1/8}\big)\Big]\cdot\int_t^{\log n}\phi(s)\,ds.$$

Solve for "$\int_t^{\log n}\phi(s)\,ds$" and then plug it in (2.8) to see that

$$1-\Phi(ct+d)=O\Big(\frac{1}{n^2}\Big)+\Big[1+O\big(n^{-1/8}\big)\Big]\cdot[1-\Phi(t)]$$

uniformly over $|c-1|\leq n^{-3/8}$, $|d|\leq n^{-3/8}$ and $|t|<\log n$ as $n\to\infty$. This together with the earlier conclusions yields the desired conclusion. □

Lemma 2.8. *Let Y_j be as in Lemma 2.5. Let $v = v_n \to \infty$ as $n \to \infty$. Fix $\delta \in (0,1)$. Then, as $n \to \infty$,*

$$P\Big(\frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j+v}} > t\Big) = (1+o(1))(1-\Phi(t)) + O\Big(\frac{1}{n^2}\Big) \tag{2.9}$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$.

Proof. Define

$$\begin{aligned}\tau(x) &= \eta(x) + \frac{1}{4v}\log(1+x^2) - \frac{2j+v-1}{v}\log x \\ &= \sqrt{1+x^2} + \log x - \log(1+\sqrt{1+x^2}) + \frac{1}{4v}\log(1+x^2) - \frac{2j+v-1}{v}\log x.\end{aligned} \tag{2.10}$$

It is easy to check that

$$\begin{aligned}\tau'(x) &= \frac{\sqrt{1+x^2}}{x} + \frac{x}{2v(1+x^2)} - \frac{2j+v-1}{vx}, \\ x\tau'(x) &= \sqrt{1+x^2} - \frac{1}{2v(1+x^2)} - \frac{2j-1.5+v}{v}\end{aligned} \tag{2.11}$$

and

$$\tau''(x) = -\frac{1}{x^2\sqrt{1+x^2}} + \frac{1-x^2}{2v(1+x^2)^2} + \frac{2j+v-1}{vx^2} \tag{2.12}$$

$$\begin{aligned}&= -\frac{1}{x^2\sqrt{1+x^2}} - \frac{1}{2v(1+x^2)} + \frac{1}{v(1+x^2)^2} + \frac{2j+v-1}{vx^2} \\ &> \frac{2j-1.5}{vx^2}\end{aligned} \tag{2.13}$$

by the trivial facts that

$$-\frac{1}{x^2\sqrt{1+x^2}} > -\frac{1}{x^2}, \quad -\frac{1}{2v(1+x^2)} > -\frac{1}{2vx^2} \quad \text{and} \quad \frac{1}{v(1+x^2)^2} > 0.$$

Note that $\tau'(0+) = -\infty$ and $\tau'(\infty) = 1$. Since $\tau''(x) > 0$ for all $x > 0$, $\tau'(x)$ is strictly increasing, and thus a unique root to the equation $\tau'(x) = 0$ exists in $(0,\infty)$. Denote this root by $\mu_{v,j}$. Then,

$$\mu_{v,j} > 0 \quad \text{and} \quad \tau'(\mu_{v,j}) = 0. \tag{2.14}$$

Define

$$\sigma_{v,j}^2 = \frac{2j+v}{v^2}, \quad \beta_{v,j}^2 = \frac{\sigma_{v,j}^2}{\mu_{v,j}^2} \quad \text{and} \quad y_{v,j}(t) = \mu_{v,j}(1+\beta_{v,j}t) \tag{2.15}$$

for $t \in \mathbb{R}$. Note that the density function of $\frac{2nY_j}{v}$ as a function of y is proportional to $y^{2j+v-1}K_v(vy)I(y>0)$. Set

$$V_j = \frac{1}{\sigma_{v,j}}\Big(\frac{2nY_j}{v} - \mu_{v,j}\Big) \tag{2.16}$$

for $1 \le j \le n$. Then the density function of V_j as a function of t is proportional to $\xi_{v,j}(t)I(1+t\beta_{v,j}>0)$, where

$$\xi_{v,j}(t) = \big(vy_{v,j}(t)\big)^{2j+v-1}K_v(vy_{v,j}(t)). \tag{2.17}$$

Hence, the density function of V_j is equal to

$$f_{v,j}(t) = \frac{1}{C_{v,j}}\frac{\xi_{v,j}(t)}{\xi_{v,j}(0)}I(1+t\beta_{v,j}>0),$$

where

$$C_{v,j} = \int_{-\infty}^{\infty}\frac{\xi_{v,j}(t)}{\xi_{v,j}(0)}I(1+t\beta_{v,m}>0)dt$$

and $\xi_{v,j}(0) = (v\mu_{v,j})^{2j+v-1}K_v(v\mu_{v,j}) > 0$ by Lemma 2.1(a) and (2.14). Review (2.17). Take $x = y_{v,j}(t)$. Then, from (2.2),

$$\begin{aligned} v^{-(2j+v-1)}\xi_{v,j}(t) &= x^{2j+v-1}K_v(vx) \\ &= \sqrt{\frac{\pi}{2v}}x^{2j+v-1}\exp\Big[-\frac{1}{4}\log(1+x^2) - v\eta(x)\Big]\Big(1+O\big(\frac{1}{v}\big)\Big) \\ &= \sqrt{\frac{\pi}{2v}}\cdot e^{-v\tau(x)}\Big(1+O\big(\frac{1}{v}\big)\Big). \end{aligned}$$

With $y_{v,j}(0) = \mu_{v,j}$ we then have

$$f_{v,j}(t) = \frac{1+O(\frac{1}{v})}{C_{v,j}}\exp\Big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\Big]I(1+t\beta_{v,j}>0) \tag{2.18}$$

and

$$C_{v,j} = \Big(1+O\big(\frac{1}{v}\big)\Big)\int_{1+t\beta_{v,j}>0}\exp\Big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\Big]dt. \tag{2.19}$$

Some properties of $\mu_{v,j}$ will be provided in Lemmas 2.9 and 2.10. Since the proof of each lemma costs a considerable length, we postpone them until Section 2.3.

Lemma 2.9. *[Estimate of $\mu_{v,j}$ from (2.14)]. Let $\delta \in (0,1)$ be fixed. Let $v = v_n \to \infty$ as $n \to \infty$. Define*

$$d_{v,j} = \frac{1}{\sigma_{v,j}}\Big(\frac{2\sqrt{j(v+j)}}{v} - \mu_{j,v}\Big),$$

$$\bar{\mu}_{v,j} = \sqrt{\Big(\frac{2j+v-1.5}{v} + \frac{v}{2(2j+v-1.5)^2}\Big)^2 - 1}$$

for $\delta n \le j \le n$. Then

$$\mu_{v,j} = \bar{\mu}_{v,j}\Big[1 + O\Big(\frac{v^4}{n(n+v)^5}\Big)\Big] \tag{2.20}$$

holds uniformly over $\delta n \le j \le n$ as $n \to \infty$. Furthermore, $\max_{\delta n \le j \le n} |d_{v,j}| = O(n^{-1/2})$.

The next result presents estimates of $\tau(y_{v,j}(t)) - \tau(\mu_{v,j})$.

Lemma 2.10. *Review the notations $\tau(x), \mu_{v,j}$, $y_{v,j}$ and $\sigma_{v,j}$, and $\beta_{v,j}$ in (2.10), (2.14) and (2.15), respectively. Let $\delta \in (0,1)$ be fixed. Then the following statements hold uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as n is sufficiently large.*

(i) $\sigma_{v,j}^2 \tau''(\mu_{v,j}) = \frac{1}{v}[1 + O(\frac{1}{n+v})]$.

(ii) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \frac{1}{2}\sigma_{v,j}^2 \tau''(\mu_{v,j}) t^2 (1 + O(\beta_{v,j}|t|))$.

(iii) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) > \frac{j}{v}\big[\beta_{v,j} t - \log(1 + \beta_{v,j} t)\big]$ for $t \ge 0$.

(iv) $\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) \ge \frac{1}{2}\sigma_{v,j}^2 \tau''(\mu_{v,j}) t^2$ for $-1/\beta_{v,j} < t < 0$.

Let V_j be as in (2.16). Let $\delta \in (0,1)$ be fixed. With Lemmas 2.9 and 2.10 at hand, we claim that V_j's satisfy a "uniform" CLT, that is,

$$P(V_j > t) = [1 - \Phi(t)](1 + o(1)) + O\Big(\frac{1}{n^2}\Big) \tag{2.21}$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$. If this is true, by (2.15) and (2.16),

$$\frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j+v}} = V_j - d_{v,j},$$

where $d_{v,j} = \big(\frac{2\sqrt{j(v+j)}}{v} - \mu_{j,v}\big)/\sigma_{v,j}$. By Lemma 2.9, $\max_{\delta n \le j \le n} |d_{v,j}| = O(n^{-1/2})$. Therefore, we have from (2.21) and Lemma 2.7 that

$$\begin{aligned} P\Big(\frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j+v}} > t\Big) &= P(V_j \ge t + d_{v,j}) \\ &= (1 + o(1))(1 - \Phi(t + d_{v,j})) + O\Big(\frac{1}{n^2}\Big) \\ &= (1 + o(1))(1 - \Phi(t)) + O\Big(\frac{1}{n^2}\Big) \end{aligned}$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$. Hence, (2.9) is obtained. So, to complete the whole proof, it remains to show (2.21). We will prove this next via the method of steepest descent.

Recall $\beta_{v,j}^2 = \sigma_{v,j}^2/\mu_{v,j}^2$ from (2.15), where $\sigma_{v,j}^2 = \frac{2j+v}{v^2}$ and $\mu_{v,j}$ is as in (2.14). Easily, from (2.20)

$$\frac{c_1}{\sqrt{n}} \le \min_{\delta n \le j \le n} \beta_{v,j} \le \max_{\delta n \le j \le n} \beta_{v,j} \le \frac{c_2}{\sqrt{n}} \tag{2.22}$$

where $c_1 > 0$ and $c_2 > 0$ are two constants depending on δ only. It follows from (i) and (ii) of Lemma 2.10 that

$$v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big) = \frac{t^2}{2}\Big[1 + O\Big(\beta_{v,j}|t| + \frac{1}{n+v}\Big)\Big] = \frac{t^2}{2} + O(n^{-1/8})$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$, which implies

$$\exp\big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\big] = e^{-t^2/2}\big[1 + O\big(n^{-1/8}\big)\big] \tag{2.23}$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$. From (2.22),

$$\max_{\delta n \le j \le n} \beta_{v,j} n^{1/8} \le c_2 n^{-3/8} < 1$$

for all large n. Thus, from Lemma 2.10(iii) and Lemma 2.3 (take $x = a$) we get

$$v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big) \ge \frac{1}{4} j n^{1/8} \beta_{v,j}^2 t \ge \frac{c_1^2 \delta}{4} n^{1/8} t$$

for any $t > n^{1/8}$. Therefore, for all large n

$$\begin{aligned}
&\int_{n^{1/8}}^{\infty} \exp\big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\big]dt \\
\le\ & \int_{n^{1/8}}^{\infty} \exp(-\frac{c_1^2\delta}{4} n^{1/8} t)dt \\
=\ & \frac{4}{c_1^2 \delta n^{1/8}} \exp\Big(-\frac{c_1^2\delta}{4} n^{1/4}\Big) \\
=\ & O\Big(\frac{1}{n^2}\Big).
\end{aligned} \tag{2.24}$$

It follows from (i) and (iv) of Lemma 2.10 that, uniformly over $\delta n \le j \le n$ and $-1/\beta_{v,j} < t < 0$, we have

$$v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big) \ge \frac{t^2}{4}$$

for all large n, which yields

$$\begin{aligned}
\int_{-1/\beta_{v,j}}^{-n^{1/8}} \exp\big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\big]dt &\le \int_{-\infty}^{-n^{1/8}} \exp\Big(-\frac{t^2}{4}\Big)dt \\
&\le \frac{2}{n^{1/8}} \exp\Big(-\frac{1}{4} n^{1/4}\Big) \\
&= O\Big(\frac{1}{n^2}\Big),
\end{aligned} \tag{2.25}$$

uniformly over $\delta n \le j \le n$, where the second integral is equal to $\sqrt{4\pi} \cdot P(N(0,1) \ge n^{1/8}/\sqrt{2})$, and the second inequality is obtained by the inequality

$$P(N(0,1) \ge x) \le \frac{1}{\sqrt{2\pi}\, x} e^{-x^2/2} \tag{2.26}$$

for all $x > 0$. Similar to (2.25), we have

$$\int_{-n^{1/8}}^{n^{1/8}} e^{-t^2/2} dt = \int_{-\infty}^{\infty} e^{-t^2/2} dt - 2\int_{n^{1/8}}^{\infty} e^{-t^2/2} dt$$
$$= \sqrt{2\pi} + O\Big(\frac{1}{n^2}\Big). \tag{2.27}$$

Consequently, from (2.23),

$$\int_{-n^{1/8}}^{n^{1/8}} \exp\big[-v\big(\tau(y_{v,j}(t)) - \tau(\mu_{v,j})\big)\big] dt = \int_{-n^{1/8}}^{n^{1/8}} e^{-t^2/2} dt \cdot \big(1 + O\big(n^{-1/8}\big)\big)$$
$$= \sqrt{2\pi} + O\big(n^{-1/8}\big)$$

uniformly over $\delta n \le j \le n$. By taking into account the above estimates we have from (2.19) that

$$C_{v,j} = \Big(1 + O\Big(\frac{1}{v}\Big)\Big)\sqrt{2\pi} + O\Big(\frac{1}{n^2}\Big) = \sqrt{2\pi} + o(1)$$

uniformly over $\delta n \le j \le n$ as $n \to \infty$. Furthermore, remember V_j has density function $f_{v,j}(t)$ as in (2.18). Easily, $V_j \ge -1/\beta_{v,j}$ from (2.16) for each j. By the expression of $f_{v,j}(t)$ from (2.18),

$$P(V_j > t) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{h(t)}^{\infty} \exp\big[-v\big(\tau(y_{v,j}(s)) - \tau(\mu_{v,j})\big)\big] ds \tag{2.28}$$

uniformly over $\delta n \le j \le n$ as $n \to \infty$, where $h(t) := \max\{-1/\beta_{v,j}, t\}$ for $t \in \mathbb{R}$. If $t < -n^{-1/8}$, then

$$0 \le \Big(\int_{h(t)}^{\infty} - \int_{-n^{-1/8}}^{\infty}\Big) \exp\big[-v\big(\tau(y_{v,j}(s)) - \tau(\mu_{v,j})\big)\big] ds$$
$$\le \int_{-1/\beta_{v,j}}^{-n^{-1/8}} \exp\big[-v\big(\tau(y_{v,j}(s)) - \tau(\mu_{v,j})\big)\big] ds$$
$$\le \frac{1}{n^2}$$

by (2.25). Hence,

$$P(V_j > t) = \frac{1 + o(1)}{\sqrt{2\pi}} \int_{-n^{-1/8}}^{\infty} \exp\big[-v\big(\tau(y_{v,j}(s)) - \tau(\mu_{v,j})\big)\big] ds + O\Big(\frac{1}{n^2}\Big)$$
$$= \frac{1 + o(1)}{\sqrt{2\pi}} \Big[\sqrt{2\pi} + O\Big(\frac{1}{n^2}\Big)\Big] + O\Big(\frac{1}{n^2}\Big)$$
$$= 1 + o(1)$$
$$= (1 + o(1))(1 - \Phi(t)) + (1 + o(1))\Phi(t)$$
$$= (1 + o(1))(1 - \Phi(t)) + O\Big(\frac{1}{n^2}\Big)$$

uniformly over $\delta n \le j \le n$ as $n \to \infty$ by (2.24) and (2.27). In the last step, we have used the inequality $\Phi(t) \le P(N(0,1) \ge n^{1/8}) \le \frac{1}{n^2}$ for all $t < -n^{-1/8}$. Therefore, (2.21) holds uniformly over $t < -n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$. If $t \ge -n^{1/8}$, we see from (2.28) that

$$P(V_j > t) = \frac{1+o(1)}{\sqrt{2\pi}} \int_t^\infty \exp\big[-v\big(\tau(y_{v,j}(s)) - \tau(\mu_{v,j})\big)\big]ds.$$

Therefore, by the same arguments as in (2.24) and (2.27), we conclude that the above integral is equal to $\int_t^\infty e^{-s^2/2} + O(n^{-2})$ uniformly over $t \ge -n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$. This combing with the earlier conclusion for $t < -n^{1/8}$ completes the proof of (2.21). □

Lemma 2.8 focuses on the CLT for Y_j as $v = v_n \to \infty$. Now we work on the same problem under the assumption that $\{v_n;\, n \ge 1\}$ is a bounded sequence. A lemma is needed first.

Lemma 2.11. *Let g_1 and g_2 be nonnegative functions defined over $(0,\infty)$ and $b_i(\theta) := \int_0^\infty t^\theta g_i(t)dt < \infty$ for all $\theta \ge \theta_0$, $i = 1,2$, where $\theta_0 \ge 0$ is a constant. Assume that $g_1(x) \sim g_2(x)$ as $x \to \infty$. The following holds.*
(a) Uniformly over $\theta \in [\theta_0, \infty)$,

$$\int_x^\infty t^\theta g_1(t)dt \sim \int_x^\infty t^\theta g_2(t)dx \qquad \text{as } x \to \infty.$$

(b) If $\lim_{\theta\to\infty} \frac{\log b_1(\theta)}{\theta} = \infty$, then $b_1(\theta) \sim b_2(\theta)$ as $\theta \to \infty$.

Proof. (a). Write

$$\int_x^\infty t^\theta g_1(t)dt = \int_x^\infty t^\theta g_2(t)dt + \int_x^\infty t^\theta g_2(t)\Big(\frac{g_1(t)}{g_2(t)} - 1\Big)I(g_2(t) \ne 0)\, dt.$$

Denote by $\epsilon(x)$ the last integral. Then

$$|\epsilon(x)| \le \max_{t\ge x}\Big\{\Big|\frac{g_1(t)}{g_2(t)} - 1\Big|I(g_2(t) \ne 0)\Big\} \cdot \int_x^\infty t^\theta g_2(t)dt.$$

The conclusion then follows from the fact that $g_1(x) \sim g_2(x)$ as $x \to \infty$.

(b). Define $x_\theta = \frac{1}{2}\big(b_1(\theta)\big)^{1/\theta}$ for large θ. Then $x_\theta = \frac{1}{2}\exp\big(\frac{\log b_1(\theta)}{\theta}\big) \to \infty$ as $\theta \to \infty$, and

$$\lim_{\theta\to\infty} \frac{(x_\theta)^\theta}{b_1(\theta)} = 0$$

which implies that

$$\int_0^{x_\theta} t^\theta g_i(t)dt \le (x_\theta)^\theta \int_0^\infty g_i(t)dt = O((x_\theta)^\theta) = o(b_1(\theta)) \tag{2.29}$$

for $i = 1, 2$. It follows from part (a) that, as $\theta \to \infty$,

$$\int_{x_\theta}^\infty t^\theta g_2(t)dt \sim \int_{x_\theta}^\infty t^\theta g_1(t)dt = b_1(\theta) - \int_0^{x_\theta} t^\theta g_1(t)dt = b_1(\theta)(1 + o(1)).$$

Therefore,

$$\int_{x_\theta}^{\infty} t^\theta g_2(t)dt = b_1(\theta)(1+o(1)).$$

This and (2.29) conclude that

$$\int_0^{\infty} t^\theta g_2(t)dt = b_1(\theta)(1+o(1))$$

as $\theta \to \infty$. □

Lemma 2.12. *Let $v_0 > 0$ be a fixed number. Let Y_j be as in Lemma 2.5. Fix $\delta \in (0,1)$. Then, as $n \to \infty$,*

$$P\Big(\frac{2nY_j - 2\sqrt{j(v+j)}}{\sqrt{2j+v}} > t\Big) = (1+o(1))(1-\Phi(t)) + O\Big(\frac{1}{n^2}\Big)$$

uniformly over $t \in \mathbb{R}$, $\delta n \le j \le n$ and $0 \le v \le v_0$.

Proof. The proof is divided into a few steps.

Step 1. Reduction of Y_j to a Gamma distribution. Let Gamma(α, β) denote a Gamma distribution with density function given by

$$\gamma_{\alpha,\beta}(t) = \frac{1}{\beta^\alpha \Gamma(\alpha)} t^{\alpha-1} e^{-t/\beta}, \quad t > 0,$$

where $\alpha > 0$ and $\beta > 0$ are parameters. It follows from part (b) and part (c) in Lemma 2.1 that

$$\int_0^{\infty} t^\theta K_v(t)dt < \infty \tag{2.30}$$

for any $\theta \ge v$. Since we consider $v \in [0, v_0]$ here, the above integral is well defined for $\theta \ge v_0 =: \theta_0$. For each $v \in [0, v_0]$, define

$$g_1(t) = t^{-1/2}e^{-t} \quad \text{and} \quad g_{v2}(t) = \sqrt{\frac{2}{\pi}} K_v(t), \quad t > 0.$$

Then, for fixed $v \in [0, v_0]$, we have from (2.1) that $g_1(t) \sim g_{v2}(t)$ as $t \to \infty$. Furthermore, we have for $\theta \ge \theta_0$ that

$$b_1(\theta) := \int_0^{\infty} t^\theta g_1(t)dt = \int_0^{\infty} t^{\theta-1/2}e^{-t}dt = \Gamma(\theta + \frac{1}{2})$$

and

$$b_{v2}(\theta) := \int_0^{\infty} t^\theta g_{v2}(t)dt = \sqrt{\frac{2}{\pi}} \int_0^{\infty} t^\theta K_v(t)dt < \infty$$

from (2.30). From Chapter 6 in Abramowitz and Stegun[33] we also have that

$$\Gamma\Big(\theta+\frac{1}{2}\Big)\sim e^{-\theta-\frac{1}{2}}\Big(\theta+\frac{1}{2}\Big)^{\theta}(2\pi)^{1/2}$$

as $\theta\to\infty$, which implies that $\lim_{\theta\to\infty}\frac{\log b_1(\theta)}{\theta}=\infty$.

In virtue of Lemma 2.11(b), for fixed $v\in[0,v_0]$,

$$b_{v2}(\theta)\sim b_1(\theta) \tag{2.31}$$

as $\theta\to\infty$; by Lemma 2.11(a), as $x\to\infty$,

$$\int_x^\infty t^\theta g_{v2}(t)dt\sim b_1(\theta)\int_x^\infty \gamma_{\theta+\frac{1}{2},1}(t)dt$$

uniformly over all $\theta\ge\theta_0$. Note that for each $t>0$, $\cosh(tv)$ is increasing in $v\in[0,\infty)$. From Lemma 2.1(a), we know $K_v(x)$ is increasing in $v\in[0,\infty)$ for fixed $x>0$, so is $g_{v2}(x)$. This implies that

$$\begin{aligned}
&\frac{b_{02}(\theta)}{b_1(\theta)}\le\frac{b_{v2}(\theta)}{b_1(\theta)}\le\frac{b_{v_02}(\theta)}{b_1(\theta)};\\
&\frac{1}{I(x,\theta)}\int_x^\infty t^\theta g_{02}(t)dt\le\frac{1}{I(x,\theta)}\int_x^\infty t^\theta g_{v2}(t)dt\le\frac{1}{I(x,\theta)}\int_x^\infty t^\theta g_{v_02}(t)dt
\end{aligned} \tag{2.32}$$

for any $0\le v\le v_0$, where $I(x,\theta):=b_1(\theta)\int_x^\infty\gamma_{\theta+\frac{1}{2},1}(t)dt$. Then (2.31) holds uniformly over $0\le v\le v_0$. Second, the left and the right expressions of (2.32) do not depend on v, and both converge uniformly in $\theta\in[\theta_0,\infty)$. This guarantees the uniform convergence of the middle term in (2.32) to one over $0\le v\le v_0$ and $\theta\ge\theta_0$ as $x\to\infty$. Thus, we conclude that

$$\begin{aligned}
\int_x^\infty\frac{t^\theta g_{v2}(t)}{b_{v2}(\theta)}dt&\sim\frac{b_1(\theta)}{b_{v2}(\theta)}\int_x^\infty\gamma_{\theta+\frac{1}{2},1}(t)dt\\
&\sim\int_x^\infty\gamma_{\theta+\frac{1}{2},1}(t)dt
\end{aligned}$$

uniformly over $0\le v\le v_0$ as $x\to\infty$ and $\theta\to\infty$, that is,

$$\int_x^\infty\frac{t^\theta g_{v2}(t)}{b_{v2}(\theta)}dt=(1+o(1))\int_x^\infty\gamma_{\theta+\frac{1}{2},1}(t)dt$$

uniformly over $0\le v\le v_0$ as $x\to\infty$ and $\theta\to\infty$. By Lemma 2.5,

$$\frac{t^{2j+v-1}g_{v2}(t)}{b_{v2}(2j+v-1)}=\sqrt{\frac{2}{\pi}}\frac{t^{2j+v-1}K_v(t)}{b_{v2}(2j+v-1)}$$

is the density of $2nY_j$ for each $1\le j\le n$. Immediately, for fixed $\delta\in(0,1)$ and any divergent sequence x_n with $\lim_{n\to\infty}x_n=\infty$, we have that

$$P(2nY_j>x)=\int_x^\infty\sqrt{\frac{2}{\pi}}\frac{t^{2j+v-1}K_v(t)}{b_{v2}(2j+v-1)}dt=(1+o(1))\int_x^\infty\gamma_{2j+v-\frac{1}{2},1}(t)dt \tag{2.33}$$

uniformly over $\delta n \le j \le n$, $0 \le v \le v_0$ and $x \ge x_n$ as $n \to \infty$.

Step 2. Estimation of the probability on the right-hand side of (2.33). Let S_m denote a random variable with density $\gamma_{m,1}(t)$. Note that S_m can be written as the sum of m i.i.d. random variables having a Gamma(1,1) distribution, and Gamma(1,1) has mean 1 and variance 1. Then it follows from Theorem 1 on page 217 of the book by Petrov[35] that, for any sequence of positive numbers τ_m such that $\tau_m = o(m^{1/6})$,

$$P(S_m > m + \sqrt{m}x) = (1 + o(1))(1 - \Phi(x)) \quad \text{uniformly over } |x| \le \tau_m \tag{2.34}$$

and

$$P(S_m < m - \sqrt{m}x) = (1 + o(1))(1 - \Phi(x)) \quad \text{uniformly over } |x| \le \tau_m \tag{2.35}$$

as $m \to \infty$. Now set $\tau_m = m^{1/7}$. Since

$$\begin{aligned}
P(S_m > m + \sqrt{m}x) &\le P(S_m > m + \sqrt{m}m^{1/7}) \\
&= (1 + o(1))(1 - \Phi(m^{1/7})) \\
&= O\Big(\frac{1}{m^2}\Big)
\end{aligned}$$

uniformly for $x \ge m^{1/7}$. We have from (2.35)

$$\begin{aligned}
P(S_m \le m + \sqrt{m}x) &\le P(S_m \le m - \sqrt{m}m^{1/7}) \\
&= (1 + o(1))(1 - \Phi(m^{1/7})) \\
&= O\Big(\frac{1}{m^2}\Big)
\end{aligned}$$

for $x \le -m^{1/7}$ and hence from (2.34)

$$\begin{aligned}
P(S_m > m + \sqrt{m}x) &= 1 - P(S_m \le m + \sqrt{m}x) \\
&= 1 + O\Big(\frac{1}{m^2}\Big) \\
&= 1 - \Phi(x) + \Phi(x) + O\Big(\frac{1}{m^2}\Big) \\
&= 1 - \Phi(x) + O\Big(\frac{1}{m^2}\Big)
\end{aligned}$$

uniformly for $x \le -m^{1/7}$ as $m \to \infty$, where in the last step we use the fact that $\Phi(x) \le \exp\{-m^{2/7}/2\}$ for all $x \le -m^{1/7}$ and $m \ge 1$ based on (2.26). Then we conclude

$$P\big(S_m > m + \sqrt{m}x\big) = (1 + o(1))(1 - \Phi(x)) + O\Big(\frac{1}{m^2}\Big) \tag{2.36}$$

uniformly over $x \in \mathbb{R}$ as $m \to \infty$.

Let m_j denote the integer such that $m_j - 1 < 2j + v - \frac{1}{2} \le m_j$, and set $m_j' = m_j - 1$. Then we have from (2.36) that

$$P\big(S_{m_j} > m_j + \sqrt{m_j}x\big) = (1 + o(1))(1 - \Phi(x)) + O\Big(\frac{1}{n^2}\Big) \tag{2.37}$$

and

$$P\big(S_{m_j'} > m_j' + \sqrt{m_j'}x\big) = (1 + o(1))(1 - \Phi(x)) + O(\frac{1}{n^2})$$

uniformly over $x \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$. Note that

$$P(S_{m_j} > 2\sqrt{j(j+v)} + t\sqrt{2j+v}) = P(S_{m_j} > m_j + \sqrt{m_j}x(t))$$

where

$$\begin{aligned} x(t) &= t\sqrt{\frac{2j+v}{m_j}} + \frac{2\sqrt{j(j+v)} - m_j}{\sqrt{m_j}} \\ &= t\Big[1 + O\Big(\frac{1}{n}\Big)\Big] + O\Big(\frac{1}{\sqrt{n}}\Big) \end{aligned}$$

uniformly over $\delta n \le j \le n$ as $n \to \infty$. Then, combining Lemma 2.7 and (2.37) we obtain

$$\begin{aligned} &P\big(S_{m_j} > 2\sqrt{j(j+v)} + t\sqrt{2j+v}\,\big) \\ &= (1 + o(1))(1 - \Phi(t)) + O\Big(\frac{1}{n^2}\Big) \end{aligned} \tag{2.38}$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$. Similarly, we have

$$\begin{aligned} &P\big(S_{m_j'} > 2\sqrt{j(j+v)} + t\sqrt{2j+v}\,\big) \\ &= (1 + o(1))(1 - \Phi(t)) + O\Big(\frac{1}{n^2}\Big) \end{aligned} \tag{2.39}$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$.

In view of Lemma 2.4 we have

$$\int_x^\infty \gamma_{\alpha_1,1}(t)dt \ge \int_x^\infty \gamma_{\alpha_2,1}(t)dt$$

whenever $\alpha_1 > \alpha_2 > 0$ and thus

$$\begin{aligned} P(S_{m_j'} > 2\sqrt{j(j+v)} + t\sqrt{2j+v}) &\le \int_{2\sqrt{j(j+v)}+t\sqrt{2j+v}}^\infty \gamma_{2j-\frac{1}{2}+v,1}(s)ds \\ &\le P(S_{m_j} > 2\sqrt{j(j+v)} + t\sqrt{2j+v}), \end{aligned}$$

which coupled with (2.38) and (2.39) yields that

$$\int_{2\sqrt{j(j+v)}+t\sqrt{2j+v}}^{\infty} \gamma_{2j+v-\frac{1}{2},1}(s)ds = (1+o(1))(1-\Phi(t)) + O(\frac{1}{n^2})$$

uniformly over $t \in \mathbb{R}$ and $\delta n \le j \le n$ as $n \to \infty$. From (2.33) we conclude

$$P\big(2nY_j > 2\sqrt{j(j+v)} + t\sqrt{2j+v}\,\big) = (1+o(1))(1-\Phi(t)) + O\Big(\frac{1}{n^2}\Big) \tag{2.40}$$

uniformly over $t \ge -n^{1/3}$ and $\delta n \le j \le n$ as $n \to \infty$, where the choice of "$-n^{1/3}$" is such that

$$\inf_{\delta n \le j \le n,\, t \ge -n^{1/3}} \big\{2\sqrt{j(j+v)} + t\sqrt{2j+v}\,\big\} \to \infty \tag{2.41}$$

as $n \to \infty$ required by (2.33). To complete the proof, we need to verify that (2.40) also holds uniformly over $t < -n^{1/3}$ and $\delta n \le j \le n$ as $n \to \infty$. In fact, by taking $t = -n^{1/3}$ in (2.41) and by the same argument as obtaining (2.40), we have

$$\begin{aligned}
P\big(2nY_j > 2\sqrt{j(j+v)} + t\sqrt{2j+v}\,\big) &\ge P\big(2nY_j > 2\sqrt{j(j+v)} - n^{1/3}\sqrt{2j+v}\,\big)\\
&= (1+o(1))\big(1-\Phi(-n^{1/3})\big) + O\Big(\frac{1}{n^2}\Big)\\
&= 1+o(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
P\big(2nY_j > 2\sqrt{j(j+v)} + t\sqrt{2j+v-1}\big) &= 1+o(1)\\
&= (1+o(1))(1-\Phi(t)) + (1+o(1))\Phi(t)\\
&= (1+o(1))(1-\Phi(t)) + O\Big(\frac{1}{n^2}\Big)
\end{aligned}$$

uniformly over $t < -n^{1/3}$ and $\delta n \le j \le n$ as $n \to \infty$. This completes the proof. □

Lemma 2.13. *(Lemma 2.2 from Jiang and Qi[20]). Let* $\{j_n, n \ge 1\}$ *and* $\{x_n, n \ge 1\}$ *be positive numbers with* $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to\infty} j_n x_n^{-1/2}(\log x_n)^{1/2} = \infty$. *Let* $a(y)$ *and* $b(y)$ *be as in (1.4). For fixed* $y \in \mathbb{R}$, *if* $\{c_{n,j}, 1 \le j \le j_n, n \ge 1\}$ *are real numbers such that* $\lim_{n\to\infty} \max_{1\le j\le j_n} |c_{n,j} x_n^{1/2} - 1| = 0$, *then*

$$\lim_{n\to\infty} \sum_{j=1}^{j_n} \big[1 - \Phi((j-1)c_{n,j} + a(x_n) + b(x_n)y)\big] = e^{-y}.$$

Lemma 2.14. *Let $v = v_n$ be a sequence of nonnegative numbers. Let $a(y)$ and $b(y)$ be as in (1.4). Recall Λ in (1.5). If $\mathbf{u}_1, \cdots, \mathbf{u}_n$ have density $f(u_1, \cdots, u_n)$ as in (2.3), then*

$$\frac{1}{b(\frac{n(n+v)}{2n+v})}\Big[\frac{2n\max_{1\le j\le n}|\mathbf{u}_j| - 2\sqrt{n(n+v)}}{\sqrt{2n+v}} - a\Big(\frac{n(n+v)}{2n+v}\Big)\Big] \xrightarrow{d} \Lambda.$$

Proof. Set $x_n = \frac{n(n+v)}{2n+v}$, $a_n = a(x_n)$ and $b_n = b(x_n)$. Let $Y_1, \cdots, Y_n$ be as in Lemma 2.5. Then from the lemma, it suffices to show that

$$\frac{1}{b_n}\Big[\frac{2n\max_{1\le j\le n} Y_j - 2\sqrt{n(n+v)}}{\sqrt{2n+v}} - a_n\Big] \xrightarrow{d} \Lambda$$

or equivalently

$$\prod_{j=1}^{n} P\big(2nY_{n-j+1} \le 2\sqrt{n(n+v)} + \sqrt{2n+v}(a_n + b_n y)\big) \to \exp(-e^{-y}) \tag{2.42}$$

for any $y \in \mathbb{R}$ as $n \to \infty$.

To prove (2.42), it suffices to verify that for every subsequence $\{n'\}$ of $\{n\}$, there exists its further subsequence, say $\{n''\}$, such that (2.42) is true along the subsequence $\{n''\}$. subsequences $\{n''\}$ are selected in such a way that $v_{n''}$ has a limit, say v_1, where $v_1 \in [0, \infty]$. Verification of (2.42) along a subsequence $\{n''\}$ with $\lim_{n''\to\infty} v_{n''} =: v_1 \in [0, \infty]$ is similar to proving (2.42) along the entire sequence with $\lim_{n\to\infty} v_n \in [0, \infty]$. Therefore, for brevity, we will show (2.42) under each of the following conditions: (i) $\lim_{n\to\infty} v_n \in [0, \infty)$; (ii) $\lim_{n\to\infty} v_n = \infty$.

Fix $y \in \mathbb{R}$. Set $a_{nj} = P(2nY_{n-j+1} > 2\sqrt{n(n+v)} + \sqrt{2n+v}(a_n + b_n y))$, $1 \le j \le n$. According to Lemma 2.6, $a_{n1} \ge a_{n2} \ge \cdots \ge a_{nn}$. We need to prove

$$\lim_{n\to\infty} \prod_{j=1}^{n}(1 - a_{nj}) = \exp(-e^{-y}),$$

which is equivalent to

$$\lim_{n\to\infty} \sum_{j=1}^{n} a_{nj} = e^{-y} \tag{2.43}$$

if we can show

$$a_{n1} = \max_{1\le j\le n} a_{nj} \to 0 \quad \text{as } n \to \infty. \tag{2.44}$$

Note that $\frac{n}{2} \le x_n \le n$. It is seen that $a_n + b_n y \to \infty$ and is of order $\sqrt{\log n}$. Under condition (i), there exists a $v_0 > 0$ such that $0 \le v_n \le v_0$ for all $n \ge 1$. Then from Lemma 2.12,

$a_{n1} = (1 + o(1))(1 - \Phi(a_n + b_n y)) + O(n^{-2}) \to 0$ as $n \to \infty$. The same is true by applying Lemma 2.8 under condition (ii) where $v_n \to \infty$. This proves (2.44).

To prove (2.43), define $j_n = [n^{2/3}] + 2$, where $[x]$ denotes the integer part of x. Set

$$\bar{Y}_j = \frac{2nY_j - 2\sqrt{j(j+v)}}{\sqrt{2j+v}}$$

for $1 \le j \le n$. Then

$$\begin{aligned} a_{nj} &= P\big(2nY_{n-j+1} > 2\sqrt{n(n+v)} + \sqrt{2n+v}(a_n + b_n y)\big) \\ &= P(\bar{Y}_{n-j+1} > d_{nj} + a_n + b_n y) \end{aligned}$$

where

$$\begin{aligned} d_{nj} :&= \frac{2\sqrt{n(n+v)} - 2\sqrt{(n-(j-1))(n+v-(j-1))}}{\sqrt{2n+v-2(j-1)}} \\ &\quad + \Big[\frac{\sqrt{2n+v}}{\sqrt{2n+v-2(j-1)}} - 1\Big](a_n + b_n y) \\ &= 2\sqrt{x_n}\frac{1 - \sqrt{(1-\frac{j-1}{n})(1-\frac{j-1}{n+v})}}{\sqrt{1-\frac{2(j-1)}{2n+v}}} + \Big[\Big(1 - \frac{2(j-1)}{2n+v}\Big)^{-1/2} - 1\Big](a_n + b_n y). \end{aligned}$$

Write $(1+x)^{\alpha} = 1 + \alpha x + \epsilon_{\alpha}(x)$. Then, there exist constants $C_{\alpha} > 0$ and $x_0 > 0$ such that $|\epsilon_{\alpha}(x)| \le C_{\alpha} x^2$ as $|x| \le x_0$. Taking $\alpha = 1/2$ and $\alpha = -1/2$, respectively, and using the trivial formula $(1+a)(1+b) = 1 + a + b + ab$, we obtain

$$d_{nj} = \sqrt{x_n}\Big[\frac{j-1}{n} + \frac{j-1}{n+v} + \epsilon_{nj}^2\Big] + \epsilon'_{nj}(a_n + b_n y)$$

where $|\epsilon_{nj}| + |\epsilon'_{nj}| \le C\frac{j-1}{n}$ uniformly over $1 \le j \le j_n$ and n is sufficiently large, and C is a constant not depending on n or j. Notice $a_n = O(\sqrt{\log n})$ and $b_n \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned} d_{nj} &= (j-1)\Big[\sqrt{x_n}\Big(\frac{2n+v}{n(n+v)} + O\Big(\frac{j_n}{n^2}\Big)\Big) + O\Big(\frac{\sqrt{\log n}}{n}\Big)\Big] \\ &= \frac{j-1}{\sqrt{x_n}}\Big[1 + O\Big(\frac{j_n x_n}{n^2}\Big) + O\Big(\frac{\sqrt{x_n \log n}}{n}\Big)\Big] \\ &= \frac{j-1}{\sqrt{x_n}}(1 + o(1)) \qquad (2.45) \end{aligned}$$

uniformly over $1 \le j \le j_n$ as $n \to \infty$. Define $c_{n1} = \sqrt{x_n}$ and $c_{nj} = \frac{d_{nj}}{j-1}$ for $2 \le j \le j_n$. Then we have

$$a_{nj} = P(\bar{Y}_{n-j+1} > (j-1)c_{nj} + a_n + b_n y)$$

and $\lim_{n\to\infty}\max_{1\le j\le j_n}|c_{nj}\sqrt{x_n}-1|=0$. By applying Lemmas 2.8 and 2.12, respectively, we have

$$a_{nj}=(1+o(1))\big[1-\Phi((j-1)c_{nj}+a_n+b_ny)\big]+O\Big(\frac{1}{n^2}\Big) \tag{2.46}$$

uniformly over $1\le j\le j_n$ as $n\to\infty$. Consequently, it follows from Lemma 2.13 that

$$\lim_{n\to\infty}\sum_{j=1}^{j_n}a_{nj}=e^{-y}. \tag{2.47}$$

Moreover, by (2.45) and (2.46),

$$\begin{aligned}\sum_{j=j_n+1}^{n}a_{nj}&\le(n-j_n)a_{nj_n}\\&=(1+o(1))\cdot n\big[1-\Phi((j_n-1)c_{nj_n}+a_n+b_ny)\big]+O\Big(\frac{1}{n}\Big)\\&\le(1+o(1))\cdot n\Big[1-\Phi\Big(\frac{j_n-1}{2\sqrt{x_n}}\Big)\Big]+O\Big(\frac{1}{n}\Big)\end{aligned}$$

where the facts $(j_n-1)c_{nj_n}=d_{nj_n}$, $a_n=O(\sqrt{\log n})$ and $b_n\to 0$ are used. Noting $(j_n-1)/\sqrt{x_n}>n^{1/6}/2$ as n is large enough, the sum above goes to zero by (2.26). This together with (2.47) yields (2.43). □

2.2. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $\mathbf{u}_1,\cdots,\mathbf{u}_n$ have density $f(u_1,\cdots,u_n)$ as in (2.3). Review $a(y)$ and $b(y)$ from (1.4). As in Lemma 2.14, set $x_n=\frac{n(n+v)}{2n+v}$, $a_n=a(x_n)$ and $b_n=b(x_n)$. Then

$$\Lambda_n:=\frac{1}{b_n}\Big[\frac{2n\max_{1\le j\le n}|\mathbf{u}_j|-2\sqrt{n(n+v)}}{\sqrt{2n+v}}-a_n\Big]\xrightarrow{d}\Lambda \tag{2.48}$$

as $n\to\infty$, where Λ is as in (1.5). Solve for $\max_{1\le j\le n}|\mathbf{u}_j|$ to get

$$\begin{aligned}\max_{1\le j\le n}|\mathbf{u}_j|&=\sqrt{\frac{n+v}{n}}+\frac{\sqrt{2n+v}}{2n}(a_n+b_n\Lambda_n)\\&=\sqrt{\frac{n+v}{n}}\left[1+\frac{1}{2}\sqrt{\frac{2n+v}{n(n+v)}}(a_n+b_n\Lambda_n)\right].\end{aligned}$$

By the formula $(1+x)^{1/2} = 1 + \frac{1}{2}x + O(x^2)$ as $x \to 0$, we have

$$\begin{aligned}
\big(\max_{1\le j\le n} |\mathbf{u}_j|\big)^{1/2} &= \Big(\frac{n+v}{n}\Big)^{1/4}\left[1 + \frac{1}{2}\sqrt{\frac{2n+v}{n(n+v)}}(a_n + b_n\Lambda_n)\right]^{1/2} \\
&= \Big(\frac{n+v}{n}\Big)^{1/4}\left[1 + \frac{1}{4}\sqrt{\frac{2n+v}{n(n+v)}}(a_n + b_n\Lambda_n) + O_p\Big(\frac{\log n}{n}\Big)\right] \\
&= \Big(\frac{n+v}{n}\Big)^{1/4} + \frac{1}{4}\frac{(2n+v)^{1/2}}{n^{3/4}(n+v)^{1/4}}(a_n + b_n\Lambda_n) + O_p\Big(\frac{(n+v)^{1/4}\log n}{n^{5/4}}\Big),
\end{aligned}$$

where we have used the following facts in the second equality:

$$a_n \sim \sqrt{\log n} \text{ and } b_n \sim (\log n)^{-1/2} \to 0 \text{ as } n \to \infty.$$

The proof of the above equation is trivial since $\frac{n}{2} \le x_n = \frac{n(n+v)}{2n+v} \le n$. For any sequence of positive numbers $\{c_n\}$, notation $O_p(c_n)$ denotes a sequence of random variables bounded by c_n in probability, that is, if $Y_n = O_p(c_n)$, then $\lim_{s\to\infty} \limsup_{n\to\infty} P(|Y_n| > sc_n) = 0$. It follows that

$$\begin{aligned}
&\frac{1}{b\big(\frac{n(n+v)}{2n+v}\big)}\Big[\frac{\big(\max_{1\le j\le n}|\mathbf{u}_j|\big)^{1/2} - (\frac{n+v}{n})^{1/4}}{\frac{1}{4}\frac{(2n+v)^{1/2}}{n^{3/4}(n+v)^{1/4}}} - a\Big(\frac{n(n+v)}{2n+v}\Big)\Big] \\
&= \Lambda_n + O_p\Big(\frac{(\log n)^{3/2}}{n^{1/2}}\Big)
\end{aligned}$$

as $n \to \infty$. Then the desired conclusion follows from Lemma 2.2 and (2.48). □

Proof of Corollary 1. Observe that

$$\begin{aligned}
&\Big(\frac{n+v}{n}\Big)^{1/4} = 1 + O\big(\frac{1}{n}\big), \qquad \Big[\frac{1}{4}\frac{(2n+v)^{1/2}}{n^{3/4}(n+v)^{1/4}}\Big]^{-1} = 2\sqrt{2n} + O\big(\frac{1}{n^{1/2}}\big), \\
&\frac{1}{b\big(\frac{n(n+v)}{2n+v}\big)} = \Big(\log\frac{n}{2}\Big)^{1/2} + O\big(\frac{1}{n}\big)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{b\big(\frac{n(n+v)}{2n+v}\big)} a\Big(\frac{n(n+v)}{2n+v}\Big) &= \log y - \log\big(\sqrt{2\pi}\log y\big)\Big|_{y=\frac{n(n+v)}{2n+v}} \\
&= \log\frac{n}{2} - \log\big(\sqrt{2\pi}\log\frac{n}{2}\big) + o(1)
\end{aligned}$$

since $\log y = \log\frac{n}{2} + O(\frac{1}{n})$ at $y = \frac{n(n+v)}{2n+v}$. Take these quantities into (1.6) to see

$$\begin{aligned}
&\Big[\Big(8n\log\frac{n}{2}\Big)^{1/2} + O\Big(\frac{(\log n)^{1/2}}{n^{1/2}}\Big)\Big]\cdot\big(\max_{1\le j\le n}|\mathbf{z}_j| - 1 + O\big(\frac{1}{n}\big)\big) \\
&-\log\frac{n}{2} + \log\big(\sqrt{2\pi}\log n\big) + o(1)
\end{aligned}$$

converges weakly to Λ. This implies

$$\Big[\Big(8n\log\frac{n}{2}\Big)^{1/2}+O\Big(\frac{(\log n)^{1/2}}{n^{1/2}}\Big)\Big]\cdot\big(\max_{1\le j\le n}|\mathbf{z}_j|-1\big)$$
$$-\log\frac{n}{2}+\log\big(\sqrt{2\pi}\log\frac{n}{2}\big)$$

converges weakly to Λ. By the Slutsky lemma, we know $\max_{1\le j\le n}|\mathbf{z}_j|-1\to 0$ in probability. Therefore

$$\Big(8n\log\frac{n}{2}\Big)^{1/2}\cdot\max_{1\le j\le n}|\mathbf{z}_j|-\Big(8n\log\frac{n}{2}\Big)^{1/2}-\log\frac{n}{2}+\log\big(\sqrt{2\pi}\log\frac{n}{2}\big)$$

converges weakly to Λ. □

Recall Lemma 2.2. The joint density of $\mathbf{u}_j=\mathbf{z}_j^2$, $1\le j\le n$, is given by

$$f(u_1,\cdots,u_n)=C\cdot\prod_{1\le j<k\le n}|u_j-u_k|^2\cdot\big[\prod_{j=1}^{n}\varphi_n(|u_j|)\big]du_1\cdots du_n$$

for all $u_j\in\mathbb{C}$, $1\le j\le n$, where $\varphi_n(y)=y^vK_v\big(2ny\big)$ for $y>0$ and $du_j=dxdy$ for $u_j=x+iy$. Then $\lambda_n(du):=\varphi_n(|u|)du$ is a finite measure, that is

$$\lambda_n(\mathbb{C})=\int_{\mathbb{C}}\varphi_n(|u|)du=\int_0^{\infty}\int_0^{2\pi}r\varphi_n(r)d\theta dr=2\pi\int_0^{\infty}r\varphi_n(r)dr<\infty$$

from (b) and (c) in Lemma 2.1.

Lemma 2.15. *(Proposition 2 from Jiang and Qi[27]). Let $\mathbf{u}_1,\cdots,\mathbf{u}_n$ have density function $f(u_1,\cdots,u_n)$ as in (2.3). For any measurable function $h:\ \mathbb{C}\to\mathbb{R}$ with $\sup_{u\in\mathbb{C}}|h(u)|\le 1$, we have*

$$\mathrm{E}\Big[\sum_{j=1}^{n}\big(h(\mathbf{u}_j)-Eh(\mathbf{u}_j)\big)\Big]^4\le Kn^2$$

for $n\ge 1$, where K is a constant not depending on n, $\varphi(u)$ or $h(u)$.

Lemma 2.16. *Let $v=v_n$ be a sequence of nonnegative numbers. Let $\mathbf{u}_1,\cdots,\mathbf{u}_n$ have density function $f(u_1,\cdots,u_n)$ as in (2.3). Let τ_n be the empirical probability measure of $[n/(n+v_n)]^{1/2}\mathbf{u}_j$, $1\le j\le n$. Assume $\lim_{n\to\infty}v_n/n=\alpha\in[0,\infty]$. Then, with probability one, $\tau_n\xrightarrow{d}\tau$ as $n\to\infty$, where τ is a probability measure on $\mathbb{C}$ with density function*

$$\Psi_\alpha(u)=\frac{1}{\pi}\cdot\frac{1}{\sqrt{a+b|u|^2}},\quad |u|\le 1,$$

$a=\alpha^2/(1+\alpha)^2$ and $b=4/(1+\alpha)$.

Proof. Let τ_n' be the empirical probability measure of $[n/(n+v_n)]^{1/2}|\mathbf{u}_j|$, $1 \le j \le n$. We first show that

$$\tau_n' \xrightarrow{d} \tau' \tag{2.49}$$

where τ' has density function $l(r) := \frac{2r}{\sqrt{a+br^2}}$, $0 \le r \le 1$. If this is true, by Theorem 1 from Jiang and Qi[27], τ_n converges weakly to the distribution of $Re^{i\Theta}$, where the random vector $(R,\Theta)'$ has the product law of τ' and the uniform distribution on $[0,2\pi]$. Therefore, for any bounded and continuous function $g(z)$ defined on $\mathbb{C}$, with probability one,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{C}} g(u)\,\tau_n(du) &= Eg(Re^{i\Theta})\\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 g(re^{i\theta})\frac{2r}{\sqrt{a+br^2}}\,dr\,d\theta\\ &= \int_0^1 g(re^{i\theta})\frac{2r}{\sqrt{a+br^2}}\,dr\,d\theta.\end{aligned}$$

On the other hand, by the polar transformation $u = re^{i\theta}$,

$$\begin{aligned}\int_{\mathbb{C}} g(u)\,\tau(du) &= \frac{1}{\pi}\int_{|z|\le 1} g(u)\frac{1}{\sqrt{a+b|u|^2}}\,du\\ &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 g(re^{i\theta})\frac{r}{\sqrt{a+br^2}}\,dr\,d\theta\\ &= \int_0^1 g(re^{i\theta})\frac{2r}{\sqrt{a+br^2}}\,dr\,d\theta.\end{aligned}$$

The two assertions assure that $\tau_n \xrightarrow{d} \tau$ as $n \to \infty$. Now we prove (2.49). To simplify notation, set

$$\varsigma_n := \Big(\frac{n}{n+v_n}\Big)^{1/2}. \tag{2.50}$$

It is enough to show

$$\frac{1}{n}\sum_{j=1}^n I\big(\varsigma_n|\mathbf{u}_j| \le y\big) \to \tau'([0,y])$$

for each $y \ge 0$. Regarding $I\big(\varsigma_n|\mathbf{u}_j| \le y\big)$ as a function of $\mathbf{u}_j$, which takes values in $[0,1]$. Recall the Markov inequality $P(|V| \ge t) \le t^{-4}E(V^4)$ for any random variable V and constant $t > 0$. By Lemma 2.15 and the Borel-Cantelli lemma, with probability one,

$$\frac{1}{n}\sum_{j=1}^n \big[I\big(\varsigma_n|\mathbf{u}_j| \le y\big) - P\big(\varsigma_n|\mathbf{u}_j| \le y\big)\big] \to 0$$

as $n \to \infty$. Therefore, to complete the proof of (2.49), by Lemma 2.5 it remains to check that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} P\big(\varsigma_n Y_j \le y\big) = \begin{cases} 1 & \text{if } y \ge 1; \\ \int_0^y \frac{2r}{\sqrt{a+br^2}}\, dr & \text{if } 0 < y < 1 \end{cases} \tag{2.51}$$

where Y_j, $1 \le j \le n$ are random variables appeared in Lemma 2.5. Now we use Lemmas 2.8 and 2.12 to estimate the above probabilities.

By using the subsequence argument similar to the proof of (2.42) in the proof of Lemma 2.14, it suffices to show that (2.51) holds under condition (I): $\lim_{n\to\infty} v_n/n = \alpha \in [0, \infty]$ with $v_n \to \infty$, or condition (II): $\lim_{n\to\infty} v_n \in [0, \infty)$. We will only prove (I) by using Lemma 2.8. The item (II) can be proved by using the same argument with "Lemma 2.8" replaced by "Lemma 2.12", and hence is omitted.

Now we assume condition (I): $v_n \to \infty$ and $\alpha_n := \frac{v_n}{n} \to \alpha \in [0, \infty]$ as $n \to \infty$. We prove this via a few steps.

Step 1. $y \ge 1$. Obviously,

$$\frac{1}{n} \sum_{j=1}^{n} P\big(\varsigma_n Y_j > y\big) \le \delta + \frac{1}{n} \sum_{n\delta \le j \le n} P\big(\varsigma_n Y_j > y\big)$$

for any $\delta \in (0, 1)$. Set

$$\begin{aligned} \varpi_{n,j} &= \frac{2n\varsigma_n^{-1} y - 2\sqrt{j(j+v)}}{\sqrt{2j+v}} \\ &= 2\frac{\sqrt{n(n+v)}\, y - \sqrt{j(j+v)}}{\sqrt{2j+v}}, \quad j = 1, 2, \cdots, n. \end{aligned}$$

From Lemma 2.8,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n\delta}^{n} P\big(\varsigma_n Y_j > y\big) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=n\delta}^{n} \big[1 - \Phi(\varpi_{n,j})\big].$$

Use the facts $\sqrt{2j+v} \le \sqrt{2(n+v)}$ and $\sqrt{j(n+v)} \le \sqrt{n(n+v)}$ to see that

$$\begin{aligned} \varpi_{n,j} &\ge 2\frac{\sqrt{n(n+v)} - \sqrt{j(n+v)}}{\sqrt{2(n+v)}} \\ &\ge \frac{1}{2}\sqrt{n}\delta' \end{aligned}$$

for each $n\delta \le j \le n(1-\delta')$ if $y \ge 1$. It follows that

$$\begin{aligned} \frac{1}{n} \sum_{j=n\delta}^{n} \big[1 - \Phi(\varpi_{n,j})\big] &\le 1 - \Phi\Big(\frac{1}{2}\sqrt{n}\delta'\Big) + \frac{1}{n} \sum_{j=n(1-\delta')}^{n} 1 \\ &\le 1 - \Phi\Big(\frac{1}{2}\sqrt{n}\delta'\Big) + \delta' + \frac{1}{n} \end{aligned}$$

for any $\delta' \in (0, 1/2)$. By sending $n \to \infty$ and then $\delta' \downarrow 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=n\delta}^{n} P\big(\varsigma_n Y_j > y\big) = 0$$

for $y \geq 1$. This shows the first identity in (2.51).

Step 2. $y \in (0, 1)$. It is easy to check that

$$\begin{aligned}\int_0^y \frac{2r}{\sqrt{a+br^2}}\, dr &= \frac{2}{b}\sqrt{a+br^2}\Big|_0^y \\ &= \frac{2}{b}\big(\sqrt{a+by^2} - \sqrt{a}\big) \\ &= \frac{1}{2}(\sqrt{\alpha^2 + 4(1+\alpha)y^2} - \alpha)\end{aligned}$$

for $\alpha \in [0, \infty)$, and

$$\int_0^y \frac{2r}{\sqrt{a+br^2}}\, dr = \int_0^y 2r\, dr = y^2$$

for $\alpha = \infty$. For $y \in [0, 1]$, define

$$F_\alpha(y) = \begin{cases} \frac{1}{2}(\sqrt{\alpha^2 + 4(1+\alpha)y^2} - \alpha), & \text{if } 0 \leq \alpha < \infty; \\ y^2, & \text{if } \alpha = \infty \end{cases}$$

and

$$G_n(y) := \frac{1}{n} \sum_{j=1}^{n} P\big(\varsigma_n Y_j \leq y\big) \tag{2.52}$$

for $y \in \mathbb{R}$. To complete the proof of (2.51), it is enough to show that

$$\lim_{n\to\infty} G_n(y) = F_\alpha(y), \quad y \in (0, 1). \tag{2.53}$$

Review $\alpha_n = \frac{v_n}{n} \to \alpha \in [0, \infty]$ as $n \to \infty$. Evidently,

$$F_{\alpha_n}(y)(F_{\alpha_n}(y) + \alpha_n) = y^2(1 + \alpha_n) \tag{2.54}$$

and $\lim_{n\to\infty} F_{\alpha_n}(y) = F_\alpha(y)$ for any $y \in (0, 1)$. Since $F_\alpha(0) < F_\alpha(y) < F_\alpha(1)$ for $y \in (0, 1)$, we have $F_\alpha(y) \in (0, 1)$ for $y \in (0, 1)$.

Now fix an $y \in (0, 1)$ and let $\varepsilon \in (0, 1)$ be any small number such that $y(1 + \varepsilon) \in (0, 1)$. Set

$$k_n^+ = [nF_{\alpha_n}(y(1+\varepsilon))] + 1 \quad \text{and} \quad k_n^- = [nF_{\alpha_n}(y(1-\varepsilon))],$$

where $[x]$ denotes the integer part of x as before. Review ς_n from (2.50). We will prove that

$$\lim_{n\to\infty} P\big(Y_{k_n^-} \le y\varsigma_n^{-1}\big) = 1 \quad\text{and}\quad \lim_{n\to\infty} P\big(Y_{k_n^+} \le y\varsigma_n^{-1}\big) = 0. \tag{2.55}$$

Assuming this is true, we have from (2.52) and Lemma 2.6 that

$$\begin{aligned}
&\limsup_{n\to\infty} G_n(y)\\
&= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P\big(Y_k \le y\varsigma_n^{-1}\big)\\
&\le \limsup_{n\to\infty} \frac{1}{n}\Big[\sum_{k=1}^{k_n^+} P\big(Y_k \le y\varsigma_n^{-1}\big) + \sum_{k=k_n^+ +1}^{n} P\big(Y_k \le y\varsigma_n^{-1}\big)\Big]\\
&\le \limsup_{n\to\infty} \frac{1}{n}\Big[k_n^+ + (n-k_n^+)P\big(Y_{k_n^+} \le y\varsigma_n^{-1}\big)\Big]\\
&\le \limsup_{n\to\infty} \Big[\frac{k_n^+}{n} + P\big(Y_{k_n^+} \le y\varsigma_n^{-1}\big)\Big]\\
&= F_\alpha(y(1+\varepsilon)).
\end{aligned}$$

This implies

$$\limsup_{n\to\infty} G_n(y) \le F_\alpha(y)$$

by letting $\varepsilon \downarrow 0$. Similarly, we have

$$\begin{aligned}
\liminf_{n\to\infty} G_n(y) &\ge \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{k_n^-} P\big(Y_k \le y\varsigma_n^{-1}\big)\\
&\ge \liminf_{n\to\infty} \Big[\frac{k_n^-}{n} P\big(Y_{k_n^-} \le y\varsigma_n^{-1}\big)\Big]\\
&= \liminf_{n\to\infty} \frac{k_n^-}{n}\\
&= F_\alpha(y(1-\varepsilon)).
\end{aligned}$$

By letting $\varepsilon \downarrow 0$ we conclude

$$\liminf_{n\to\infty} G_n(y) \ge F_\alpha(y).$$

Therefore, we get (2.53).

Now we proceed to prove (2.55). We will prove the first limit only. The second one can be proved in the same manner. In fact, from (2.54),

$$\begin{aligned}
k_n^-(k_n^- + v_n) &\le nF_{\alpha_n}(y(1-\varepsilon))\big[nF_{\alpha_n}(y(1-\varepsilon)) + v_n\big]\\
&= n^2 F_{\alpha_n}(y(1-\varepsilon))\big[F_{\alpha_n}(y(1-\varepsilon)) + \alpha_n\big] \qquad (2.56)\\
&= n^2 y^2 (1-\varepsilon)^2 (1+\alpha_n)
\end{aligned}$$

and

$$2k_n^- + v_n \leq 2n(1+\alpha_n). \tag{2.57}$$

Set

$$L_n = \frac{2nY_{k_n^-} - 2\sqrt{k_n^-(k_n^- + v_n)}}{\sqrt{2k_n^- + v_n}} \quad \text{and} \quad l_n = \frac{2y\big(n(n+v_n)\big)^{1/2} - 2\sqrt{k_n^-(k_n^- + v_n)}}{\sqrt{2k_n^- + v_n}}.$$

By (2.56) and (2.57),

$$l_n \geq \frac{2ny(1+\alpha_n)^{1/2} - 2ny(1-\varepsilon)\sqrt{1+\alpha_n}}{\sqrt{2n(1+\alpha_n)}} = \varepsilon y\sqrt{2n}.$$

Hence,

$$\begin{aligned} P\big(Y_{k_n^-} \leq y\varsigma_n^{-1}\big) &= P(L_n \leq l_n) \\ &\geq P\big(L_n \leq \varepsilon y\sqrt{2n}\,\big) \\ &= 1 - (1+o(1))\big(1 - \Phi(\varepsilon y\sqrt{2n}\,)\big) + O\Big(\frac{1}{n^2}\Big) \\ &= 1 + o(1) \end{aligned}$$

by Lemma 2.8. This completes the proof of the first limit in (2.55). □

Proof of Theorem 2. Let $v = v_n$ be any sequence of nonnegative numbers. Our assumption is that $\lim_{n\to\infty} \frac{v_n}{n} = \alpha \in [0, \infty]$. Recall (2.50) that

$$\varsigma_n = \Big(\frac{n}{n+v_n}\Big)^{1/2}.$$

Let ρ be the probability distribution with density function Φ_α appeared in the statement of Theorem 2. To prove that $\rho_n \xrightarrow{d} \rho$, it suffices to verify that, with probability one,

$$\frac{1}{n}\sum_{j=1}^{n} h\big(\varsigma_n^{1/2}\mathbf{z}_j\big) \to \int_{\mathbb{C}_1} h(z)\rho(dz) \tag{2.58}$$

for any continuous function $h(z)$ defined on $\mathbb{C}_1 = \{z \in \mathbb{C}; \operatorname{Re}(z) > 0\}$ with $0 \leq h(z) \leq 1$ for all $z \in \mathbb{C}_1$. Here and later on "dz" stands for "$dxdy$" when $z = x + iy$.

Let $\mathbf{u}_1, \cdots, \mathbf{u}_n$ have density function $f(u_1, \cdots, u_n)$ as in (2.3). Assume ρ_1 is a probability measure with density function

$$\Psi_\alpha(z) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{a + b|u|^2}}, \quad |u| \leq 1,$$

where $a = \alpha^2/(1+\alpha)^2$ and $b = 4/(1+\alpha)$. By Lemma 2.16, with probability one,

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{\varsigma_n \mathbf{u}_j} \to \rho_1 \tag{2.59}$$

as $n\to\infty$. Recall $\mathbb{C}_2=\mathbb{C}\backslash\{z\in\mathbb{R};\ z\le 0\}$. Obviously the Lebesgue measure of $\{z\in\mathbb{R};\ z\le 0\}$ is zero. This implies that (2.59) also holds if we restrict $\mathbf{u}_1,\cdots,\mathbf{u}_n$ and ρ_1 on $\mathbb{C}_2$ (one way to show this is the approximation of $\mathbb{C}_2$ via $\mathbb{C}_{2,\epsilon}:=\mathbb{C}\backslash\{s+it;\ s\le\epsilon,\ |t|\le\epsilon\}$ as $\epsilon\downarrow 0$). Therefore, with probability one,

$$\frac{1}{n}\sum_{j=1}^{n}h_1\big(\varsigma_n\mathbf{u}_j\big)\to\int_{\mathbb{C}_2}h_1(z)\rho_1(dz). \tag{2.60}$$

for any continuous function $h_1(z)$ defined on $\mathbb{C}_2$ with $0\le h_1(z)\le 1$ for all $z\in\mathbb{C}_2$. Review Lemma 2.2. Let $u=z^2$ be the one-to-one and onto transform from $\mathbb{C}_1$ to $\mathbb{C}_2$. Denote by $u^{1/2}$ the inverse transform from $\mathbb{C}_2$ to $\mathbb{C}_1$. As seen from (2.4), $u^{1/2}$ is also continuous. Take $h_1=h\circ u^{1/2}:\mathbb{C}_2\to[0,1]$. Then $h_1:\mathbb{C}_2\to[0,1]$ is a continuous functuion. By (2.60),

$$\frac{1}{n}\sum_{j=1}^{n}h\big(\varsigma_n^{1/2}\mathbf{z}_j\big)\to\int_{\mathbb{C}_2}h(z^{1/2})\rho_1(dz).$$

For $z=re^{i\theta}\in\mathbb{C}_2$, it is easy to check that $z^{1/2}=\sqrt{r}e^{i\theta/2}$ with $0<r\le 1$ and $\theta\in(-\pi,\pi)$. Then

$$\begin{aligned}\int_{\mathbb{C}_2}h(z^{1/2})\rho_1(dz)&=\frac{1}{\pi}\int_{-\pi}^{\pi}\int_0^1 h(\sqrt{r}e^{i\theta/2})\frac{1}{\sqrt{a+br^2}}\cdot r\,dr\,d\theta\\&=\frac{4}{\pi}\int_{-\pi/2}^{\pi/2}\int_0^1 h(se^{i\eta})\frac{s^2}{\sqrt{a+bs^4}}\cdot s\,ds\,d\eta\\&=\frac{4}{\pi}\int_{\mathbb{C}_1}h(z)\frac{|z|^2}{\sqrt{a+b|z|^4}}\,dz\end{aligned}$$

where we make a transform that $s=\sqrt{r}$ and $\eta=\theta/2$ in the second equality and last step follows from making a similar polar transformation. The last two assertions yield (2.58). $\square$

2.3. Proofs of Lemmas 2.9 and 2.10

Proof of Lemma 2.9. First,

$$\sqrt{1+\bar{\mu}_{v,j}^2}=\frac{2j+v-1.5}{v}+\frac{v}{2(2j+v-1.5)^2}; \tag{2.61}$$

$$\frac{1}{C_\delta}\cdot\Big(\frac{v+n}{v}\Big)^2\le 1+\bar{\mu}_{v,j}^2\le C_\delta\cdot\Big(\frac{v+n}{v}\Big)^2 \tag{2.62}$$

uniformly over $\delta n\le j\le n$ as $n\to\infty$, where $C_\delta>0$ is a constant depending on δ but not depending on n. Second, write $\frac{2j+v-1.5}{v}=\frac{2j-1.5}{v}+1$. Then

$$\Big(\frac{2j+v-1.5}{v}\Big)^2=\frac{(2j-1.5)(2j+2v-1.5)}{v^2}+1. \tag{2.63}$$

Consequently, from the definition of $\bar{\mu}_{v,j}$,

$$\begin{aligned}
&\bar{\mu}_{v,j}^2 \\
&= \frac{(2j-1.5)(2j+2v-1.5)}{v^2} + \frac{1}{2j+v-1.5} + O\Big(\frac{v^2}{(n+v)^4}\Big) \\
&= \frac{(2j-1.5)(2v+2j-1.5)}{v^2}\Big[1 + \frac{v^2}{(2j-1.5)(2j+v-1.5)(2j+2v-1.5)} \\
&\qquad +O\Big(\frac{v^4}{n(n+v)^5}\Big)\Big].
\end{aligned} \tag{2.64}$$

By (2.11) and (2.61), this again implies

$$\begin{aligned}
\bar{\mu}_{v,j}\tau'(\bar{\mu}_{v,j}) &= \frac{v}{2(2j+v-1.5)^2} - \frac{1}{2v(1+\bar{\mu}_{v,j}^2)} \\
&= \frac{v}{2\kappa^2} - \frac{1}{2v}\Big(\frac{\kappa}{v} + \frac{v}{2\kappa^2}\Big)^{-2} \\
&= \frac{v}{2\kappa^2} - \frac{v}{2\kappa^2}\Big(1 + \frac{v^2}{2\kappa^3}\Big)^{-2} \\
&= O\Big(\frac{v^3}{(n+v)^5}\Big)
\end{aligned} \tag{2.65}$$

where $\kappa := 2j+v-1.5$ and in the last step we use the fact $\frac{v^2}{2\kappa^3} \to 0$ and the fact that $(1+a)^{-2} = 1 + O(a)$ as $a \to 0$. From the definition of $\bar{\mu}_{v,j}$ and (2.63), we see that

$$\frac{\delta^2}{v^2}n(n+v) \le \bar{\mu}_{v,j}^2 \le \frac{4}{v^2}n(n+v) \tag{2.66}$$

uniformly for $\delta n \le j \le n$ as n is sufficiently large. It follows from (2.65) that

$$\frac{\tau'(\bar{\mu}_{v,j})}{\bar{\mu}_{v,j}} = O\Big(\frac{v^5}{n(n+v)^6}\Big) \tag{2.67}$$

holds uniformly for $\delta n \le j \le n$ as $n \to \infty$.

Now, define $x_{v,j}(t) = \bar{\mu}_{v,j}(1 + \bar{\beta}_{v,j}t)$, where

$$\bar{\beta}_{v,j} := \sqrt{\frac{2j+v-1.5}{(2j-1.5)(2v+2j-1.5)}} \tag{2.68}$$

is of order $n^{-1/2}$ uniformly over $\delta n \le j \le n$ as $n \to \infty$. Then

$$x_{v,j}^2(t) = \bar{\mu}_{v,j}^2(1 + \bar{\beta}_{v,j}t)^2, \tag{2.69}$$

$$\begin{aligned}
1 + x_{v,j}^2(t) &= 1 + \bar{\mu}_{v,j}^2 + \bar{\mu}_{v,j}^2(2\bar{\beta}_{v,j}t + \bar{\beta}_{v,j}^2t^2) \\
&= (1 + \bar{\mu}_{v,j}^2)\Big[1 + \frac{\bar{\mu}_{v,j}^2}{1+\bar{\mu}_{v,j}^2}(2\bar{\beta}_{v,j}t + \bar{\beta}_{v,j}^2t^2)\Big].
\end{aligned} \tag{2.70}$$

By the formula $\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2)$ as $x \to 0$, we have

$$\sqrt{1+x_{v,j}^2(t)} = \sqrt{1+\bar{\mu}_{v,j}^2} + \frac{\bar{\mu}_{v,j}^2 \bar{\beta}_{v,j} t}{\sqrt{1+\bar{\mu}_{v,j}^2}}\big(1 + O(n^{-3/8})\big)$$

holds uniformly over $|t| \leq n^{1/8}$ and $\delta n \leq j \leq n$ as $n \to \infty$. Similarly, from the fact that $\frac{1}{1+x} = 1 - x(1+o(1))$ as $x \to 0$, we obtain

$$\begin{aligned}
\frac{1}{1+x_{v,j}^2(t)} &= \frac{1}{1+\bar{\mu}_{v,j}^2} \cdot \Big[1 + \frac{\bar{\mu}_{v,j}^2}{1+\bar{\mu}_{v,j}^2}(2\bar{\beta}_{v,j}t + \bar{\beta}_{v,j}^2 t^2)\Big]^{-1} \\
&= \frac{1}{1+\bar{\mu}_{v,j}^2} \cdot \Big[1 - \frac{\bar{\mu}_{v,j}^2}{1+\bar{\mu}_{v,j}^2}(2\bar{\beta}_{v,j}t + \bar{\beta}_{v,j}^2 t^2)(1+o(1))\Big] \\
&= \frac{1}{1+\bar{\mu}_{v,j}^2} - \frac{\bar{\mu}_{v,j}^2}{(1+\bar{\mu}_{v,j}^2)^2}(2\bar{\beta}_{v,j}t + \bar{\beta}_{v,j}^2 t^2)(1+o(1)).
\end{aligned}$$

Hence

$$\begin{aligned}
&-\frac{1}{2v} \cdot \frac{1}{1+x_{v,j}^2(t)} \\
&= -\frac{1}{2v} \cdot \frac{1}{1+\bar{\mu}_{v,j}^2} + \frac{\bar{\mu}_{v,j}^2 \bar{\beta}_{v,j} t}{\sqrt{1+\bar{\mu}_{v,j}^2}} \cdot \frac{\big(1 + \frac{1}{2}\bar{\beta}_{v,j}t\big)(1+o(1))}{v(1+\bar{\mu}_{v,j}^2)^{3/2}} \\
&= -\frac{1}{2v} \cdot \frac{1}{1+\bar{\mu}_{v,j}^2} + \frac{\bar{\mu}_{v,j}^2 \bar{\beta}_{v,j} t}{\sqrt{1+\bar{\mu}_{v,j}^2}} \cdot O\Big(\frac{v^2}{(v+n)^3}\Big)
\end{aligned}$$

uniformly over $|t| \leq n^{1/8}$ and $\delta n \leq j \leq n$ as $n \to \infty$, where the fact (2.62) is applied in the last step. Then we conclude from (2.11) that

$$\begin{aligned}
x_{v,j}(t)\tau'(x_{v,j}(t)) &= \sqrt{1+x_{v,j}^2} - \frac{1}{2v(1+x_{v,j}^2)} - \frac{2j+v-1.5}{v} \\
&= \bar{\mu}_{v,j}\tau'(\bar{\mu}_{v,j}) + \frac{\bar{\mu}_{v,j}^2 \bar{\beta}_{v,j} t}{\sqrt{1+\bar{\mu}_{v,j}^2}}\big(1 + O(n^{-3/8})\big)
\end{aligned}$$

uniformly over $|t| \leq n^{1/8}$ and $\delta n \leq j \leq n$ as $n \to \infty$. The assertion from (2.61) implies $\sqrt{1+\bar{\mu}_{v,j}^2} = \frac{2j+v-1.5}{v}(1 + O\big(\frac{v^2}{(n+v)^3}\big))$. Furthermore, $x_{v,j}^2(t) = \bar{\mu}_{v,j}^2(1+O(n^{-3/8}))$ from (2.69). With these at hand, we conclude from (2.67) and (2.68) that

$$\begin{aligned}
&\frac{\tau'(x_{v,j}(t))}{x_{v,j}(t)} \\
&= \frac{\bar{\beta}_{v,j} t}{\sqrt{1+\bar{\mu}_{v,j}^2}}(1+O(n^{-3/8})) + \frac{\tau'(\bar{\mu}_{v,j})}{\bar{\mu}_{v,j}}(1+O(n^{-3/8})) \qquad (2.71) \\
&= \frac{vt}{\sqrt{(2j-1.5)(2j+v-1.5)(2j+2v-1.5)}}(1+O(n^{-3/8})) + O\Big(\frac{v^5}{n(n+v)^6}\Big)
\end{aligned}$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$.

Now, set $t = t_\alpha = \frac{\alpha v^4}{\sqrt{n(n+v)^5}}$ in (2.71). If α is large enough, the first term in (2.71) dominates the last one. Then $\tau'(x_{v,j}(t_\alpha)) > 0$ and $\tau'(x_{v,j}(-t_\alpha)) < 0$ uniformly over all $\delta n \le j \le n$ and large n. We obtain (2.20) by the continuity of function $\tau(x)$ and the definitions of $\mu_{v,j}$ from (2.14) and $x_{v,j}(t)$.

Finally, by (2.64),

$$\bar{\mu}_{v,j}^2 = \frac{4j(v+j)+O(n+v)}{v^2}\Big[1+O\Big(\frac{v^2}{n(n+v)^2}\Big)\Big],$$

which combing with (2.20) implies

$$\mu_{v,j}^2 = \frac{4j(v+j)+O(n+v)}{v^2}\Big[1+O\Big(\frac{v^2}{n(n+v)^2}\Big)\Big].$$

Then, use (2.15) and the fact $|a-b| \le |a^2-b^2|\cdot a^{-1}$ for any $a>0$, $b>0$ to see that

$$\begin{aligned}
|d_{v,j}| &\le \frac{1}{\sigma_{v,j}}\Big|\frac{4j(v+j)}{v^2} - \mu_{j,v}^2\Big| \cdot \Big(\frac{2\sqrt{j(v+j)}}{v}\Big)^{-1} \\
&= \frac{v}{\sqrt{2j+v}} \cdot O\Big(\frac{n+v}{v^2} + \frac{1}{n+v}\Big) \cdot \frac{v}{2\sqrt{j(v+j)}} \\
&= O\Big(\frac{v}{\sqrt{n+v}}\Big) \cdot O\Big(\frac{n+v}{v^2} + \frac{1}{n+v}\Big) \cdot O\Big(\frac{v}{\sqrt{n(n+v)}}\Big) \\
&= O(n^{-1/2})
\end{aligned}$$

uniformly over all $\delta n \le j \le n$ as $n \to \infty$. The proof is completed. □

Proof of Lemma 2.10. It follows from (2.15) that

$$\begin{aligned}
1 + y_{v,j}^2(t) &= 1 + \mu_{v,j}^2 + \mu_{v,j}^2(2\beta_{v,j}t + \beta_{v,j}^2 t^2) \\
&= (1+\mu_{v,j}^2)\Big[1 + \frac{\mu_{v,j}^2}{1+\mu_{v,j}^2}(2\beta_{v,j}t + \beta_{v,j}^2 t^2)\Big]. \qquad (2.72)
\end{aligned}$$

From (2.20) and (2.66),

$$\frac{\delta^2}{2v^2} n(n+v) \le \mu_{v,j}^2 \le \frac{5}{v^2} n(n+v) \qquad (2.73)$$

uniformly for $\delta n \le j \le n$ as n is sufficiently large. Similar to the argument between (2.70) and (2.71), we see

$$\sqrt{1+y_{v,j}^2(t)} = \sqrt{1+\mu_{v,j}^2} + \frac{\mu_{v,j}^2}{\sqrt{1+\mu_{v,j}^2}}\beta_{v,j} t\big(1+O(n^{-3/8})\big);$$

$$-\frac{1}{2v}\cdot\frac{1}{1+y_{v,j}^2(t)} = -\frac{1}{2v}\cdot\frac{1}{1+\mu_{v,j}^2} + \frac{\mu_{v,j}^2\beta_{v,j}t}{\sqrt{1+\mu_{v,j}^2}} \cdot O\Big(\frac{v^2}{(v+n)^3}\Big)$$

hold uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$. Therefore, we get from (2.11) and the fact $\tau'(\mu_{v,j}) = 0$ that

$$\begin{aligned} y_{v,j}\tau'(y_{v,j}(t)) &= \sqrt{1+y_{v,j}^2} - \frac{1}{2v(1+y_{v,j}^2)} - \frac{2j+v-1-0.5}{v} \\ &= \frac{\mu_{v,j}^2\beta_{v,j}t}{\sqrt{1+\mu_{v,j}^2}}\big(1+O\big(n^{-3/8}\big)\big) + \mu_{v,j}\tau'(\mu_{v,j}) \\ &= \frac{\mu_{v,j}^2\beta_{v,j}t}{\sqrt{1+\mu_{v,j}^2}}\big(1+O\big(n^{-3/8}\big)\big) \end{aligned}$$

and

$$\frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)} = \frac{\beta_{v,j}t}{\sqrt{1+\mu_{v,j}^2}}\big(1+O\big(n^{-3/8}\big)\big). \tag{2.74}$$

Notice $1+\bar{\mu}_{v,j}^2$ and $\big((n+v)/v\big)^2$ are of the same order from (2.61). From (2.20) and the fact $\sqrt{1+x} = 1+O(x)$ as $x \to 0$ we have

$$\begin{aligned} \sqrt{1+\mu_{v,j}^2} &= \frac{2j+v-1.5}{v} + \frac{v}{2(2j+v-1.5)^2} + O\Big(\frac{v^3}{n(n+v)^4}\Big) \\ &= \frac{2j+v-1.5}{v}\Big[1+O\Big(\frac{v^2}{(n+v)^3}\Big)\Big]. \end{aligned}$$

By the identity between (2.12) and (2.13),

$$\tau''(x) = \frac{1}{\sqrt{1+x^2}} + \frac{1}{v(1+x^2)^2} - \frac{\tau'(x)}{x}. \tag{2.75}$$

It follows that

$$\begin{aligned} \tau''(y_{v,j}(0)) = \tau''(\mu_{v,j}) &= \frac{1}{\sqrt{1+\mu_{v,j}^2}} + \frac{1}{v(1+\mu_{v,j}^2)^2} \\ &= \frac{v}{2j+v-1.5}\Big[1+O\Big(\frac{v^2}{(n+v)^3}\Big)\Big] \\ &= \frac{v}{2j+v}\Big[1+O\Big(\frac{1}{n+v}\Big)\Big] \end{aligned} \tag{2.76}$$

and

$$\mu_{v,j}^2\beta_{v,j}^2\tau''(\mu_{v,j}) = \sigma_{v,j}^2\tau''(\mu_{v,j}) = \frac{1}{v}\Big[1+O\Big(\frac{1}{n+v}\Big)\Big] \tag{2.77}$$

from (2.15). We obtain (i). Furthermore, by (2.72)-(2.74),

$$\tau''(y_{v,j}(t)) = \frac{1}{\sqrt{1+y_{v,j}^2(t)}} + \frac{1}{v(1+y_{v,j}^2(t))^2} - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)} \tag{2.78}$$

$$= \frac{1}{\sqrt{1+\mu_{v,j}^2}}\Big[1+\frac{\mu_{v,j}^2\beta_{v,j}|t|}{1+\mu_{v,j}^2}O(1)\Big]$$

$$+\frac{1}{v(1+\mu_{v,j}^2)^2}\Big[1+\frac{\mu_{v,j}^2\beta_{v,j}|t|}{1+\mu_{v,j}^2}O(1)\Big] - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)}$$

$$= \tau''(y_{v,j}(0)) + \frac{\mu_{v,j}^2\beta_{v,j}|t|}{(1+\mu_{v,j}^2)^{3/2}}O(1) - \frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)} \tag{2.79}$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$, where (2.75) is used in the first equality; (2.72) is used in the second equality; (2.76) is used in the third equality. From (2.79) we claim that

$$\tau''(y_{v,j}(t)) = \tau''(y_{v,j}(0))\big[1+O(\beta_{v,j}|t|)\big] \tag{2.80}$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$. In fact, we know $1+\mu_{v,j}^2$ has the same order as $\frac{(n+v)^2}{v^2}$ from (2.73). Therefore,

$$\tau''(y_{v,j}(0)) = \frac{1}{\sqrt{1+\mu_{v,j}^2}}\Big[1+O\Big(\frac{v}{(n+v)^2}\Big)\Big]$$

by (2.76). This is equivalent to

$$\frac{1}{\sqrt{1+\mu_{v,j}^2}} = \tau''(y_{v,j}(0))\Big[1+O\Big(\frac{v}{(n+v)^2}\Big)\Big].$$

This yields that

$$\frac{\mu_{v,j}^2\beta_{v,j}|t|}{(1+\mu_{v,j}^2)^{3/2}} = \frac{1}{\sqrt{1+\mu_{v,j}^2}}\cdot\frac{\mu_{v,j}^2}{1+\mu_{v,j}^2}\beta_{v,j}|t| = \tau''(y_{v,j}(0))\cdot O(\beta_{v,j}|t|);$$

$$\frac{\tau'(y_{v,j}(t))}{y_{v,j}(t)} = \tau''(y_{v,j}(0))\cdot O(\beta_{v,j}|t|)$$

by (2.74). These two facts conclude (2.80).

Review $y_{v,j}(t) = \mu_{v,j}(1+\beta_{v,j}t)$ and $y_{v,j}(0) = \mu_{v,j}$. We have

$$\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \mu_{v,j}\beta_{v,j}\int_0^t \tau'(y_{v,j}(s))ds = \mu_{v,j}^2\beta_{v,j}^2\int_0^t\int_0^s \tau''(y_{v,j}(w))dwds.$$

From (2.80) we have

$$\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) = \frac{\mu_{v,j}^2\beta_{v,j}^2\tau''(\mu_{v,j})}{2}t^2[1+O(\beta_{v,j}|t|)]$$

uniformly over $|t| \le n^{1/8}$ and $\delta n \le j \le n$ as $n \to \infty$. We get (ii) by the notation $\beta_{v,j}^2 = \sigma_{v,j}^2/\mu_{v,j}^2$ in (2.15). It is seen from (2.13) that

$$\begin{aligned}\tau(y_{v,j}(t)) - \tau(\mu_{v,j}) &> \beta_{v,j}^2 \cdot \frac{2j-1.5}{v} \cdot \int_0^t \int_0^s \frac{1}{(1+\beta_{v,j}w)^2} dwds \\ &= \frac{2j-1.5}{v}\big[\beta_{v,j}t - \log(1+\beta_{v,j}t)\big]\end{aligned}$$

for any $t > -1/\beta_{v,j}$. This implies (iii) since $2j - 1.5 > j$ for all $j \ge 1$.

Finally, from (2.15), $y_{v,j}(t) = \mu_{v,j}(1+\beta_{v,j}t)$ with $\mu_{v,j} > 0$ and $\beta_{v,j} > 0$. Thus, $y_{v,j}(t)$ is an increasing function in $t > -1/\beta_{v,j}$. Keep in mind that we will interchange "$y_{v,j}(0)$" and"$\mu_{v,j}$" next. By (2.13), $\tau'(x)$ is increasing in $x > 0$ and $\tau'(\mu_{v,j}) = \tau'(y_{v,j}(0)) = 0$. This says that $\tau'(y_{v,j}(t)) \le \tau'(y_{v,j}(0)) = 0$ for $-1/\beta_{v,j} < t < 0$. Observe that the first two terms on the right hand side of (2.78) are decreasing in $-1/\beta_{v,j} < t \le 0$. The two facts show that $\tau''(y_{v,j}(t)) \ge \tau''(y_{v,j}(0)) = \tau''(\mu_{v,j})$ for $-1/\beta_{v,j} < t \le 0$. Note that $\frac{d}{dt}\tau(y_{v,j}(t)) = \mu_{v,j}\beta_{v,j}\tau'(y_{v,j}(t))$ and $\frac{d^2}{dt^2}\tau(y_{v,j}(t)) = \mu_{v,j}^2\beta_{v,j}^2\tau''(y_{v,j}(t))$ by the chain rule. For any t with $-1/\beta_{v,j} < t < 0$, by using Taylor's theorem, there exists $w \in (t, 0)$ such that

$$\begin{aligned}\tau(y_{v,j}(t)) - \tau(y_{v,j}(0)) &= \mu_{v,j}\beta_{v,j}t\tau'(y_{v,j}(0)) + \frac{1}{2}\mu_{v,j}^2\beta_{v,j}^2t^2\tau''(y_{v,j}(w)) \\ &\ge \frac{t^2}{2}\mu_{v,j}^2\beta_{v,j}^2\tau''(\mu_{v,j}).\end{aligned}$$

This leads to (iv). □

2.4. Proof of (1.9)

From p. 163 from Do Carmo[36], for a surface $(x, y, h(x, y))$ on an open set $(x, y) \in U \subset \mathbb{R}^2$, the Gaussian curvature K is given by

$$K = \frac{h_{xx}h_{yy} - h_{xy}^2}{(1 + h_x^2 + h_y^2)^2}, \ (x, y) \in U.$$

In our case,

$$h(x, y) = \frac{c(x^2+y^2)}{\sqrt{a + b(x^2+y^2)^2}}, \quad x^2 + y^2 < 1, x > 0$$

where $a = \alpha^2/(1+\alpha)^2$, $b = 4/(1+\alpha)$ and $c = 4/\pi$. To simplify notation, set $A = a + b(x^2 + y^2)^2$. Then we have from the formula $(f/g)' = (f'g - fg')/g^2$ that

$$\begin{aligned} h_x &= \frac{2cx\sqrt{A} - c(x^2+y^2)\frac{4bx(x^2+y^2)}{2\sqrt{A}}}{A} \\ &= \frac{2cxA - 2bcx(x^2+y^2)^2}{A^{3/2}} \\ &= \frac{2acx}{A^{3/2}}. \end{aligned} \tag{2.81}$$

So

$$\begin{aligned} h_{xx} &= (2ac)\frac{A^{3/2} - x \cdot \frac{3}{2}\sqrt{A} \cdot 4bx(x^2+y^2)}{A^3} \\ &= (2ac)\frac{(a + b(x^2+y^2)^2) - 6bx^2(x^2+y^2)}{A^{5/2}} \\ &= (2ac)\frac{a + b(x^2+y^2)(y^2-5x^2)}{A^{5/2}} \end{aligned} \tag{2.82}$$

and

$$\begin{aligned} h_{xy} &= 2acx \cdot \Big(-\frac{3}{2}\Big)A^{-5/2} \cdot 4by(x^2+y^2) \\ &= -12abcxy(x^2+y^2)A^{-5/2}. \end{aligned}$$

By symmetry, we get the expressions for h_y and h_{yy} by replacing "x" with "y" in (2.81) and (2.82). In particular,

$$(1 + h_x^2 + h_y^2)^2 = \Big[1 + \frac{4a^2c^2(x^2+y^2)}{A^3}\Big]^2$$

and

$$\begin{aligned} &h_{xx}h_{yy} - h_{xy}^2 \\ &= 4a^2c^2A^{-5}\big[a + b(x^2+y^2)(y^2-5x^2)\big] \cdot \big[a + b(x^2+y^2)(x^2-5y^2)\big] \\ &\qquad\qquad - 144a^2b^2c^2x^2y^2(x^2+y^2)^2A^{-5} \\ &= 4a^2c^2A^{-5}\big\{\big[a + b(x^2+y^2)(y^2-5x^2)\big] \cdot \big[a + b(x^2+y^2)(x^2-5y^2)\big] \\ &\qquad\qquad - 36b^2x^2y^2(x^2+y^2)^2\big\}. \end{aligned}$$

It is a bit tedious but easy to check that

$$\begin{aligned} &\big[a + b(x^2+y^2)(y^2-5x^2)\big] \cdot \big[a + b(x^2+y^2)(x^2-5y^2)\big] - 36b^2x^2y^2(x^2+y^2)^2 \\ &= a^2 - 4ab(x^2+y^2)^2 - 5b^2(x^2+y^2)^4 \\ &= \big[a + b(x^2+y^2)^2\big]\big[a - 5b(x^2+y^2)^2\big]. \end{aligned}$$

This says that

$$h_{xx}h_{yy} - h_{xy}^2 = 4a^2c^2A^{-5}\big[a + b(x^2+y^2)^2\big]\big[a - 5b(x^2+y^2)^2\big]. \tag{2.83}$$

Therefore, by the notation $A = a + b(x^2+y^2)^2$,

$$\begin{aligned} K &= \frac{A^6}{[A^3 + 4a^2c^2(x^2+y^2)]^2} \cdot \frac{4a^2c^2}{A^5} \cdot A\big[a - 5b(x^2+y^2)^2\big] \\ &= (4c^2) \cdot \frac{a^2(a+bR^4)^2(a-5bR^4)}{\big[(a+bR^4)^3 + 4a^2c^2R^2\big]^2}. \end{aligned}$$

where $R := \sqrt{x^2+y^2} \in (0,1)$. So $K > 0$ for all (x,y) with $x^2+y^2 < 1$, $|x| \le 1$ if $a > 5b$, and $K < 0$ for all (x,y) with $x^2+y^2 > \sqrt{a/5b}$ if $a < 5b$. Now $a > 5b$ if and only if

$$\frac{\alpha^2}{1+\alpha} > 20,$$

which again holds if and only if $\alpha > 10+\sqrt{120}$. Similarly, $a < 5b$ if and only if $\alpha < 10+\sqrt{120}$.

To make $h(x,y)$ be a convex function it suffices to check that the Hessian of h is non-negative definite. That is, that matrix

$$H = \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}$$

has to be non-negative definite. This holds if $h_{xx} \ge 0$, $h_{yy} \ge 0$ and $\det(H) \ge 0$. It is easy to see that $\inf_{x^2+y^2<1,\,|x|\le 1}(x^2+y^2)(y^2-5x^2) = -5$ by taking $x \uparrow 1$ and $y \downarrow 0$. So $h_{xx} \ge 0$ and $h_{yy} \ge 0$ if $a \ge 5b$.

From (2.83) we know $K \ge 0$ for all (x,y) with $x^2+y^2 < 1$, $|x| \le 1$ if and only if $\alpha > 10+\sqrt{120}$. □

ACKNOWLEDGMENTS

The authors would like to thank the referee for his/her careful reading of the manuscript and valuable comments that have led to improvements on the paper. Shuhua Chang's research was supported in part by NSFC 91430108 and 11771322.

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