The Annals of Applied Probability
2010, Vol. 20, No. 6, 2086–2117
DOI: 10.1214/10-AAP677

SPECTRAL DISTRIBUTIONS OF ADJACENCY AND LAPLACIAN MATRICES OF RANDOM GRAPHS

BY XUE DING AND TIEFENG JIANG[1]

Jilin University and University of Minnesota, and University of Minnesota

In this paper, we investigate the spectral properties of the adjacency and the Laplacian matrices of random graphs. We prove that:

(i) the law of large numbers for the spectral norms and the largest eigenvalues of the adjacency and the Laplacian matrices;

(ii) under some further independent conditions, the normalized largest eigenvalues of the Laplacian matrices are dense in a compact interval almost surely;

(iii) the empirical distributions of the eigenvalues of the Laplacian matrices converge weakly to the free convolution of the standard Gaussian distribution and the Wigner's semi-circular law;

(iv) the empirical distributions of the eigenvalues of the adjacency matrices converge weakly to the Wigner's semi-circular law.

1. Introduction. The theory of random graphs was founded in the late 1950s by Erdös and Rényi [19–22]. The work of Watts and Strogatz [46] and Barabási and Albert [3] at the end of the last century initiated new interest in this field. The subject is at the intersection between graph theory and probability theory. One can see, for example, [10, 14–16, 18, 23, 30, 34, 40] for book-length treatments.

The spectral graph theory is the study of the properties of a graph in relationship to the characteristic polynomial, eigenvalues and eigenvectors of its adjacency matrix or Laplacian matrix. For reference, one can see books [14, 42] for the deterministic case and [15] for the random case, and literatures therein. The spectral graph theory has applications in chemistry [9] where eigenvalues were relevant to the stability of molecules. Also, graph spectra appear naturally in numerous questions in theoretical physics and quantum mechanics (see, e.g., [24–26, 38, 39, 43, 44]). For connections between the eigenvalues of the adjacency matrices and the Laplacian matrices of graphs and Cheeger constants, diameter bounds, paths and routing in graphs, one can see [15].

Although there are many matrices for a given graph with n vertices, the most studied are their adjacency matrices and the Laplacian matrices. Typically, random graphs are considered with the number of vertices n tending to infinity. Many

Received April 2009; revised November 2009.

[1]Supported in part by NSF Grant DMS-04-49365.

AMS 2000 subject classifications. 05C80, 05C50, 15A52, 60B10.

Key words and phrases. Random graph, random matrix, adjacency matrix, Laplacian matrix, largest eigenvalue, spectral distribution, semi-circle law, free convolution.

geometrical and topological properties can be deduced for a large class of random graph ensembles, but the spectral properties of the random graphs are still uncovered to a large extent.

In this paper, we will investigate the spectral properties of the adjacency and the Laplacian matrices of some random graphs. The framework of the two matrices will be given next.

Let $n \geq 2$ and $\Gamma_n = (\mathcal{V}_n, E_n)$ be a graph, where $\mathcal{V}_n$ denotes a set of n vertices $v_1, v_2, \dots, v_n$, and E_n is the set of edges. In this paper, we assume that the edges in E_n are always nonoriented. For basic definitions of graphs, one can see, for example, [11]. The adjacency matrix and the Laplacian matrix of the graph are of the form

$$\mathbf{A}_n = \begin{pmatrix} 0 & \xi_{12}^{(n)} & \xi_{13}^{(n)} & \cdots & \xi_{1n}^{(n)} \\ \xi_{21}^{(n)} & 0 & \xi_{23}^{(n)} & \cdots & \xi_{2n}^{(n)} \\ \xi_{31}^{(n)} & \xi_{32}^{(n)} & 0 & \cdots & \xi_{3n}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{n1}^{(n)} & \xi_{n2}^{(n)} & \xi_{n3}^{(n)} & \cdots & 0 \end{pmatrix} \tag{1.1}$$

and

$$\mathbf{\Delta}_n = \begin{pmatrix} \sum_{j\neq 1} \xi_{1j}^{(n)} & -\xi_{12}^{(n)} & -\xi_{13}^{(n)} & \cdots & -\xi_{1n}^{(n)} \\ -\xi_{21}^{(n)} & \sum_{j\neq 2} \xi_{2j}^{(n)} & -\xi_{23}^{(n)} & \cdots & -\xi_{2n}^{(n)} \\ -\xi_{31}^{(n)} & -\xi_{32}^{(n)} & \sum_{j\neq 3} \xi_{3j}^{(n)} & \cdots & -\xi_{3n}^{(n)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -\xi_{n1}^{(n)} & -\xi_{n2}^{(n)} & -\xi_{n3}^{(n)} & \cdots & \sum_{j\neq n} \xi_{nj}^{(n)} \end{pmatrix} \tag{1.2}$$

with relationship

$$\mathbf{\Delta}_n = \mathbf{D}_n - \mathbf{A}_n, \tag{1.3}$$

where $\mathbf{D}_n = (\sum_{l\neq i}^n \xi_{il}^{(n)})_{1\leq i\leq n}$ is a diagonal matrix.

As mentioned earlier, we will focus on nonoriented random graphs in this paper. Thus, the adjacency matrix $\mathbf{A}_n$ is always symmetric. If the graph is also simple, the entry $\xi_{ij}^{(n)}$ for $i \neq j$ only takes value 1 or 0 with 1 for an edge between v_i and v_j, and 0 for no edge between them.

The Laplacian matrix $\mathbf{\Delta}_n$ for graph Γ_n is also called the admittance matrix or the Kirchhoff matrix in literature. If Γ_n is a simple random graph, the (i,i)-entry of $\mathbf{\Delta}_n$ represents the degree of vertex v_i, that is, the number of vertices connected to v_i. $\mathbf{\Delta}_n$ is always nonnegative (this is also true for $\mathbf{\Delta}_n$ as long as the entries

$\{\xi_{ij}^{(n)}; 1 \le i \ne j \le n\}$ are nonnegative); the smallest eigenvalue of $\mathbf{\Delta}_n$ is zero; the second smallest eigenvalue stands for the algebraic connectivity; the Kirchhoff theorem establishes the relationship between the number of spanning trees of Γ_n and the eigenvalues of $\mathbf{\Delta}_n$.

An *Erdös–Rényi random graph* $G(n, p)$ has n vertices. For each pair of vertices v_i and v_j with $i \ne j$, an edge between them is formed randomly with chance p_n and independently of other edges (see [19–22]). This random graph corresponds to Bernoulli entries $\{\xi_{ij}^{(n)}; 1 \le i < j \le n\}$, which are independent random variables with $P(\xi_{ij}^{(n)} = 1) = 1 - P(\xi_{ij}^{(n)} = 0) = p_n$ for all $1 \le i < j \le n$.

For weighted random graphs, $\{\xi_{ij}^{(n)}; 1 \le i < j \le n\}$ are independent random variables and $\xi_{ij}^{(n)}$ is a product of a Bernoulli random variable $\mathrm{Ber}(p_n)$ and a nice random variable, for instance, a Gaussian random variable or a random variable with all finite moments (see, e.g., [32, 33]). For the sign model studied in [7, 33, 43, 44], $\xi_{ij}^{(n)}$ are independent random variables taking three values: $0, 1, -1$. In this paper, we will study the spectral properties of $\mathbf{A}_n$ and $\mathbf{\Delta}_n$ under more general conditions on $\{\xi_{ij}^{(n)}; 1 \le i < j \le n\}$ [see (1.5)].

Now we need to introduce some notation about the eigenvalues of matrices. Given an $n \times n$ symmetric matrix $\mathbf{M}$. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of $\mathbf{M}$, we sometimes also write this as $\lambda_1(\mathbf{M}) \ge \lambda_2(\mathbf{M}) \ge \cdots \ge \lambda_n(\mathbf{M})$ for clarity. The notation $\lambda_{\max} = \lambda_{\max}(\mathbf{M})$, $\lambda_{\min} = \lambda_{\min}(\mathbf{M})$ and $\lambda_k(\mathbf{M})$ stand for the largest eigenvalue, the smallest eigenvalue and the kth largest eigenvalue of $\mathbf{M}$, respectively. Set

$$\begin{aligned} &\hat{\mu}(\mathbf{M}) = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i} \quad \text{and} \\ &F^{\mathbf{M}}(x) = \frac{1}{n}\sum_{i=1}^{n} I(\lambda_i \le x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1.4}$$

Then, $\hat{\mu}(\mathbf{M})$ and $F^{\mathbf{M}}(x)$ are the empirical spectral distribution of $\mathbf{M}$ and the empirical spectral cumulative distribution function of $\mathbf{M}$, respectively.

In this paper, we study $\mathbf{A}_n$ and $\mathbf{\Delta}_n$ not only for random graphs but also study them in the context of random matrices. Therefore, we allow the entries $\xi_{ij}^{(n)}$'s to take real values and possibly with mean zero. It will be clear in our theorems if the framework is in the context of random graphs or that of of random matrices.

Under general conditions on $\{\xi_{ij}^{(n)}\}$, we prove in this paper that a suitably normalized $\hat{\mu}(\mathbf{A}_n)$ converges to the semi-circle law; a suitably normalized $\hat{\mu}(\mathbf{\Delta}_n)$ converges weakly to the free convolution of the standard normal distribution and the semi-circle law. Besides, the law of large numbers for largest eigenvalues and the spectral norms of $\mathbf{A}_n$ and $\mathbf{\Delta}_n$ are obtained. Before stating these results, we need to give the assumptions on the entries of $\mathbf{A}_n$ in (1.1) and $\mathbf{\Delta}_n$ in (1.2).

$$\text{Let } \{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\} \text{ be random variables defined on the same probability space and } \{\xi_{ij}^{(n)}; 1 \le i < j \le n\} \text{ be independent for each } n \ge 2 \text{ (not necessarily identically distributed) with } \xi_{ij}^{(n)} = \xi_{ji}^{(n)}, E(\xi_{ij}^{(n)}) = \mu_n, \operatorname{Var}(\xi_{ij}^{(n)}) = \sigma_n^2 > 0 \text{ for all } 1 \le i < j \le n \text{ and } n \ge 2 \text{ and } \sup_{1\le i<j\le n, n\ge 2} E|(\xi_{ij}^{(n)} - \mu_n)/\sigma_n|^p < \infty \text{ for some } p > 0. \tag{1.5}$$

The values of p above will be specified in each result later. In what follows, for an $n \times n$ matrix $\mathbf{M}$, let $\|\mathbf{M}\| = \sup_{\mathbf{x}\in\mathbb{R}^n:\,\|\mathbf{x}\|=1} \|\mathbf{M}\mathbf{x}\|$ be the spectral norm of $\mathbf{M}$, where $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$ for $\mathbf{x} = (x_1, \ldots, x_n)' \in \mathbb{R}^n$. Now we state the main results of this paper.

THEOREM 1. *Suppose* (1.5) *holds for some* $p > 6$. *Assume* $\mu_n = 0$ *and* $\sigma_n = 1$ *for all* $n \ge 2$. *Then*:

(a) $\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sqrt{n \log n}} \to \sqrt{2}$ *in probability as* $n \to \infty$.

Furthermore, if $\{\mathbf{\Delta}_2, \mathbf{\Delta}_3, \ldots\}$ *are independent, then*:

(b) $\liminf_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sqrt{n \log n}} = \sqrt{2}$ *a.s. and* $\limsup_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sqrt{n \log n}} = 2$ *a.s., and the sequence* $\{\lambda_{\max}(\mathbf{\Delta}_n)/\sqrt{n \log n}; n \ge 2\}$ *is dense in* $[\sqrt{2}, 2]$ *a.s.*;

(c) *the conclusions in* (a) *and* (b) *still hold if* $\lambda_{\max}(\mathbf{\Delta}_n)$ *is replaced by* $\|\mathbf{\Delta}_n\|$.

For typically-studied random matrices such as the Hermite ensembles and the Laguerre ensembles, if we assume the sequence of $n \times n$ matrices for all $n \ge 1$ are independent as in Theorem 1, the conclusions (b) and (c) in Theorem 1 do not hold. In fact, for Gaussian Unitary Ensemble (GUE), which is a special case of the Hermite ensemble, there is a large deviation inequality $P(|n^{-1/2}\lambda_{\max} - \sqrt{2}| \ge \varepsilon) \le e^{-nC_\varepsilon}$ for any $\varepsilon > 0$ as n is sufficiently large, where $C_\varepsilon > 0$ is some constant (see (1.24) and (1.25) from [36] or [8]). With or without the independence assumption, this inequality implies from the Borel–Cantelli lemma that $n^{-1/2}\lambda_{\max} \to \sqrt{2}$ a.s. as $n \to \infty$. Similar large deviation inequalities also hold for Wishart and sample covariance matrices (see, e.g., [27, 45]).

For two sequence of real numbers $\{a_n; n \ge 1\}$ and $\{b_n; n \ge 1\}$, we write $a_n \ll b_n$ if $a_n/b_n \to 0$ as $n \to \infty$, and $a_n \gg b_n$ if $a_n/b_n \to +\infty$ as $n \to \infty$. We use $n \gg 1$ to denote that n is sufficiently large.

COROLLARY 1.1. *Suppose* (1.5) *holds for some* $p > 6$. *Then, as* $n \to \infty$:

(a1) $\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n \log n}} \to \sqrt{2}$ *in probability if* $|\mu_n| \ll \sigma_n(\frac{\log n}{n})^{1/2}$;

(a2) $\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{n\mu_n} \to 1$ *in probability if* $\mu_n > 0$ *for* $n \gg 1$ *and* $\mu_n \gg \sigma_n(\frac{\log n}{n})^{1/2}$;

(a3) $\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{n\mu_n} \to 0$ *in probability if* $\mu_n < 0$ *for* $n \gg 1$ *and* $|\mu_n| \gg \sigma_n(\frac{\log n}{n})^{1/2}$.

Furthermore, *assume* $\{\mathbf{\Delta}_2, \mathbf{\Delta}_3, \dots\}$ *are independent*, *then*:

(b1) $\liminf_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}} = \sqrt{2}$ *a.s. and* $\limsup_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}} = 2$ *a.s.*, *and the sequence* $\{\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sigma_n\sqrt{n\log n}}; n \ge 2\}$ *is dense in* $[\sqrt{2}, 2]$ *a.s. if* $|\mu_n| \ll \sigma_n(\frac{\log n}{n})^{1/2}$;

(b2) $\lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{n\mu_n} = 1$ *a.s. if* $\mu_n > 0$ *for* $n \gg 1$ *and* $\mu_n \gg \sigma_n(\frac{\log n}{n})^{1/2}$;

(b3) $\lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{\Delta}_n)}{n\mu_n} = 0$ *a.s. if* $\mu_n < 0$ *for* $n \gg 1$ *and* $|\mu_n| \gg \sigma_n(\frac{\log n}{n})^{1/2}$.

Finally, (a1) *and* (b1) *still hold if* $\lambda_{\max}(\mathbf{\Delta}_n)$ *is replaced by* $\|\mathbf{\Delta}_n\|$; *if* $\xi_{ij}^{(n)} \ge 0$ *for all* i, j, n, *then* (a2) *and* (b2) *still hold if* $\lambda_{\max}(\mathbf{\Delta}_n)$ *is replaced by* $\|\mathbf{\Delta}_n\|$.

REMARK 1. For the Erdös–Rényi random graph, the condition "(1.5) holds for some $p > p_0$" with $p_0 > 2$ is true only when p_n is bounded away from zero and one. So, under this condition of p_n, Corollary 1.1 holds. Moreover, under the same restriction of p_n, Theorems 2 and 4, that will be given next, also hold.

Let $\{\nu, \nu_1, \nu_2, \dots\}$ be a sequence of probability measures on $\mathbb{R}$. We say that ν_n converges weakly to ν if $\int_{\mathbb{R}} f(x)\nu_n(dx) \to \int_{\mathbb{R}} f(x)\nu(dx)$ for any bounded and continuous function $f(x)$ defined on $\mathbb{R}$. The Portmanteau lemma says that the weak convergence can also be characterized in terms of open sets or closed sets (see, e.g., [17]).

Now we consider the empirical distribution of the eigenvalues of the Laplacian matrix $\mathbf{\Delta}_n$. Bauer and Golinelli [7] simulate the eigenvalues for the Erdös–Rényi random graph with p fixed. They observe that the limit ν of the empirical distribution of $\lambda_i(\mathbf{\Delta}_n)$, $1 \le i \le n$, has a shape between the Gaussian and the semicircular curves. Further, they conclude from their simulations that m_4/m_2^2 is between 2 and 3, where m_i is the ith moment of probability measure ν. In fact, we have the following result.

THEOREM 2. *Suppose* (1.5) *holds for some* $p > 4$. *Set* $\tilde{F}_n(x) = \frac{1}{n} \times \sum_{i=1}^n I\{\frac{\lambda_i(\mathbf{\Delta}_n) - n\mu_n}{\sqrt{n}\sigma_n} \le x\}$ *for* $x \in \mathbb{R}$. *Then*, *as* $n \to \infty$, *with probability one*, $\tilde{F}_n$ *converges weakly to the free convolution* γ_M *of the semicircular law and the standard normal distribution. The measure* γ_M *is a nonrandom symmetric probability measure with smooth bounded density*, *does not depend on the distribution of* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ *and has an unbounded support.*

More information on γ_M can be found in [12]. For the Erdös–Rényi random graphs, the weighted random graphs in [32, 33] and the sign models in [7, 33, 43, 44], if p_n is bounded away from 0 and 1 as n is large, then (1.5) holds for all $p > 4$; thus Theorem 2 holds for all of these graphs.

It is interesting to notice that the limiting curve appeared in Theorem 2 is indeed a hybrid between the standard Gaussian distribution and the semi-circular law, as observed in [7]. Moreover, for the limiting distribution, it is shown in [12] that $m_4/m_2^2 = 8/3 \in (2,3)$, which is also consistent with the numerical result in [7].

Before introducing the next theorem, we now make a remark. It is proved in [12] that the conclusion in the above theorem holds when $\xi_{ij}^{(n)} = \xi_{ij}$ for all $1 \le i < j \le n$ and $n \ge 2$, where $\{\xi_{ij}; 1 \le i < j < \infty\}$ are independent and identically distributed random variables with $E\xi_{12} = 0$ and $E(\xi_{12})^2 = 1$. The difference is that the current theorem holds for any independent, but not necessarily identically distributed, random variables with arbitrary mean μ_n and variance σ_n^2.

Now we consider the adjacency matrices. Recall $\mathbf{A}_n$ in (1.1). Wigner [47] establishes the celebrated semi-circle law for matrix $\mathbf{A}_n$ with entries $\{\xi_{ij}^{(n)} = \xi_{ij} : 1 \le i < j < \infty\}$ being i.i.d. $N(0,1)$-distributed random variables (for its extensions, one can see, e.g., [5] and literatures therein). Arnold [1, 2] proves that Wigner's result holds also for the entries being i.i.d. random variables with a finite sixth moment. In particular, this implies that, for the adjacency matrix $\mathbf{A}_n$ of the Erdös–Rényi random graph with p fixed, the empirical distribution of the eigenvalues of $\mathbf{A}_n$ converges to the semi-circle law (see also Bollobas [10]). In the next result we show that, under a condition slightly stronger than a finite second moment, the semicircular law still holds for $\mathbf{A}_n$.

THEOREM 3. *Let* $\omega_{ij}^{(n)} := (\xi_{ij}^{(n)} - \mu_n)/\sigma_n$ *for all* i, j, n. *Assume* (1.5) *with* $p = 2$ *and*

$$\max_{1 \le i < j \le n} E\{(\omega_{ij}^{(n)})^2 I(|\omega_{ij}^{(n)}| \ge \varepsilon\sqrt{n})\} \to 0$$

as $n \to \infty$ *for any* $\varepsilon > 0$, *which is particularly true when* (1.5) *holds for some* $p > 2$. *Set*

$$\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} I\left\{\frac{\lambda_i(\mathbf{A}_n) + \mu_n}{\sqrt{n}\sigma_n} \le x\right\}, \qquad x \in \mathbb{R}.$$

Then, almost surely, $\tilde{F}_n$ *converges weakly to the semicircular law with density* $\frac{1}{2\pi}\sqrt{4 - x^2} I(|x| \le 2)$.

Applying Theorem 3 to the Erdös–Rényi random graph, we have the following result.

COROLLARY 1.2. *Assume* (1.5) *with* $P(\xi_{ij}^{(n)} = 1) = p_n = 1 - P(\xi_{ij}^{(n)} = 0)$ *for all* $1 \le i < j \le n$ *and* $n \ge 2$. *If* $\alpha_n := (np_n(1 - p_n))^{1/2} \to \infty$ *as* $n \to \infty$, *then, almost surely,* $F^{\mathbf{A}_n/\alpha_n}$ *converges weakly to the semicircular law with density* $\frac{1}{2\pi}\sqrt{4 - x^2} I(|x| \le 2)$. *In particular, if* $1/n \ll p_n \to 0$ *as* $n \to \infty$, *then, almost surely,* $F^{\mathbf{A}_n/\sqrt{np_n}}$ *converges weakly to the same semicircular law.*

The condition "$\alpha_n := (np_n(1-p_n))^{1/2} \to \infty$ as $n \to \infty$" cannot be relaxed to that "$np_n \to \infty$." This is because, as p_n is very close to 1, say, $p_n = 1$, then $\xi_{ij}^{(n)} = 1$ for all $i \neq j$. Thus $\mathbf{A}_n$ has eigenvalue $n-1$ with one fold and -1 with $n-1$ fold. This implies that $F^{\mathbf{A}_n} \to \delta_{-1}$ weakly as $n \to \infty$.

Corollary 1.2 shows that the semicircular law holds not only for p being a constant as in Arnold [1, 2], it also holds for the dilute Erdös–Rényi graph, that is, $1/n \ll p_n \to 0$ as $n \to \infty$. A result in Rogers and Bray [43] (see also a discussion for it in Khorunzhy et al. [33]) says that, if $P(\xi_{ij}^{(n)} = \pm 1) = p_n/2$ and $P(\xi_{ij}^{(n)} = 0) = 1 - p_n$, the semicircular law holds for the corresponding $\mathbf{A}_n$ with $1/n \ll p_n \to 0$. It is easy to check that their result is a corollary of Theorem 3.

Now we study the spectral norms and the largest eigenvalues of $\mathbf{A}_n$. For the Erdös–Rényi random graph, the largest eigenvalue of $\mathbf{A}_n$ is studied in [28, 35]. In particular, following Juház [31], Füredi and Komló [28] showed that the largest eigenvalue has asymptotically a normal distribution when $p_n = p$ is a constant; Krivelevich and Sudakov [35] proved a weak law of large numbers for the largest eigenvalue for the full range of $p_n \in (0, 1)$. In the following, we give a result for $\mathbf{A}_n$ whose entries do not necessarily take values of 0 or 1 only. Recall $\lambda_k(\mathbf{A}_n)$ and $\|\mathbf{A}_n\|$ are the kth largest eigenvalue and the spectral norm of $\mathbf{A}_n$, respectively.

THEOREM 4. *Assume* (1.5) *holds for some* $p > 6$. *Let* $\{k_n; n \geq 1\}$ *be a sequence of positive integers such that* $k_n = o(n)$ *as* $n \to \infty$. *The following hold*:

(i) *If* $\lim_{n\to\infty} \mu_n/(n^{-1/2}\sigma_n) = 0$, *then* $\|\mathbf{A}_n\|/\sqrt{n}\sigma_n \to 2$ *a.s. and* $\lambda_{k_n}(\mathbf{A}_n)/(\sqrt{n}\sigma_n) \to 2$ *a.s. as* $n \to \infty$.

(ii) *If* $\lim_{n\to\infty} \mu_n/(n^{-1/2}\sigma_n) = +\infty$, *then* $\lambda_{\max}(\mathbf{A}_n)/(n\mu_n) \to 1$ *a.s. as* $n \to \infty$.

(iii) *If* $\lim_{n\to\infty} |\mu_n|/(n^{-1/2}\sigma_n) = +\infty$, *then* $\|\mathbf{A}_n\|/(n|\mu_n|) \to 1$ *a.s. as* $n \to \infty$.

REMARK 2. The conclusion in (ii) cannot be improved in general to that $\lambda_{k_n}(\mathbf{A}_n)/(n\mu_n) \to 1$ a.s. as $n \to \infty$. This is because when σ_n is extremely small, $\mathbf{A}_n$ roughly looks like $\mu_n(\mathbf{J}_n - \mathbf{I}_n)$, where all the entries of $\mathbf{J}_n$ are equal to one, and $\mathbf{I}_n$ is the $n \times n$ identity matrix. It is easy to see that the largest eigenvalue of $\mu_n(\mathbf{J}_n - \mathbf{I}_n)$ is $(n-1)\mu_n > 0$, and all of the remaining $n-1$ eigenvalues are identical to -1.

From the above results, we see two probability distributions related to the spectral properties of the random graphs: the Wigner's semi-circle law and the free convolution of the standard normal distribution and the semi-circle law. The Kesten–McKay law is another one. It is the limit of the empirical distributions of the eigenvalues of the random d-regular graphs (see [37]).

The proofs of Theorems 1 and 2 rely on the moment method and some tricks developed in [12]. Theorems 3 and 4 are derived through a general result from [6] and certain truncation techniques in probability theory.

The rest of the paper is organized as follows: we will prove the theorems stated above in the next section; several auxiliary results for the proofs are collected in the Appendix.

2. Proofs.

LEMMA 2.1. *Let* $\mathbf{U}_n = (u_{ij}^{(n)})$ *be an* $n \times n$ *symmetric random matrix, and* $\{u_{ij}^{(n)}; 1 \le i \le j \le n, n \ge 1\}$ *are defined on the same probability space. Suppose, for each* $n \ge 1$, $\{u_{ij}^{(n)}; 1 \le i \le j \le n\}$ *are independent random variables with* $Eu_{ij}^{(n)} = 0$, $\mathrm{Var}(u_{ij}^{(n)}) = 1$ *for all* $1 \le i, j \le n$, *and* $\sup_{1\le i,j\le n, n\ge 1} E|u_{ij}^{(n)}|^{6+\delta} < \infty$ *for some* $\delta > 0$. *Then*:

(i) $\lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{U}_n)}{\sqrt{n}} = 2$ *a.s. and* $\lim_{n\to\infty} \frac{\|\mathbf{U}_n\|}{\sqrt{n}} = 2$ *a.s.*;

(ii) *the statements in* (i) *still hold if* $\mathbf{U}_n$ *is replaced by* $\mathbf{U}_n - \mathrm{diag}(u_{ii}^{(n)})_{1\le i\le n}$.

The proof of this lemma is a combination of Lemmas A.2 and A.3 in the Appendix and some truncation techniques. It is postponed and will be given later in this section.

PROOF OF THEOREM 1. First, assume (a) and (b) hold. Since $\mu_n = 0$ for all $n \ge 2$, (a) and (b) also hold if $\lambda_{\max}(\mathbf{\Delta}_n)$ is replaced by $\lambda_{\max}(-\mathbf{\Delta}_n)$. From the symmetry of $\mathbf{\Delta}_n$, we know that

$$\|\mathbf{\Delta}_n\| = \max\{-\lambda_{\min}(\mathbf{\Delta}_n), \lambda_{\max}(\mathbf{\Delta}_n)\} = \max\{\lambda_{\max}(-\mathbf{\Delta}_n), \lambda_{\max}(\mathbf{\Delta}_n)\}.$$

Now the function $h(x, y) := \max\{x, y\}$ is continuous in $(x, y) \in \mathbb{R}^2$, applying the two assertions

$$\limsup_{n\to\infty} \max\{a_n, b_n\} = \max\Big\{\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n\Big\}$$

and

$$\liminf_{n\to\infty} \max\{a_n, b_n\} \ge \max\Big\{\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n\Big\}$$

for any $\{a_n \in \mathbb{R}; n \ge 1\}$ and $\{b_n \in \mathbb{R}; n \ge 1\}$, we obtain $\|\mathbf{\Delta}_n\|/\sqrt{n\log n}$ converges to $\sqrt{2}$ in probability, and

$$\liminf_{n\to\infty} \frac{\|\mathbf{\Delta}_n\|}{\sqrt{n\log n}} \ge \sqrt{2} \qquad \text{a.s.} \quad \text{and} \quad \limsup_{n\to\infty} \frac{\|\mathbf{\Delta}_n\|}{\sqrt{n\log n}} = 2 \qquad \text{a.s.}$$

and

$$\text{the sequence } \left\{\frac{\|\mathbf{\Delta}_n\|}{\sqrt{n\log n}}; n \ge 2\right\} \text{ is dense in } [\sqrt{2}, 2] \qquad \text{a.s.}$$

Thus (c) is proved. Now we turn to prove (a) and (b).

Recall (1.3), $\mathbf{\Delta}_n = \mathbf{D}_n - \mathbf{A}_n$. First, $\lambda_{\max}(\mathbf{D}_n) - \|\mathbf{A}_n\| \le \lambda_{\max}(\mathbf{\Delta}_n) \le \lambda_{\max}(\mathbf{D}_n) + \|\mathbf{A}_n\|$ for all $n \ge 2$. Second, by (ii) of Lemma 2.1, $\|\mathbf{A}_n\|/\sqrt{n} \to 2$ a.s. as $n \to \infty$. Thus, to prove (a) and (b) in the theorem, it is enough to show that

$$\frac{T_n}{\sqrt{n\log n}} \to \sqrt{2} \qquad \text{in probability;} \tag{2.1}$$

$$\liminf_{n\to\infty} \frac{T_n}{\sqrt{n\log n}} = \sqrt{2} \qquad \text{a.s.} \quad \text{and} \quad \limsup_{n\to\infty} \frac{T_n}{\sqrt{n\log n}} = 2 \qquad \text{a.s.;} \tag{2.2}$$

$$\text{the sequence } \left\{\frac{T_n}{\sqrt{n\log n}}; n \ge 2\right\} \text{ is dense in } [\sqrt{2}, 2] \qquad \text{a.s.,} \tag{2.3}$$

where $T_n = \lambda_{\max}(\mathbf{D}_n) = \max_{1\le i\le n} \sum_{j\ne i} \xi_{ij}^{(n)}$ for $n \ge 2$.

PROOF OF (2.1). By Lemma A.1, for each $1 \le i \le n$ and $n \ge 2$, there exist i.i.d. $N(0,1)$-distributed random variables $\{\eta_{ij}^{(n)}; 1 \le j \le n, j \ne i\}$ for each $n \ge 2$ such that

$$\begin{aligned}
&\max_{1\le i\le n} P\left(\left|\sum_{j\ne i} \xi_{ij}^{(n)} - \sum_{j\ne i} \eta_{ij}^{(n)}\right| \ge \varepsilon\sqrt{n\log n}\right) \\
&\quad \le \frac{C}{1 + (\varepsilon\sqrt{n\log n})^6} \sum_{j\ne i} E|\xi_{ij}^{(n)}|^6 \\
&\quad \le \frac{C}{n^2(\log n)^3},
\end{aligned} \tag{2.4}$$

where here and later in all proofs, C stands for a constant not depending on i, j or n, and may be different from line to line. It is well known that

$$\frac{x}{\sqrt{2\pi}(1+x^2)} e^{-x^2/2} \le P(N(0,1) \ge x) \le \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \tag{2.5}$$

for any $x > 0$. Since $\sum_{j\ne i} \xi_{ij}^{(n)} \le \sum_{j\ne i} \eta_{ij}^{(n)} + |\sum_{j\ne i} \xi_{ij}^{(n)} - \sum_{j\ne i} \eta_{ij}^{(n)}|$, then

$$\begin{aligned}
&P(T_n \ge (\alpha + 2\varepsilon)\sqrt{n\log n}) \\
&\quad \le n \cdot \max_{1\le i\le n} P\left(\sum_{j\ne i} \xi_{ij}^{(n)} \ge (\alpha + 2\varepsilon)\sqrt{n\log n}\right) \\
&\quad \le n \cdot \max_{1\le i\le n} P\left(\sum_{j\ne i} \eta_{ij}^{(n)} \ge (\alpha + \varepsilon)\sqrt{n\log n}\right) \\
&\qquad + n \cdot \max_{1\le i\le n} P\left(\left|\sum_{j\ne i} \xi_{ij}^{(n)} - \sum_{j\ne i} \eta_{ij}^{(n)}\right| \ge \varepsilon\sqrt{n\log n}\right)
\end{aligned} \tag{2.6}$$

for any $\alpha > 0$ and $\varepsilon > 0$. Noticing $\sum_{j\neq i} \eta_{ij}^{(n)} \sim \sqrt{n-1}\cdot N(0,1)$ for any $1 \le i \le n$, by (2.5) and then (2.4),

$$P\big(T_n \ge (\alpha+2\varepsilon)\sqrt{n\log n}\big) \le nP\big(N(0,1) \ge (\alpha+\varepsilon)\sqrt{\log n}\big) + \frac{C}{n(\log n)^3} \tag{2.7}$$
$$\le Cn^{1-(\alpha+\varepsilon)^2/2} + \frac{C}{n(\log n)^3}$$

for n sufficiently large. In particular, taking $\alpha = \sqrt{2}$, we obtain that

$$P\left(\frac{T_n}{\sqrt{n\log n}} \ge \sqrt{2}+2\varepsilon\right) = O\left(\frac{1}{n^{\varepsilon}}\right) \tag{2.8}$$

as $n \to \infty$ for any $\varepsilon \in (0, 1]$, since the last term in (2.7) is of order $n^{-1}(\log n)^{-3}$ as $n \to \infty$.

Define $k_n = [n/\log n]$ and $V_n = \max_{1\le i\le k_n} |\sum_{1\le j\le k_n} \xi_{ij}^{(n)}|$ with $\xi_{ii}^{(n)} = 0$ for all $1 \le i \le n$. By the same argument as in obtaining (2.7), we have that, for any fixed $\alpha > 0$,

$$P\left(\frac{V_n}{\sqrt{k_n\log k_n}} \ge \alpha+2\varepsilon\right) \le C(k_n)^{1-(\alpha+\varepsilon)^2/2} + \frac{C}{k_n(\log k_n)^3} \tag{2.9}$$

as n is sufficiently large. Noticing $n/k_n \to \infty$, and taking $\alpha+\varepsilon = 10$ above, we have

$$P\big(V_n \ge \varepsilon\sqrt{n\log n}\big) \le \frac{1}{n(\log n)^{3/2}} \tag{2.10}$$

as n is sufficiently large. Observe that

$$T_n \ge \max_{1\le i\le k_n} \sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} - V_n. \tag{2.11}$$

Similarly to (2.4), by Lemma A.1, for each $1 \le i \le n$ and $n \ge 2$, there exist i.i.d. $N(0, 1)$-distributed random variables $\{\zeta_{ij}^{(n)}; 1 \le i \le n, j \ne i\}$ such that

$$\max_{1\le i\le k_n} P\left(\left|\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} - \sum_{j=k_n+1}^{n} \zeta_{ij}^{(n)}\right| \ge \varepsilon\sqrt{n\log n}\right)$$
$$\le \frac{C}{1+(\varepsilon\sqrt{n\log n})^6} \sum_{j=k_n+1}^{n} E|\xi_{ij}^{(n)}|^6 \tag{2.12}$$
$$\le \frac{C}{n^2(\log n)^3}$$

as n is sufficiently large for any $\varepsilon > 0$. Fix $\beta > 0$. By (2.11), (2.10) and then independence

$$\begin{aligned}
&P\big(T_n \le (\beta - 2\varepsilon)\sqrt{n\log n}\big)\\
&\quad\le P\Bigg(\max_{1\le i\le k_n}\sum_{j=k_n+1}^{n}\xi_{ij}^{(n)} \le (\beta-\varepsilon)\sqrt{n\log n}\Bigg) + P\big(V_n \ge \varepsilon\sqrt{n\log n}\big) \qquad (2.13)\\
&\quad\le \max_{1\le i\le k_n} P\Bigg(\sum_{j=k_n+1}^{n}\xi_{ij}^{(n)} \le (\beta-\varepsilon)\sqrt{n\log n}\Bigg)^{k_n} + \frac{1}{n(\log n)^{3/2}}
\end{aligned}$$

as n is sufficiently large. Observe that

$$\begin{aligned}
&P\Bigg(\sum_{j=k_n+1}^{n}\xi_{ij}^{(n)} \le (\beta-\varepsilon)\sqrt{n\log n}\Bigg)\\
&\quad\le P\Bigg(\sum_{j=k_n+1}^{n}\zeta_{ij}^{(n)} \le \bigg(\beta-\frac{\varepsilon}{2}\bigg)\sqrt{n\log n}\Bigg) \qquad (2.14)\\
&\qquad + P\Bigg(\Bigg|\sum_{j=k_n+1}^{n}\xi_{ij}^{(n)} - \sum_{j=k_n+1}^{n}\zeta_{ij}^{(n)}\Bigg| \ge \frac{\varepsilon}{2}\sqrt{n\log n}\Bigg).
\end{aligned}$$

Use the fact that $\sum_{j=k_n+1}^{n}\zeta_{ij}^{(n)} \sim \sqrt{n-k_n}\cdot N(0,1)$ and (2.5) to have

$$\begin{aligned}
&P\Bigg(\sum_{j=k_n+1}^{n}\zeta_{ij}^{(n)} > \bigg(\beta-\frac{\varepsilon}{2}\bigg)\sqrt{n\log n}\Bigg)\\
&\quad= P\Bigg(N(0,1) > \bigg(\beta-\frac{\varepsilon}{2}\bigg)\sqrt{\frac{n}{n-k_n}}\cdot\sqrt{\log n}\Bigg)\\
&\quad\ge \frac{C}{n^{(\beta-\varepsilon/3)^2/2}\log n}
\end{aligned}$$

uniformly for all $1 \le i \le k_n$ as n is sufficiently large and as $0 < \varepsilon/3 < \beta$, where in the last inequality we use the fact that $(\beta - (\varepsilon/2))\sqrt{n/(n-k_n)} \le (\beta - (\varepsilon/3))$ as n is sufficiently large. This, (2.12) and (2.14) imply

$$\begin{aligned}
\max_{1\le i\le k_n} P\Bigg(\sum_{j=k_n+1}^{n}\xi_{ij}^{(n)} \le (\beta-\varepsilon)\sqrt{n\log n}\Bigg) &\le 1 - \frac{C_1}{n^{(\beta-\varepsilon/3)^2/2}\log n} + \frac{C_2}{n^2(\log n)^3}\\
&\le 1 - \frac{C_3}{n^{(\beta-\varepsilon/3)^2/2}\log n}
\end{aligned}$$

as n is sufficiently large for any $0 < \varepsilon/3 < \beta \le 2$. Use inequality $1 - x \le e^{-x}$ for any $x > 0$ to obtain

$$\max_{1\le i\le k_n} P\Bigg(\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \le (\beta-\varepsilon)\sqrt{n\log n}\Bigg)^{k_n} \le \exp\{-Cn^{1-(\beta-\varepsilon/4)^2/2}\} \tag{2.15}$$

as n is sufficiently large for any $0 < \varepsilon/4 < \beta \le 2$. From (2.13), we conclude that

$$P\big(T_n \le (\beta-2\varepsilon)\sqrt{n\log n}\big) \le \exp\{-Cn^{1-(\beta-\varepsilon/4)^2/2}\} + \frac{1}{n(\log n)^{3/2}} \tag{2.16}$$

as n is sufficiently large for any $0 < \varepsilon/4 < \beta \le 2$. Now, take $\beta = \sqrt{2}$, and we get

$$P\bigg(\frac{T_n}{\sqrt{n\log n}} \le \sqrt{2} - 2\varepsilon\bigg) = O\bigg(\frac{1}{n(\log n)^{3/2}}\bigg) \tag{2.17}$$

as $n \to \infty$ for sufficiently small $\varepsilon > 0$. This and (2.8) imply (2.1).

PROOF OF (2.2) AND (2.3). To prove these, it suffices to show

$$\limsup_{n\to\infty} \frac{T_n}{\sqrt{n\log n}} \le 2 \quad \text{a.s.} \quad \text{and} \quad \liminf_{n\to\infty} \frac{T_n}{\sqrt{n\log n}} \ge \sqrt{2} \quad \text{a.s.} \tag{2.18}$$

and

$$P\bigg(\frac{T_n}{\sqrt{n\log n}} \in [a,b) \text{ for infinitely many } n \ge 2\bigg) = 1 \tag{2.19}$$

for any $(a,b) \subset (\sqrt{2}, 2)$.

First, choosing $\alpha = 2$ in (2.7), we have that $P(T_n \ge (2+2\varepsilon)\sqrt{n\log n}) = O(n^{-1}(\log n)^{-3})$ as $n \to \infty$ for any $\varepsilon \in (0,1)$. Thus, $\sum_{n\ge 2} P(T_n \ge (2+2\varepsilon) \times \sqrt{n\log n}) < \infty$. By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \frac{T_n}{\sqrt{n\log n}} \le 2 + 2\varepsilon \quad \text{a.s.}$$

for any $\varepsilon \in (0,1)$. This gives the first inequality in (2.18). By the same reasoning, the second inequality follows from (2.17). To prove (2.19), since $\{T_n, n \ge 2\}$ are independent from assumption, by the second Borel–Cantelli lemma, it is enough to show

$$\sum_{n\ge 2} P\bigg(\frac{T_n}{\sqrt{n\log n}} \in [a,b)\bigg) = \infty \tag{2.20}$$

for any $(a,b) \subset (\sqrt{2}, 2)$. By (2.7), we have that

$$P\bigg(\frac{T_n}{\sqrt{n\log n}} \ge b\bigg) \le \frac{C}{n^{(b-\varepsilon)^2/2-1}} \tag{2.21}$$

as n is sufficiently large and $\varepsilon > 0$ is sufficiently small. By (2.11),

$$\max_{1\le i\le k_n} \sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \le T_n + V_n$$

for $n \ge 2$. Thus, by independence and (2.10),

$$\begin{aligned}
& P\big(T_n \ge a\sqrt{n\log n}\big) \\
& \quad \ge P\Bigg(\max_{1\le i\le k_n} \sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \ge (a+\varepsilon)\sqrt{n\log n}\Bigg) \\
& \qquad - P\big(V_n \ge \varepsilon\sqrt{n\log n}\big) \qquad (2.22) \\
& \quad \ge 1 - \Bigg(1 - \min_{1\le i\le k_n} P\Bigg(\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \ge (a+\varepsilon)\sqrt{n\log n}\Bigg)\Bigg)^{k_n} \\
& \qquad - \frac{1}{n(\log n)^{3/2}}
\end{aligned}$$

as n is sufficiently large. By (2.12)

$$\begin{aligned}
& P\Bigg(\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \ge (a+\varepsilon)\sqrt{n\log n}\Bigg) \\
& \quad \ge P\Bigg(\sum_{j=k_n+1}^{n} \zeta_{ij}^{(n)} \ge (a+2\varepsilon)\sqrt{n\log n}\Bigg) \\
& \qquad - P\Bigg(\Bigg|\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} - \sum_{j=k_n+1}^{n} \zeta_{ij}^{(n)}\Bigg| \ge \varepsilon\sqrt{n\log n}\Bigg) \\
& \quad \ge P\big(N(0,1) \ge (a+3\varepsilon)\sqrt{\log n}\big) - \frac{1}{n^2}
\end{aligned}$$

uniformly for all $1 \le i \le k_n$ as n is sufficiently large. From (2.5), for any $\varepsilon > 0$,

$$P\big(N(0,1) \ge (a+3\varepsilon)\sqrt{\log n}\big) \sim \frac{C}{n^{(a+3\varepsilon)^2/2}\sqrt{\log n}}$$

as n is sufficiently large. Noting that $a \in (\sqrt{2}, 2)$, we have

$$P\Bigg(\sum_{j=k_n+1}^{n} \xi_{ij}^{(n)} \ge (a+\varepsilon)\sqrt{n\log n}\Bigg) \ge \frac{C}{n^{(a+3\varepsilon)^2/2}\sqrt{\log n}}$$

uniformly for all $1 \le i \le k_n$ as n is sufficiently large and ε is sufficiently small. Thus, since $k_n = [n/\log n]$, relate the above to (2.22) to give us that

$$\begin{aligned} P(T_n \ge a\sqrt{n\log n}) &\ge 1 - \left(1 - \frac{C}{n^{(a+3\varepsilon)^2/2}\sqrt{\log n}}\right)^{k_n} - \frac{1}{n(\log n)^{3/2}} \\ &\sim \frac{Ck_n}{n^{(a+3\varepsilon)^2/2}\sqrt{\log n}}(1+o(1)) - \frac{1}{n(\log n)^{3/2}} \\ &\ge \frac{C}{n^{(a+3\varepsilon)^2/2-1}(\log n)^2} \end{aligned}$$

as n is sufficiently large and $\varepsilon > 0$ is small enough, where in the "$\sim$" step above we use the fact that $1-(1-x_n)^{k_n} \sim k_n x_n$ if $x_n \to 0$, $k_n \to +\infty$ and $k_n x_n \to 0$ as $n \to \infty$. Combining this and (2.21), we eventually arrive at

$$\begin{aligned} P\left(\frac{T_n}{\sqrt{n\log n}} \in [a,b)\right) &= P\left(\frac{T_n}{\sqrt{n\log n}} \ge a\right) - P\left(\frac{T_n}{\sqrt{n\log n}} \ge b\right) \\ &\ge \frac{C_3}{n^{(a+3\varepsilon)^2/2-1}(\log n)^2} - \frac{C_4}{n^{(b-\varepsilon)^2/2-1}} \\ &\sim \frac{C_3}{n^{(a+3\varepsilon)^2/2-1}(\log n)^2} \end{aligned}$$

as n is sufficiently large and $\varepsilon > 0$ is sufficiently small, where $[a,b) \subset (\sqrt{2}, 2)$. Finally, choosing $\varepsilon > 0$ so small that $(a+3\varepsilon)^2/2 - 1 \in (0,1)$, we get (2.20). $\square$

PROOF OF COROLLARY 1.1. Recalling (1.2), let $\tilde{\xi}_{ij}^{(n)} = (\xi_{ij}^{(n)} - \mu_n)/\sigma_n$ for all $1 \le i < j \le n$ and $n \ge 2$. Then $\{\tilde{\xi}_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ satisfies (1.5) with $\mu_n = 0, \sigma_n = 1$ and $p > 6$. Let $\tilde{\mathbf{\Delta}}_n$ be generated by $\{\tilde{\xi}_{ij}^{(n)}\}$ as in (1.2). By Theorem 1, the conclusions there hold if $\lambda_{\max}(\mathbf{\Delta}_n)$ is replaced by $\lambda_{\max}(\tilde{\mathbf{\Delta}}_n)$. Notice

$$\mathbf{\Delta}_n = \sigma_n \tilde{\mathbf{\Delta}}_n + \mu_n \cdot (n\mathbf{I}_n - \mathbf{J}_n), \tag{2.23}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{J}_n$ is the $n \times n$ matrix with all of its entries equal to 1. It is easy to check that the eigenvalues of $n\mathbf{I}_n - \mathbf{J}_n$ are 0 with one fold and n with $n-1$ folds, respectively. First, apply the triangle inequality to (2.23) to have that $|\lambda_{\max}(\mathbf{\Delta}_n) - \sigma_n\lambda_{\max}(\tilde{\mathbf{\Delta}}_n)| \le \|\mu_n \cdot (n\mathbf{I}_n - \mathbf{J}_n)\| \le n|\mu_n|$. It follows that

$$\left|\frac{\lambda_{\max}(\mathbf{\Delta}_n)}{\sqrt{n\log n}\,\sigma_n} - \frac{\lambda_{\max}(\tilde{\mathbf{\Delta}}_n)}{\sqrt{n\log n}}\right| \le \frac{|\mu_n|}{(\log n)^{1/2} n^{-1/2}\sigma_n} \to 0$$

provided $|\mu_n| \ll \sigma_n\sqrt{\log n/n}$. Then (a1) and (b1) follow from Theorem 1. By the same argument

$$\left|\lambda_{\max}(\mathbf{\Delta}_n) - \lambda_{\max}(\mu_n \cdot (n\mathbf{I}_n - \mathbf{J}_n))\right| \le \sigma_n\|\tilde{\mathbf{\Delta}}_n\| = O(\sigma_n\sqrt{n\log n}) \qquad \text{a.s.}$$

as $n \to \infty$. Note that $\lambda_{\max}(\mu_n \cdot (n\mathbf{I}_n - \mathbf{J}_n)) = 0$ if $\mu_n < 0$, and is equal to $n\mu_n$ if $\mu_n > 0$ for any $n \ge 2$. Thus, if $\mu_n \gg \sigma_n\sqrt{\log n/n}$, we have $\lambda_{\max}(\mathbf{\Delta}_n)/(n\mu_n) \to 1$ a.s. as $n \to \infty$. If $\mu_n < 0$ for all $n \ge 2$, and $|\mu_n| \gg \sigma_n\sqrt{\log n/n}$, we obtain $\lambda_{\max}(\mathbf{\Delta}_n)/(n\mu_n) \to 0$ a.s. as $n \to \infty$. Then (a2), (a3), (b2) and (b3) are yielded.

Finally, since $E(-\xi_{ij}^{(n)}) = -\mu_n$ and $\mathrm{Var}(-\xi_{ij}^{(n)}) = \mathrm{Var}(\xi_{ij}^{(n)}) = \sigma_n^2$ for all i, j, n, by using the proved (a1) and (b1), we know that (a1) and (b1) are also true if $\lambda_{\max}(\mathbf{\Delta}_n)$ is replaced by $\lambda_{\max}(-\mathbf{\Delta}_n)$. Now, use the same arguments as in the proof of part (c) in Theorem 1 to get (a1) and (b1) when $\lambda_{\max}(\mathbf{\Delta}_n)$ is replaced with $\|\mathbf{\Delta}_n\|$. On the other hand, it is well known that $\mathbf{\Delta}_n$ is nonnegative definite if $\xi_{ij}^{(n)} \ge 0$ for all i, j, n (see, e.g., page 5 in [14]). Thus $\|\mathbf{\Delta}_n\| = \lambda_{\max}(\mathbf{\Delta}_n)$. Consequently (a2) and (b2) follow when $\lambda_{\max}(\mathbf{\Delta}_n)$ is replaced with $\|\mathbf{\Delta}_n\|$. □

To prove Theorem 2, we need some preliminary results.

LEMMA 2.2. *Let* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ *be defined on the same probability space. For each* $n \ge 2$, *let* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n\}$ *be independent r.v.s with* $E\xi_{ij}^{(n)} = 0$. *Define* $\xi_{ji}^{(n)} = \xi_{ij}^{(n)}$ *for all* i, j, n *and* $S_{n,1} = \sum_{1\le i\ne j\le n}(\xi_{ij}^{(n)})^2$ *and* $S_{n,2} = \sum_{i=1}^n(\sum_{j\ne i}\xi_{ij}^{(n)})^2$. *If* $\sup_{1\le i<j\le n, n\ge 2} E|\xi_{ij}^{(n)}|^{4+\delta} < \infty$ *for some* $\delta > 0$, *then*

$$\lim_{n\to\infty}\frac{S_{n,k} - ES_{n,k}}{n^2} = 0 \qquad a.s. \qquad \text{for } k = 1, 2. \tag{2.24}$$

PROOF. To make notation simple, we write $\xi_{ij} = \xi_{ij}^{(n)}$ for all $1 \le i \le j \le n$ when there is no confusion.

Case 1: $k = 1$. Recall the Marcinkiewicz–Zygmund inequality (see, e.g., Corollary 2 and its proof on page 368 in [13]), for any $p \ge 2$, there exists a constant C_p depending on p only such that

$$E\left|\sum_{i=1}^n X_i\right|^p \le C_p n^{p/2-1}\sum_{i=1}^n E|X_i|^p \tag{2.25}$$

for any sequence of independent random variables $\{X_i; 1 \le i \le n\}$ with $EX_i = 0$ and $E(|X_i|^p) < \infty$ for all $1 \le i \le n$. Taking $p = 2 + (\delta/2)$ in (2.25), we have from the Hölder inequality that

$$\begin{aligned} E(|\xi_{ij}^2 - E\xi_{ij}^2|^p) &\le 2^{p-1}E|\xi_{ij}|^{2p} + 2^{p-1}(E|\xi_{ij}|^2)^p \\ &\le 2^p \cdot \sup_{1\le i,j\le n, n\ge 1} E\left|\xi_{ij}^{(n)}\right|^{4+\delta} \\ &< \infty \end{aligned} \tag{2.26}$$

uniformly for all $1 \le i < j \le n, n \ge 2$. Write $S_{n,1} - ES_{n,1} = 2\sum_{1\le i<j\le n}(\xi_{ij}^2 - E\xi_{ij}^2)$. By (2.25),

$$E|S_{n,1} - ES_{n,1}|^p \le C \cdot \left(\frac{n(n-1)}{2}\right)^{\delta/4} \cdot \sum_{1\le i<j\le n} E(|\xi_{ij}^2 - E\xi_{ij}^2|^p) \tag{2.27}$$
$$\le C \cdot n^{2+(\delta/2)},$$

where C here and later, as earlier, is a constant not depending on n, and may be different from line to line. Then $P(|S_{n,1} - ES_{n,1}| \ge n^2\varepsilon) \le (n^2\varepsilon)^{-p} E|S_{n,1} - ES_{n,1}|^p = O(n^{-2-(\delta/2)})$ for any $\varepsilon > 0$ by the Markov inequality. Then (2.24) holds for $k = 1$ by the Borel–Cantelli lemma.

Case 2: $k = 2$. For $n \ge 2$, set $u_1 = v_n = 0$ and

$$u_i = \sum_{j=1}^{i-1} \xi_{ij} \qquad \text{for } 2 \le i \le n+1 \quad \text{and} \quad v_i = \sum_{j=i+1}^{n} \xi_{ij} \qquad \text{for } 0 \le i \le n-1.$$

Then, $\sum_{j\ne i} \xi_{ij} = u_i + v_i$ for all $1 \le i \le n$. Clearly, $S_{n,2} = \sum_{i=1}^n u_i^2 + \sum_{i=1}^n v_i^2 + 2\sum_{i=1}^n u_i v_i$ for all $n \ge 1$. Since $E(u_i v_i) = (Eu_i)Ev_i = 0$ by independence, to prove the lemma for $k = 2$, it suffices to show

$$\frac{1}{n^2}\sum_{i=1}^n (u_i^2 - Eu_i^2) \to 0 \qquad \text{a.s.,} \qquad \frac{1}{n^2}\sum_{i=1}^n (v_i^2 - Ev_i^2) \to 0 \qquad \text{a.s.}$$

and

$$\frac{1}{n^2}\sum_{i=1}^n u_i v_i \to 0 \qquad \text{a.s.} \tag{2.28}$$

as $n \to \infty$. We will only prove the first and the last assertions in two steps. The proof of the middle one is almost the same as that of the first and, therefore, is omitted.

Step 1. Similarly to the discussion in (2.26) and (2.27), we have $E|u_i|^{4+\delta} \le Ci^{2+(\delta/2)}$ for all $1 \le i \le n$ and $n \ge 2$. Now set $Y_{n,i} = (u_i^2 - Eu_i^2)/i$ for $i = 1, 2, \dots, n$. Then, $\{Y_{n,i}; 1 \le i \le n\}$ are independent random variables with

$$EY_{n,i} = 0, \qquad \sup_{1\le i,j\le n, n\ge 1} E|Y_{n,i}|^{2+\delta'} < \infty \tag{2.29}$$

and

$$\frac{1}{n^2}\sum_{i=1}^n (u_i^2 - Eu_i^2) = \frac{1}{n^2}\sum_{i=1}^n iY_{n,i} \tag{2.30}$$

for all $1 \le i \le n$ and $n \ge 2$, where $\delta' = \delta/2$. By (2.25) and (2.29),

$$E\left|\sum_{i=1}^n iY_{n,i}\right|^{2+\delta'} \le C \cdot n^{(2+\delta')/2-1} \sum_{i=1}^n i^{2+\delta'} = O\big(n^{3+(3\delta'/2)}\big)$$

as $n \to \infty$, where the inequality $\sum_{i=1}^{n} i^{2+\delta'} \le \sum_{i=1}^{n} n^{2+\delta'} \le n^{3+\delta'}$ is used in the above inequality. For any $t > 0$,

$$P\left(\frac{1}{n^2}\left|\sum_{i=1}^{n} iY_{n,i}\right| \ge t\right) \le \frac{E|\sum_{i=1}^{n} iY_{n,i}|^{2+\delta'}}{(n^2t)^{2+\delta'}} = O\left(\frac{1}{n^{1+(\delta'/2)}}\right)$$

as $n \to \infty$. This together with (2.30) concludes the first limit in (2.28) by the Borel–Cantelli lemma.

Step 2. We will prove the last assertion in (2.28) in this step. Define σ-algebra

$$\mathcal{F}_{n,0} = \{\varnothing, \Omega\} \quad \text{and} \quad \mathcal{F}_{n,k} = \sigma\big(\xi_{ij}^{(n)}; 1 \le i \le k, i+1 \le j \le n\big)$$

for $1 \le k \le n-1$. Obviously, $\mathcal{F}_{n,0} \subset \mathcal{F}_{n,1} \subset \cdots \subset \mathcal{F}_{n,n-1}$. It is easy to verify that

$$E\left(\sum_{i=1}^{k+1} u_i v_i | \mathcal{F}_{n,k}\right) = \sum_{i=1}^{k} u_i v_i$$

for $k = 1, 2, \dots, n-1$. Therefore, $\{\sum_{i=1}^{k} u_i v_i, \mathcal{F}_{n,k}, 1 \le k \le n-1\}$ is a martingale. By the given moment condition, $\tau := \sup_{1 \le i,j \le n, n \ge 1} E|\xi_{ij}^{(n)}|^4 < \infty$. From (2.25), $E(u_i^4) \le Ci^2 \le Cn^2$ and $E(v_i^4) \le C(n-i)^2 \le Cn^2$ for $1 \le i \le n$ and $n \ge 2$. By applying the Burkholder inequality (see, e.g., Theorem 2.10 from [29] or Theorem 1 on page 396 and the proof of Corollary 2 on page 268 from [13]), we have

$$E\left(\sum_{i=1}^{n-1} u_i v_i\right)^4 \le Cn^{(4/2)-1} \sum_{i=1}^{n-1} E((u_i v_i)^4) = Cn \sum_{i=1}^{n-1} E(u_i)^4 \cdot E(v_i)^4 = O(n^6)$$

as $n \to \infty$. By the Markov inequality,

$$P\left(\frac{1}{n^2}\left|\sum_{i=1}^{n} u_i v_i\right| \ge \delta\right) \le \frac{E|\sum_{i=1}^{n-1} u_i v_i|^4}{n^8\delta^4} = O\left(\frac{1}{n^2}\right)$$

as $n \to \infty$. The Borel–Cantelli says that $\sum_{i=1}^{n} u_i v_i / n^2 \to 0$ a.s. as $n \to \infty$. $\square$

For any two probability measures μ and ν on $\mathbb{R}$, define

$$d_{\mathrm{BL}}(\mu, \nu) = \sup\left\{\int f\, d\mu - \int f\, d\nu : \|f\|_\infty + \|f\|_L \le 1\right\}, \tag{2.31}$$

where $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$, $\|f\|_L = \sup_{x \ne y} |f(x) - f(y)|/|x-y|$. It is well known (see, e.g., Section 11.3 from [17]), that $d_{\mathrm{BL}}(\cdot, \cdot)$ is called the bounded Lipschitz metric, which characterizes the weak convergence of probability measures. Reviewing (1.4), for the spectral measures of $n \times n$ real and symmetric matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, we have (see, e.g., (2.16) from [12])

$$d_{\mathrm{BL}}^2(\hat{\mu}(\mathbf{M}_1), \hat{\mu}(\mathbf{M}_2)) \le \frac{1}{n} \operatorname{tr}\big((\mathbf{M}_1 - \mathbf{M}_2)^2\big). \tag{2.32}$$

To prove Theorem 2, we first reduce it to the case that all random variables in the matrices are uniformly bounded. This step will be carried out through a truncation argument by using (2.32).

LEMMA 2.3. *If Theorem 2 holds for all uniformly bounded r.v.s* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ *satisfying* (1.5) *with* $\mu_n = 0$ *and* $\sigma_n = 1$ *for all* $n \ge 2$, *then it also holds for all r.v.s* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ *satisfying* (1.5) *with* $p = 4 + \delta$ *for some* $\delta > 0$, *and* $\mu_n = 0$ *and* $\sigma_n = 1$ *for all* $n \ge 2$.

PROOF. As in the proof of Lemma 2.2, we write ξ_{ij} for $\xi_{ij}^{(n)}$ if there is no danger of confusion. Fix $u > 0$. Let

$$\tilde{\xi}_{ij} = \xi_{ij} I\{|\xi_{ij}| \le u\} - E(\xi_{ij} I\{|\xi_{ij}| \le u\})$$

and

$$\sigma_{ij}(u) = \sqrt{\mathrm{Var}(\tilde{\xi}_{ij})}$$

for all i and j. Note that

$$\left|\sigma_{ij}(u) - \sqrt{\mathrm{Var}(\xi_{ij})}\right| \le \sqrt{\mathrm{Var}(\xi_{ij} - \tilde{\xi}_{ij})} \le \sqrt{E\xi_{ij}^2 I\{|\xi_{ij}| > u\}}$$

by the triangle inequality. Thus, with condition that $\sup_{1 \le i < j \le n, n \ge 2} E|\xi_{ij}^{(n)}|^{4+\delta} < \infty$, we see that

$$\sup_{1 \le i < j \le n, n \ge 2} |\sigma_{ij}(u) - 1| \to 0 \quad \text{and} \quad \sup_{1 \le i < j \le n, n \ge 2} E(\xi_{ij} - \tilde{\xi}_{ij})^2 \to 0 \tag{2.33}$$

as $u \to +\infty$. Take $u > 0$ large enough such that $\sigma_{ij}(u) > 1/2$ for all $1 \le i \ne j \le n$ and $n \ge 2$. Write

$$\xi_{ij} = \underbrace{\frac{\tilde{\xi}_{ij}}{\sigma_{ij}(u)}}_{x_{ij}^{(n)}} + \underbrace{\frac{\sigma_{ij}(u) - 1}{\sigma_{ij}(u)} \cdot \tilde{\xi}_{ij}}_{y_{ij}^{(n)}} + \underbrace{(\xi_{ij} - \tilde{\xi}_{ij})}_{z_{ij}^{(n)}}$$

for all $1 \le i \ne j \le n, n \ge 2$. Obviously, for $a_{ij}^{(n)} = x_{ij}^{(n)}, y_{ij}^{(n)}$ or $z_{ij}^{(n)}$, we know $\{a_{ij}^{(n)}; 1 \le i < j \le n\}$ are independent for each $n \ge 2$, and

$$Ea_{ij}^{(n)} = 0 \quad \text{and} \quad \sup_{1 \le i < j \le n, n \ge 2} E\left|a_{ij}^{(n)}\right|^{4+\delta} < \infty. \tag{2.34}$$

Again, for convenience, write x_{ij}, y_{ij} and z_{ij} for $x_{ij}^{(n)}, y_{ij}^{(n)}$ and $z_{ij}^{(n)}$. Clearly, $\{x_{ij}; 1 \le i < j \le n, n \ge 2\}$ are uniformly bounded. Besides, it is easy to see from (2.33) that

$$\sup_{1 \le i < j \le n, n \ge 2} \left(E(y_{ij}^2) + E(z_{ij}^2)\right) \to 0 \tag{2.35}$$

as $u \to +\infty$.

Let $\mathbf{X}_n$, $\mathbf{Y}_n$ and $\mathbf{Z}_n$ be the Laplacian matrices generated by $\{x_{ij}\}$, $\{y_{ij}\}$ and $\{z_{ij}\}$ as in (1.2), respectively. Then $\boldsymbol{\Delta}_n = \mathbf{X}_n + \mathbf{Y}_n + \mathbf{Z}_n$. With (2.32), use the inequality that $\operatorname{tr}((\mathbf{M}_1 + \mathbf{M}_2)^2) \le 2\operatorname{tr}(\mathbf{M}_1^2) + 2\operatorname{tr}(\mathbf{M}_2^2)$ for any symmetric matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ to obtain that

$$\begin{aligned} d_{\mathrm{BL}}^2\biggl(\frac{\boldsymbol{\Delta}_n}{\sqrt{n}}, \frac{\mathbf{X}_n}{\sqrt{n}}\biggr) &\le \frac{1}{n^2}\operatorname{tr}\bigl((\mathbf{Y}_n + \mathbf{Z}_n)^2\bigr) \\ &\le \frac{2}{n^2}\sum_{1\le i\ne j\le n}\bigl((y_{ij})^2 + (z_{ij})^2\bigr) \\ &\quad + \frac{2}{n^2}\sum_{i=1}^n\biggl\{\biggl(\sum_{j\ne i} y_{ij}\biggr)^2 + \biggl(\sum_{j\ne i} z_{ij}\biggr)^2\biggr\}. \end{aligned}$$

By independence and symmetry,

$$E\biggl(\biggl(\sum_{j\ne i} y_{ij}\biggr)^2 + \biggl(\sum_{j\ne i} z_{ij}\biggr)^2\biggr) = 2\sum_{j\ne i}\{E(y_{ij})^2 + E(z_{ij})^2\}.$$

Recalling (2.34), by applying Lemma 2.2, we have

$$\begin{aligned} &\limsup_{n\to\infty} d_{\mathrm{BL}}^2\biggl(\frac{\boldsymbol{\Delta}_n}{\sqrt{n}}, \frac{\mathbf{X}_n}{\sqrt{n}}\biggr) \\ &\quad \le C\cdot \sup_{1\le i<j\le n, n\ge 2}\bigl(E(y_{ij}^2) + E(z_{ij}^2)\bigr) \to 0 \qquad \text{a.s.} \end{aligned} \tag{2.36}$$

as $u \to +\infty$ thanks to (2.35). Noticing $Ex_{ij} = 0$, $Ex_{ij}^2 = 1$ for all i, j, and $\{x_{ij}; 1 \le i < j \le n, n \ge 2\}$ are uniformly bounded. By assumption, $d_{\mathrm{BL}}(\hat{\mu}(n^{-1/2} \times \mathbf{X}_n), \gamma_M) \to 0$ as $n \to \infty$, where γ_M is the probability measure mentioned in Theorem 2. With this, (2.36) and the triangle inequality of metric d_{BL}, we see that $d_{\mathrm{BL}}(\hat{\mu}(n^{-1/2}\boldsymbol{\Delta}_n), \gamma_M) \to 0$ as $n \to \infty$. □

Given $n \ge 2$, let $\Gamma_n = \{(i, j); 1 \le j < i \le n\}$ be a graph. We say $a = (i_1, j_1)$ and $b = (i_2, j_2)$ form an edge and denote it by $a \sim b$, if one of i_1 and j_1 is identical to one of i_2 and j_2. For convenience of notation, from now on, we write $a = (a^+, a^-)$ for any $a \in \Gamma_n$. Of course, $a^+ > a^-$. Given $a, b \in \Gamma_n$, define an $n \times n$ matrix

$$\mathbf{Q}_{a,b}[i, j] = \begin{cases} -1, & \text{if } i = a^+, j = b^+ \text{ or } i = a^-, j = b^-; \\ 1, & \text{if } i = a^+, j = b^- \text{ or } i = a^-, j = b^+; \\ 0, & \text{otherwise.} \end{cases}$$

With this notation, we rewrite $\mathbf{M}_n$ as follows

$$-\boldsymbol{\Delta}_n = \sum_{a\in\Gamma_n} \xi_a^{(n)}\mathbf{Q}_{a,a}, \tag{2.37}$$

where $\xi_a^{(n)} = \xi_{a^+a^-}^{(n)}$ for $a \in \Gamma_n$. Let $t_{a,b} = \operatorname{tr}(\mathbf{Q}_{a,b})$. We summarize some facts from [12] in the following lemma.

LEMMA 2.4. *Let $a, b \in \Gamma_n$. The following assertions hold*:

(i) $t_{a,b} = t_{b,a}$.

(ii)
$$t_{a,b} = \begin{cases} -2, & \text{if } a = b; \\ -1, & \text{if } a \neq b \text{ and } a^- = b^- \text{ or } a^+ = b^+; \\ 1, & \text{if } a \neq b \text{ and } a^- = b^+ \text{ or } a^+ = b^-; \\ 0, & \text{otherwise.} \end{cases}$$

(iii) $\mathbf{Q}_{a,b} \times \mathbf{Q}_{c,d} = t_{b,c}\mathbf{Q}_{a,d}$. *Therefore*, $\operatorname{tr}(\mathbf{Q}_{a_1,a_1} \times \mathbf{Q}_{a_2,a_2} \times \cdots \times \mathbf{Q}_{a_r,a_r}) = \prod_{j=1}^r t_{a_j,a_{j+1}}$, *where $a_1, \dots, a_r \in \Gamma_n$, and $a_{r+1} = a_1$.*

We call $\pi = (a_1, \dots, a_r)$ a circuit of length r if $a_1 \sim \cdots \sim a_r \sim a_1$. For such a circuit, let

$$\xi_\pi^{(n)} = \prod_{j=1}^r t_{a_j,a_{j+1}} \prod_{j=1}^r \xi_{a_j}^{(n)}. \tag{2.38}$$

From (2.37), we know

$$\operatorname{tr}(\mathbf{\Delta}_n^r) = (-1)^r \sum_\pi \xi_\pi^{(n)} \quad \text{and} \quad E \operatorname{tr}(\mathbf{\Delta}_n^r) = (-1)^r \sum_\pi E\xi_\pi^{(n)}, \tag{2.39}$$

where the sum is taken over all circuits of length r in Γ_n.

DEFINITION 2.1. We say that a circuit $\pi = (a_1 \sim \cdots \sim a_r \sim a_1)$ of length r in Γ_n is vertex-matched if for each $i = 1, \dots, r$ there exists some $j \neq i$ such that $a_i = a_j$, and that it has a match of order 3 if some value is repeated at least three times among $\{a_j, j = 1, \dots, r\}$.

Clearly, by independence, the only possible nonzero terms in the second sum in (2.39) come from vertex-matched circuits. For $x \geq 0$, denote by $\lfloor x \rfloor$ the integer part of x. The following two lemmas will be used later.

LEMMA 2.5 (Propositions 4.10 and 4.14 from [12]). *Fix $r \in \mathbb{N}$.*

(i) *Let N denote the number of vertex-matched circuits in Γ_n with vertices having at least one match of order 3. Then $N = O(n^{\lfloor (r+1)/2 \rfloor})$ as $n \to \infty$.*

(ii) *Let N denote the number of vertex-matched quadruples of circuits in Γ_n with r vertices each, such that none of them is self-matched. Then $N = O(n^{2r+2})$ as $n \to \infty$.*

Let $\mathbf{U}_n$ be a symmetric matrix of form

$$\mathbf{U_n} = \begin{pmatrix} \sum_{j\neq 1} Y_{1j} & -Y_{12} & -Y_{13} & \cdots & -Y_{1n} \\ -Y_{21} & \sum_{j\neq 2} Y_{2j} & -Y_{23} & \cdots & -Y_{2n} \\ -Y_{31} & -Y_{32} & \sum_{j\neq 3} Y_{3j} & \cdots & -Y_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -Y_{n1} & -Y_{n2} & -Y_{n3} & \cdots & \sum_{j\neq n} Y_{nj} \end{pmatrix}, \tag{2.40}$$

where $\{Y_{ij}; 1 \le i < j < \infty\}$ are i.i.d. standard normal random variables not depending on n.

LEMMA 2.6. *Suppose the conditions in Theorem 2 hold with $\mu_n = 0$ and $\sigma_n = 1$ for all $n \ge 2$. Furthermore, assume $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ are uniformly bounded. Then*:

(i) $\lim_{n\to\infty} \frac{1}{n^{k+1/2}} \mathbb{E}\,\mathrm{tr}(\boldsymbol{\Delta}_n^{2k-1}) = 0$;

(ii) $\lim_{n\to\infty} \frac{1}{n^{k+1}} (\mathbb{E}\,\mathrm{tr}(\boldsymbol{\Delta}_n^{2k}) - \mathbb{E}\,\mathrm{tr}(\mathbf{U}_n^{2k})) = 0$

for any integer $k \ge 1$, where $\mathbf{U}_n$ is as in (2.40).

PROOF. (i) As remarked earlier, all nonvanishing terms in the representation of $\mathbb{E}\,\mathrm{tr}(\boldsymbol{\Delta}_n^{2k-1})$ in (2.39) are of form $\mathbb{E}\xi_\pi^{(n)}$ with the vertices of the path $a_1 \sim a_2 \sim \cdots \sim a_{2k-1} \sim a_1$ in π repeating at least two times. Since $2k-1$ is an odd number, there exists a vertex such that it repeats at least three times. Also, in view of (2.38) and that $|t_{a,b}| \le 2$ for any $a, b \in \Gamma_n$, thus all such terms $\mathbb{E}\xi_\pi^{(n)}$ are uniformly bounded. Therefore, by (i) of Lemma 2.5,

$$\left|\frac{1}{n^{k+1/2}} \cdot \mathbb{E}\,\mathrm{tr}(\boldsymbol{\Delta}_n^{2k-1})\right| \le \frac{C}{\sqrt{n}} \to 0$$

as $n \to \infty$, where C is a constant not depending on n.

(ii) Recall (2.40). Define $Y_\pi^{(n)}$ similarly to $\xi_\pi^{(n)}$ in (2.38). We then have that

$$\begin{aligned} |\mathbb{E}\,\mathrm{tr}(\boldsymbol{\Delta}_n^{2k}) - \mathbb{E}\,\mathrm{tr}(\mathbf{U}_n^{2k})| &= \left|\sum_\pi (\mathbb{E}\xi_\pi^{(n)} - \mathbb{E}Y_\pi^{(n)})\right| \\ &\le \left|\sum_{\pi\in A_1} (\mathbb{E}\xi_\pi^{(n)} - \mathbb{E}Y_\pi^{(n)})\right| + \left|\sum_{\pi\in A_2} (\mathbb{E}\xi_\pi^{(n)} - \mathbb{E}Y_\pi^{(n)})\right| \\ &:= I_1 + I_2, \end{aligned}$$

where A_1 denotes the set of the vertex-matched circuits with match of order 3, and A_2 denotes the set of the vertex-matched circuits in Γ_n such that there are exactly

k distinct matches. Observe that each vertex of any circuit in A_2 matches exactly two times. From the independence assumption and that $E|\xi_{ij}^{(n)}|^2 = 1$ for all $1 \le i < j \le n$ and $n \ge 2$, we know $\mathbb{E}\xi_\pi^{(n)} = \mathbb{E}Y_\pi^{(n)} = 1$ for $\pi \in A_2$. This gives $I_2 = 0$. By Lemma 2.5, the cardinality of $A_1 \le n^k$. Since $\xi_{ij}^{(n)}$ are uniformly bounded and Y_{ij} are standard normal random variables, we have $I_1 \le Cn^k$ for some constant $C > 0$ not depending on n. In summary

$$\frac{1}{n^{k+1}} |\mathbb{E}\operatorname{tr}(\mathbf{\Delta}_n^{2k}) - \mathbb{E}\operatorname{tr}(\mathbf{U}_n^{2k})| = O\left(\frac{1}{n}\right)$$

as $n \to \infty$. The proof is complete. □

LEMMA 2.7. *Suppose* (1.5) *holds for some* $p > 4$. *Assume* $\mu_n = 0, \sigma_n = 1$ *for all* $n \ge 2$. *Then, as* $n \to \infty$, $F^{\mathbf{\Delta}_n/\sqrt{n}}$ *converges weakly to the free convolution* γ_M *of the semicircular law and standard normal distribution. The measure* γ_M *is a nonrandom symmetric probability measure with smooth bounded density, does not depend on the distribution of* $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ *and has an unbounded support.*

PROOF. By Lemma 2.3, without loss of generality, we now assume that $\{\xi_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ are uniformly bounded random variables with mean zero and variance one, and $\{\xi_{ij}^{(n)}; 1 \le i < j \le n\}$ are independent for each $n \ge 2$.

Proposition A.3 from [12] says that γ_M is a symmetric distribution and uniquely determined by its moments. Thus, to prove the theorem, it is enough to show that

$$\begin{aligned}\frac{1}{n}\operatorname{tr}(n^{-1/2}\mathbf{\Delta}_n)^r &= \frac{1}{n^{r/2+1}}\operatorname{tr}(\mathbf{\Delta}_n^r) = \int x^r \, dF^{n^{-1/2}\mathbf{\Delta}_n} \\ &\to \int x^r \, d\gamma_M \qquad \text{as } n \to \infty \qquad \text{a.s.}\end{aligned} \tag{2.41}$$

for any integer $r \ge 1$. First, we claim that

$$\mathbb{E}\big[\big(\operatorname{tr}(\mathbf{\Delta}_n^r) - \mathbb{E}\operatorname{tr}(\mathbf{\Delta}_n^r)\big)^4\big] = O(n^{2r+2}) \tag{2.42}$$

as $n \to \infty$. In fact, by (2.39), we have

$$\mathbb{E}\big[\big(\operatorname{tr}(\mathbf{\Delta}_n^r) - \mathbb{E}\operatorname{tr}(\mathbf{\Delta}_n^r)\big)^4\big] = \sum_{\pi_1,\pi_2,\pi_3,\pi_4} \mathbb{E}\left[\prod_{j=1}^4 \big(\xi_{\pi_j} - \mathbb{E}(\xi_{\pi_j})\big)\right], \tag{2.43}$$

where the sum runs over all circuits π_j, $j = 1, 2, 3, 4$ in Γ_n, each having r vertices. From the assumption, we know $\{\xi_{ij}^{(n)}, 1 \le i < j \le n\}$ are independent random variables of mean zero, and it is enough to consider the terms in (2.43) with all vertex-matched quadruples of circuits on Γ_n, such that none of them is self-matched. By assumption, $\{\xi_{ij}^{(n)}; 1 \le i < j \le n; n \ge 2\}$ are uniformly bounded, so all terms $\mathbb{E}[\prod_{j=1}^4(\xi_{\pi_j} - \mathbb{E}(\xi_{\pi_j}))]$ in the sum are uniformly bounded. By (ii) of Lemma 2.5, we obtain (2.42).

By the Markov inequality,

$$
\begin{aligned}
&P\left(\frac{1}{n}|\operatorname{tr}((n^{-1/2}\mathbf{\Delta}_n)^r) - \mathbb{E}\operatorname{tr}((n^{-1/2}\mathbf{\Delta}_n)^r)| \ge \varepsilon\right) \\
&\quad \le \frac{E|\operatorname{tr}(\mathbf{\Delta}_n^r) - \mathbb{E}\operatorname{tr}(\mathbf{\Delta}_n^r)|^4}{(n^{1+(r/2)}\varepsilon)^4} = O\left(\frac{1}{n^2}\right)
\end{aligned}
\tag{2.44}
$$

as $n \to \infty$. It follows from the Borel–Cantelli lemma that

$$
\frac{1}{n}\bigl(\operatorname{tr}((n^{-1/2}\mathbf{\Delta}_n)^r) - \mathbb{E}\operatorname{tr}((n^{-1/2}\mathbf{\Delta}_n)^r)\bigr) \to 0 \qquad \text{a.s.} \tag{2.45}
$$

as $n \to \infty$. Recalling $\mathbf{U}_n$ in (2.40), Proposition 4.13 in [12] says that

$$
\frac{1}{n}\mathbb{E}\operatorname{tr}((n^{-1/2}\mathbf{U}_n)^{2k}) \to \int_{\mathbb{R}} x^{2k}\, d\gamma_M
$$

as $n \to \infty$ for any $k \ge 1$. This, (ii) of Lemma 2.6 and (2.45) imply (2.41) for any even number $r \ge 1$. For odd number r, (i) of Lemma 2.6 and (2.45) yield (2.41) since γ_M is symmetric, hence its odd moments are equal to zero. $\square$

PROOF OF THEOREM 2. Recalling (1.2), let $\tilde{\xi}_{ij}^{(n)} = (\xi_{ij}^{(n)} - \mu_n)/\sigma_n$ for all $1 \le i < j \le n$ and $n \ge 2$. Then $\{\tilde{\xi}_{ij}^{(n)}; 1 \le i < j \le n, n \ge 2\}$ satisfies (1.5) with $\mu_n = 0, \sigma_n = 1$ and $p > 4$. Let $\mathbf{\Delta}_{n,1}$ be generated by $\{\tilde{\xi}_{ij}^{(n)}\}$ as in (1.2). By Lemma 2.7, almost surely,

$$
F^{\mathbf{\Delta}_{n,1}/\sqrt{n}} \qquad \text{converges weakly to } \gamma_M \tag{2.46}
$$

as $n \to \infty$. It is easy to verify that

$$
\mathbf{\Delta}_n = \underbrace{\sigma_n \mathbf{\Delta}_{n,1} + (n\mu_n)\mathbf{I}_n}_{\mathbf{\Delta}_{n,2}} - \mu_n \mathbf{J}_n, \tag{2.47}
$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and $\mathbf{J}_n$ is the $n \times n$ matrix with all of its entries equal to 1. Obviously, the eigenvalues of $\mathbf{\Delta}_{n,2}$ are $\sigma_n \cdot \lambda_i(\mathbf{\Delta}_{n,1}) + n\mu_n$, $1 \le i \le n$. By (2.46),

$$
\frac{1}{n}\sum_{i=1}^n I\left(\frac{\lambda_i(\mathbf{\Delta}_{n,2}) - n\mu_n}{\sqrt{n}\sigma_n} \le x\right) \qquad \text{converges weakly to } \gamma_M \tag{2.48}
$$

almost surely as $n \to \infty$. By (2.47) and the rank inequality (see Lemma 2.2 from [6]),

$$
\begin{aligned}
&\|F^{(\mathbf{\Delta}_n - n\mu_n\mathbf{I}_n)/\sqrt{n}\sigma_n} - F^{(\mathbf{\Delta}_{n,2} - n\mu_n\mathbf{I}_n)/\sqrt{n}\sigma_n}\| \\
&\quad \le \frac{1}{n}\cdot\operatorname{rank}\left(\frac{\mathbf{\Delta}_n}{\sqrt{n}\sigma_n} - \frac{\mathbf{\Delta}_{n,2}}{\sqrt{n}\sigma_n}\right) = \frac{1}{n}\cdot\operatorname{rank}\left(\frac{\mu_n}{\sqrt{n}\sigma_n}\mathbf{J}_n\right) \le \frac{1}{n} \to 0,
\end{aligned}
\tag{2.49}
$$

where $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ for any bounded, measurable function $f(x)$ defined on $\mathbb{R}$. Finally, (2.48) and (2.49) lead to the desired conclusion. □

PROOF OF THEOREM 3. Let $\mathbf{V}_n = (v_{ij}^{(n)})$ be defined by

$$v_{ii}^{(n)} = 0 \quad \text{and} \quad v_{ij}^{(n)} = \frac{\xi_{ij}^{(n)} - \mu_n}{\sigma_n} \tag{2.50}$$

for any $1 \le i \ne j \le n$ and $n \ge 2$, where $\mathbf{A}_n = (\xi_{ij}^{(n)})_{n\times n}$ as in (1.1) with $\xi_{ii}^{(n)} = 0$ for all $1 \le i \le n$ and $n \ge 2$. It is easy to check that $\mathbf{A}_n = \mu_n(\mathbf{J}_n - \mathbf{I}_n) + \sigma_n \mathbf{V}_n$, where all the entries of $\mathbf{J}_n$ are equal to one, and $\mathbf{I}_n$ is the $n \times n$ identity matrix. Thus

$$\frac{\mathbf{A}_n + \mu_n \mathbf{I}_n}{\sqrt{n}\sigma_n} - \frac{\mathbf{V}_n}{\sqrt{n}} = \frac{\mu_n \mathbf{J}_n}{\sqrt{n}\sigma_n} \qquad \text{where all entries of } \mathbf{J}_n \text{ are equal to 1.}$$

By the rank inequality (see Lemma 2.2 from [6]),

$$\|F^{(\mathbf{A}_n+\mu_n\mathbf{I})/\sqrt{n}\sigma_n} - F^{n^{-1/2}\mathbf{V}_n}\| \le \frac{1}{n} \cdot \operatorname{rank}\left(\frac{\mathbf{A}_n + \mu_n \mathbf{I}_n}{\sqrt{n}\sigma_n} - \frac{\mathbf{V}_n}{\sqrt{n}}\right) \le \frac{1}{n} \to 0,$$

where $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ for any bounded, measurable function $f(x)$ defined on $\mathbb{R}$ as in (2.49). So, to prove the theorem, it is enough to show that $F^{n^{-1/2}\mathbf{V}_n}$ converges weakly to the semicircular law with the density given in statement of the theorem. In view of normalization (2.50), without loss of the generality, we only need to prove the theorem under the conditions that

$$E\omega_{ij}^{(n)} = 0, \qquad E(\omega_{ij}^{(n)})^2 = 1$$

and

$$\max_{1\le i<j\le n} E\{(\omega_{ij}^{(n)})^2 I(|\omega_{ij}^{(n)}| \ge \varepsilon\sqrt{n})\} \to 0 \qquad \text{as } n \to \infty$$

for all $1 \le i, j \le n$ and $n \ge 2$. Given $\delta > 0$, note that

$$\begin{aligned} &\frac{1}{n^2\delta^2} \sum_{1\le i,j\le n} E\{(\omega_{ij}^{(n)})^2 I(|\omega_{ij}^{(n)}| \ge \delta\sqrt{n})\} \\ &\quad \le \frac{2}{\delta^2} \cdot \max_{1\le i<j\le n} E\{(\omega_{ij}^{(n)})^2 I(|\omega_{ij}^{(n)}| \ge \delta\sqrt{n})\} \to 0 \end{aligned}$$

as $n \to \infty$. By Lemma A.2 in the Appendix, $\tilde{F}_n := F^{n^{-1/2}\mathbf{V}_n}$, and hence $F^{n^{-1/2}(\mathbf{A}_n+\mu_n\mathbf{I})}$, converges weakly to the semicircular law. □

PROOF OF COROLLARY 1.2. To apply Theorem 3, we first need to verify

$$\max_{1\le i<j\le n} E\{(\omega_{ij}^{(n)})^2 I(|\omega_{ij}^{(n)}| \ge \varepsilon\sqrt{n})\} \to 0 \tag{2.51}$$

as $n \to \infty$ for any $\varepsilon > 0$, where $\omega_{ij}^{(n)} := (\xi_{ij}^{(n)} - \mu_n)/\sigma_n$. Note that $\mu_n = p_n$ and $\sigma_n^2 = p_n(1-p_n)$. Now, use the fact that $\xi_{ij}^{(n)}$ take values one and zero only, and then the condition $np_n(1-p_n) \to \infty$ to see that $|\omega_{ij}^{(n)}| \le 1/\sigma_n = o(\sqrt{n})$ as $n \to \infty$. Then (2.51) follows. By Theorem 3,

$$\frac{1}{n}\sum_{i=1}^{n} I\left\{\frac{\lambda_i(\mathbf{A}_n) + p_n}{\sqrt{np_n(1-p_n)}} \le x\right\} \tag{2.52}$$

converges weakly to the distribution with density $\frac{1}{2\pi}\sqrt{4-x^2}I(|x| \le 2)$ almost surely. Notice

$$\left\{\frac{\lambda_i(\mathbf{A}_n) + p_n}{\sqrt{np_n(1-p_n)}} \le x\right\} = \left\{\frac{\lambda_i(\mathbf{A}_n)}{\sqrt{np_n(1-p_n)}} \le x - \frac{p_n}{\sqrt{np_n(1-p_n)}}\right\}$$

and $p_n/\sqrt{np_n(1-p_n)} \to 0$ as $n \to \infty$. By using a standard analysis, we obtain that, with probability one, $F^{\mathbf{A}_n/\alpha_n}$ converges weakly to the semicircular law with density $\frac{1}{2\pi}\sqrt{4-x^2}I(|x| \le 2)$, where $\alpha_n = \sqrt{np_n(1-p_n)}$. Further, assume now $1/n \ll p_n \to 0$ as $n \to \infty$. Write

$$\left\{\frac{\lambda_i(\mathbf{A}_n) + p_n}{\sqrt{np_n(1-p_n)}} \le x\right\} = \left\{\frac{\lambda_i(\mathbf{A}_n)}{\sqrt{np_n}} \le x\sqrt{1-p_n} - \sqrt{\frac{p_n}{n}}\right\}.$$

Clearly, $x\sqrt{1-p_n} - \sqrt{p_n/n} \to x$ as $n \to \infty$. Thus, by (2.52), we have $\frac{1}{n} \times \sum_{i=1}^{n} I\{\frac{\lambda_i(\mathbf{A}_n)}{\sqrt{np_n}} \le x\}$ converges weakly to the semicircular law with density $\frac{1}{2\pi} \times \sqrt{4-x^2}I(|x| \le 2)$. $\square$

We need the following lemma to prove Theorem 4.

LEMMA 2.8. *For $n \ge 2$, let $\lambda_{n,1} \ge \lambda_{n,2} \ge \cdots \ge \lambda_{n,n}$ be real numbers. Set $\mu_n = (1/n)\sum_{i=1}^{n}\delta_{\lambda_{n,i}}$. Suppose μ_n converges weakly to a probability measure μ. Then, for any sequence of integers $\{k_n; n \ge 2\}$ satisfying $k_n = o(n)$ as $n \to \infty$, we have* $\liminf_{n\to\infty}\lambda_{n,k_n} \ge \alpha$, *where* $\alpha = \inf\{x \in \mathbb{R} : \mu([x, +\infty]) = 0\}$ *with* $\inf\varnothing = +\infty$.

PROOF. Since μ is a probability measure, we know that $\alpha > -\infty$. Without loss of the generality, assume that $\alpha > 0$. For brevity of notation, write $k_n = k$. Set $\tilde{\mu}_n = (n-k+1)^{-1}\sum_{i=k}^{n}\delta_{\lambda_{n,i}}$ for $n \ge k$. Observe that

$$\mu_n(B) - \tilde{\mu}_n(B) = \frac{1}{n}\sum_{i=1}^{k-1} I(\lambda_{n,i} \in B) - \frac{k-1}{n(n-k+1)}\sum_{i=k}^{n} I(\lambda_{n,i} \in B)$$

for any set $B \subset \mathbb{R}$, where $\sum_{i=1}^{k-1} I(\lambda_{n,i} \in B)$ is understood to be zero if $k = 1$. Thus, $|\mu_n(B) - \tilde{\mu}_n(B)| \le 2k/n$. Therefore,

$$\tilde{\mu}_n \qquad \text{converges weakly to } \mu \tag{2.53}$$

since $k = k_n = o(n)$ as $n \to \infty$. Easily,

$$\lambda_{n,k}^m I(\lambda_{n,k} > 0) \geq \frac{1}{n-k+1} \sum_{i=k}^{n} \lambda_{n,i}^m I(\lambda_{n,i} > 0) = \int_0^\infty x^m \tilde{\mu}_n(dx)$$

for any integer $m \geq 1$. Write the last term above as $\int_{\mathbb{R}} g(x)\tilde{\mu}_n(dx)$, where $g(x) := x^m I(x \geq 0)$, $x \in \mathbb{R}$, is a continuous and nonnegative function. By (2.53) and the Fatou lemma,

$$\liminf_{n\to\infty} \lambda_{n,k}^m I(\lambda_{n,k} > 0) \geq \liminf_{n\to\infty} \int_{\mathbb{R}} g(x)\tilde{\mu}_n(dx) \geq \int_0^\infty x^m \mu(dx) \tag{2.54}$$

for any $m \geq 1$. If $\alpha < \infty$, then

$$\int_0^\infty x^m \mu(dx) \geq \int_{\alpha-\varepsilon}^{\alpha} x^m \mu(dx) \geq (\alpha - \varepsilon)^m \mu([\alpha - \varepsilon, \alpha]) > 0 \tag{2.55}$$

for any $\varepsilon \in (0, \alpha)$. Take the $(1/m)$th power for each term in (2.54) and (2.55), and let $m \to \infty$ to get

$$\liminf_{n\to\infty} \{\lambda_{n,k} I(\lambda_{n,k} > 0)\} \geq \alpha - \varepsilon$$

for any $\varepsilon \in (0, \alpha)$. By sending $\varepsilon \downarrow 0$ and using the fact $\alpha > 0$, the conclusion is yielded.

If $\alpha = +\infty$, notice

$$\int_0^\infty x^m \mu(dx) \geq \int_\rho^\infty x^m \mu(dx) \geq \rho^m \mu([\rho, \infty)) > 0$$

for any $\rho > 0$. Using the same argument as above and then letting $\rho \to +\infty$, we get the desired assertion. $\square$

PROOF OF LEMMA 2.1. (i) By Theorem 3, $F^{n^{-1/2}\mathbf{U}_n}$ converges weakly to the semicircular law with density function $\frac{1}{2\pi}\sqrt{4 - x^2} I(|x| \leq 2)$. Use Lemma 2.8 to have that

$$\liminf_{n\to\infty} \frac{\lambda_{\max}(\mathbf{U}_n)}{\sqrt{n}} \geq 2 \qquad \text{a.s.} \tag{2.56}$$

Now we prove the upper bound, that is,

$$\limsup_{n\to\infty} \frac{\lambda_{\max}(\mathbf{U}_n)}{\sqrt{n}} \leq 2 \qquad \text{a.s.} \tag{2.57}$$

Define

$$\delta_n = \frac{1}{\log(n+1)}, \qquad \tilde{u}_{ij}^{(n)} = u_{ij}^{(n)} I\big(|u_{ij}^{(n)}| \leq \delta_n \sqrt{n}\big) \quad \text{and} \quad \tilde{\mathbf{U}}_n = (\tilde{u}_{ij}^{(n)})_{1\leq i,j\leq n}$$

for $1 \le i \le j \le n$ and $n \ge 1$. By the Markov inequality,

$$\begin{aligned} P(\mathbf{U}_n \neq \tilde{\mathbf{U}}_n) &\le P\big(|u_{ij}^{(n)}| > \delta_n\sqrt{n} \text{ for some } 1 \le i, j \le n\big) \\ &\le n^2 \max_{1\le i,j\le n} P\big(|u_{ij}^{(n)}| > \delta_n\sqrt{n}\big) \\ &\le \frac{K(\log(n+1))^{6+\delta}}{n^{1+(\delta/2)}}, \end{aligned}$$

where $K = \sup_{1\le i,j\le n,n\ge 1} E|u_{ij}^{(n)}|^{6+\delta} < \infty$. Therefore, by the Borel–Cantelli lemma,

$$P(\mathbf{U}_n = \tilde{\mathbf{U}}_n \text{ for sufficiently large } n) = 1. \tag{2.58}$$

From $Eu_{ij}^{(n)} = 0$, we have that

$$\big|Eu_{ij}^{(n)} I\big(|u_{ij}^{(n)}| \le \delta_n\sqrt{n}\big)\big| = \big|Eu_{ij}^{(n)} I\big(|u_{ij}^{(n)}| > \delta_n\sqrt{n}\big)\big| \le \frac{K}{(\delta_n\sqrt{n})^{5+\delta}} \tag{2.59}$$

for any $1 \le i \le j \le n, n \ge 1$. Note that $\lambda_{\max}(\mathbf{A}+\mathbf{B}) \le \lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B})$, and $\lambda_{\max}(\mathbf{A}) \le \|\mathbf{A}\| \le n \cdot \max_{1\le i,j\le n} |a_{ij}|$ for any $n \times n$ symmetric matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B}$. We have from (2.59) that

$$\begin{aligned} &\lambda_{\max}(\tilde{\mathbf{U}}_n) - \lambda_{\max}\big(\tilde{\mathbf{U}}_n - E(\tilde{\mathbf{U}}_n)\big) \\ &\quad \le \lambda_{\max}(E\tilde{\mathbf{U}}_n) \le n \max_{1\le i,j\le n}\big|Eu_{ij}^{(n)} I\big(u_{ij}^{(n)} \le \delta_n\sqrt{n}\big)\big| \\ &\quad \le \frac{K}{\delta_n^{5+\delta}(\sqrt{n})^{3+\delta}} \end{aligned}$$

for any $n \ge 1$. This and (2.58) imply that

$$\limsup_{n\to\infty} \frac{\lambda_{\max}(\mathbf{U}_n)}{\sqrt{n}} = \limsup_{n\to\infty} \frac{\lambda_{\max}(\tilde{\mathbf{U}}_n)}{\sqrt{n}} \le \limsup_{n\to\infty} \frac{\lambda_{\max}(\tilde{\mathbf{U}}_n - E\tilde{\mathbf{U}}_n)}{\sqrt{n}}$$

almost surely. Note that $|\tilde{u}_{ij}^{(n)}| \le |u_{ij}^{(n)}|$ and $\mathrm{Var}(\tilde{u}_{ij}^{(n)}) \le E(u_{ij}^{(n)})^2 = 1$, to save notation, without loss of generality, we will prove (2.57) by assuming that

$$E\big(u_{ij}^{(n)}\big) = 0, \qquad E\big(u_{ij}^{(n)}\big)^2 \le 1, \qquad |u_{ij}^{(n)}| \le \frac{2\sqrt{n}}{\log(n+1)}$$

and

$$\max_{1\le i,j\le n,n\ge 1} E\big|u_{ij}^{(n)}\big|^{6+\delta} < \infty$$

for all $1 \le i, j \le n$ and $n \ge 1$. Now,

$$\max_{i,j,n} E|u_{ij}^{(n)}|^3 \le \max_{i,j,n}\big(E|u_{ij}^{(n)}|^{6+\delta}\big)^{3/(6+\delta)} = K^{3/(6+\delta)}$$

by the Hölder inequality. Hence,

$$\max_{1\le i,j\le n} E|u_{ij}^{(n)}|^l \le K^{3/(6+\delta)} \cdot \left(\frac{2\sqrt{n}}{\log(n+1)}\right)^{l-3} \tag{2.60}$$

for all $n \ge 1$ and $l \ge 3$, where K is a constant. The inequality in (2.57) follows from Lemma A.3 in the Appendix. Thus the first limit in the lemma is proved. Applying this result to $-\mathbf{U}_n$, we obtain

$$\lim_{n\to\infty} \frac{\lambda_{\min}(\mathbf{U}_n)}{\sqrt{n}} = -\lim_{n\to\infty} \frac{\lambda_{\max}(-\mathbf{U}_n)}{\sqrt{n}} = -2 \qquad \text{a.s.} \tag{2.61}$$

Since $\|\mathbf{U}_n\| = \max\{\lambda_{\max}(\mathbf{U}_n), -\lambda_{\min}(\mathbf{U}_n)\}$, the above and the first limit in the lemma yield the second limit.

(ii) Let $\hat{\mathbf{U}}_n = \mathbf{U}_n - \operatorname{diag}(u_{ii}^{(n)})_{1\le i\le n}$. It is not difficult to check that both $|\lambda_{\max}(\hat{\mathbf{U}}_n) - \lambda_{\max}(\mathbf{U}_n)|$ and $|\|\hat{\mathbf{U}}_n\| - \|\mathbf{U}_n\||$ are bounded by $\|\operatorname{diag}(u_{ii}^{(n)})_{1\le i\le n}\| = \max_{1\le i\le n} |u_{ii}^{(n)}|$. By (i), it is enough to show

$$\max_{1\le i\le n} |u_{ii}^{(n)}|/n^{1/3} \to 0 \qquad \text{a.s.} \tag{2.62}$$

as $n \to \infty$. In fact, by the Markov inequality

$$\begin{aligned} \sum_{n=1}^{\infty} P\Bigl(\max_{1\le i\le n} |u_{ii}^{(n)}| \ge n^{1/3}t\Bigr) &\le \sum_{n=1}^{\infty} n \cdot \max_{1\le i\le n} P(|u_{ii}^{(n)}| \ge n^{1/3}t) \\ &\le \sum_{n=1}^{\infty} \frac{t^{-6-\delta}}{n^{1+(\delta/3)}} \cdot \sup_{1\le i,j\le n, n\ge 1} E|u_{ij}^{(n)}|^{6+\delta} < \infty \end{aligned}$$

for any $t > 0$. Thus, (2.62) is concluded by the Borel–Cantelli lemma. $\square$

PROOF OF THEOREM 4. Let $\mathbf{J}_n$ be the $n \times n$ matrix whose n^2 entries are all equal to 1. Let $\mathbf{V}_n$ be defined as in (2.50). Then $\mathbf{B}_n := \mathbf{A}_n + \mu_n \mathbf{I}_n = \sigma_n \mathbf{V}_n + \mu_n \mathbf{J}_n$. First, by Lemma 2.1,

$$\lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{V}_n)}{\sqrt{n}} = 2 \qquad \text{a.s.} \quad \text{and} \quad \lim_{n\to\infty} \frac{\|\mathbf{V}_n\|}{\sqrt{n}} = 2 \qquad \text{a.s.} \tag{2.63}$$

Since $\mathbf{V}_n$ is symmetric, $\|\mathbf{V}_n\| = \sup_{x\in\mathbb{R}^n : \|x\|=1} \|\mathbf{V}_n x\| = \sup_{\|x\|=1} |x^T \mathbf{V}_n x|$. By definition

$$\begin{aligned} \lambda_{\max}(\mathbf{B}_n) &= \sup_{\|x\|=1} \{\sigma_n(x^T \mathbf{V}_n x) + \mu_n(x^T \mathbf{J}_n x)\} \\ &= \sup_{\|x\|=1} \{\sigma_n(x^T \mathbf{V}_n x) + \mu_n(\mathbf{1}'x)^2\}, \end{aligned} \tag{2.64}$$

because $\mathbf{J} = \mathbf{1} \cdot \mathbf{1}^T$, where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$. Second, by Theorem 3, $F^{\mathbf{B}_n/\sqrt{n}\sigma_n}$ converges weakly to the semicircular law $\frac{1}{2\pi}\sqrt{4-x^2}I(|x| \le 2)$. From Lemma 2.8, we know that

$$\liminf_{n\to\infty} \frac{\lambda_{k_n}(\mathbf{B}_n)}{\sqrt{n}\sigma_n} \ge 2 \qquad \text{a.s.} \tag{2.65}$$

Now we are ready to prove the conclusions.

(i) It is easy to check that $\sup_{\|x\|=1}\{(\mathbf{1}'x)^2\} = n$. By (2.64), $\lambda_{\max}(\mathbf{B}_n) \le \sigma_n\|\mathbf{V}_n\| + n|\mu_n|$. Thus $\limsup_{n\to\infty} \lambda_{\max}(\mathbf{B}_n)/\sqrt{n}\sigma_n \le 2$ a.s. by (2.63) under the assumption $\mu_n/(n^{-1/2}\sigma_n) \to 0$ as $n \to \infty$. Since $\lambda_{\max}(\mathbf{B}_n) = \mu_n + \lambda_{\max}(\mathbf{A}_n)$. From (2.65) we see that $\lim_{n\to\infty} \lambda_{k_n}(\mathbf{A}_n)/\sqrt{n}\sigma_n = 2$ a.s. when $\mu_n/(n^{-1/2}\sigma_n) \to 0$ as $n \to \infty$. In particular, $\lim_{n\to\infty} \lambda_{\max}(\mathbf{A}_n)/\sqrt{n}\sigma_n = 2$ a.s. Under the same condition, we also have $\lim_{n\to\infty} \lambda_{\max}(-\mathbf{A}_n)/\sqrt{n}\sigma_n = 2$ a.s. Finally, using $\|\mathbf{A}_n\| = \max\{\lambda_{\max}(\mathbf{A}_n), \lambda_{\max}(-\mathbf{A}_n)\}$, we obtain that $\lim_{n\to\infty} \|\mathbf{A}_n\|/\sqrt{n}\sigma_n = 2$ a.s.

(ii) Without loss of generality, assume $\mu_n > 0$ for all $n \ge 2$. From (2.64) we see that

$$\begin{aligned} &\mu_n \sup_{\|x\|=1}\{(\mathbf{1}'x)^2\} - \sigma_n \sup_{\|x\|=1}\{|x^T\mathbf{V}_n x|\} \\ &\quad \le \lambda_{\max}(\mathbf{B}_n) \le \mu_n \sup_{\|x\|=1}\{(\mathbf{1}'x)^2\} + \sigma_n \sup_{\|x\|=1}\{|x^T\mathbf{V}_n x|\}. \end{aligned}$$

Hence, $n\mu_n - \sigma_n\|\mathbf{V}_n\| \le \lambda_{\max}(\mathbf{B}_n) \le n\mu_n + \sigma_n\|\mathbf{V}_n\|$. Consequently, if $\mu_n \gg n^{-1/2}\sigma_n$, by (2.63), we have

$$\lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{A}_n)}{n\mu_n} = \lim_{n\to\infty} \frac{\lambda_{\max}(\mathbf{B}_n)}{n\mu_n} = 1 \qquad \text{a.s.}$$

since $\lambda_{\max}(\mathbf{B}_n) = \mu_n + \lambda_{\max}(\mathbf{A}_n)$.

(iii) Since $\mathbf{B}_n = \sigma_n\mathbf{V}_n + \mu_n\mathbf{J}_n$ and $\|\mathbf{J}_n\| = n$, by the triangle inequality of $\|\cdot\|$,

$$n|\mu_n| - \sigma_n\|\mathbf{V}_n\| \le \|\mathbf{B}_n\| \le n|\mu_n| + \sigma_n\|\mathbf{V}_n\|.$$

By (2.63) and the definition that $\mathbf{A}_n = \mathbf{B}_n - \mu_n\mathbf{I}_n$, we obtain

$$\lim_{n\to\infty} \frac{\|\mathbf{A}_n\|}{n|\mu_n|} = \lim_{n\to\infty} \frac{\|\mathbf{B}_n\|}{n|\mu_n|} = 1 \qquad \text{a.s.}$$

as $|\mu_n| \gg n^{-1/2}\sigma_n$. $\square$

APPENDIX

LEMMA A.1 (Sakhanenko). *Let $\{\xi_i; i = 1, 2, \ldots\}$ be a sequence of independent random variables with mean zero and variance σ_i^2. If $E|\xi_i|^p < \infty$ for some $p > 2$, then there exists a constant $C > 0$ and $\{\eta_i; i = 1, 2, \ldots\}$, a sequence of independent normally distributed random variables with $\eta_i \sim N(0, \sigma_i^2)$ such that*

$$P\Big(\max_{1\le k\le n} |S_k - T_k| > x\Big) \le \frac{C}{1+|x|^p} \sum_{i=1}^n E|\xi_i|^p$$

for any n *and* $x > 0$, *where* $S_k = \sum_{i=1}^{k} \xi_i$ *and* $T_k = \sum_{i=1}^{k} \eta_i$.

Let $\mathbf{W}_n = (\omega_{ij}^n)_{1\le i,j\le n}$ be an $n \times n$ symmetric matrix, where $\{\omega_{ij}^n; 1 \le i \le j \le n\}$ are random variables defined on the same probability space. We need the following two results from Bai [6].

LEMMA A.2 (Theorem 2.4 in [6]). *For each* $n \ge 2$, *let* $\{\omega_{ij}^n; 1 \le i \le j \le n\}$ *be independent random variables* (*not necessarily identically distributed*) *with* $\omega_{ii}^n = 0$ *for all* $1 \le i \le n$, $E(\omega_{ij}^n) = 0$ *and* $E(\omega_{ij}^n)^2 = \sigma^2 > 0$ *for all* $1 \le i < j \le n$, *and*

$$\lim_{n\to\infty} \frac{1}{n^2\delta^2} \sum_{1\le i,j\le n} E(\omega_{ij}^n)^2 I(|\omega_{ij}^n| \ge \delta\sqrt{n}) = 0$$

for any $\delta > 0$. *Then* $F^{n^{-1/2}\mathbf{W}_n}$ *converges weakly to the semicircular law of scale-parameter* σ *with density function*

$$p_\sigma(x) = \begin{cases} \dfrac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}, & \text{if } |x| \le 2\sigma; \\ 0, & \text{otherwise}. \end{cases} \tag{A.1}$$

Some recent results in [4, 41] are in the realm of the above lemma.

LEMMA A.3 (Remark 2.7 in [6]). *Suppose, for each* $n \ge 1$, $\{\omega_{ij}^{(n)}; 1 \le i \le j \le n\}$ *are independent random variables* (*not necessarily identically distributed*) *with mean* $\mu = 0$ *and variance no larger than* σ^2. *Assume there exist constants* $b > 0$ *and* $\delta_n \downarrow 0$ *such that* $\sup_{1\le i,j\le n} E|\omega_{ij}^{(n)}|^l \le b(\delta_n\sqrt{n})^{l-3}$ *for all* $n \ge 1$ *and* $l \ge 3$. *Then*

$$\limsup_{n\to\infty} \frac{\lambda_{\max}(\mathbf{W}_n)}{n^{1/2}} \le 2\sigma \qquad a.s.$$

REFERENCES

[1] ARNOLD, L. (1967). On the asymptotic distribution of the eigenvalues of random matrices. *J. Math. Anal. Appl.* **20** 262–268. MR0217833

[2] ARNOLD, L. (1971). On Wigner's semicircle law for the eigenvalues of random matrices. *Z. Wahrsch. Verw. Gebiete* **19** 191–198. MR0348820

[3] BARABÁSI, A.-L. and ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286** 509–512. MR2091634

[4] BAI, Z. and ZHOU, W. (2008). Large sample covariance matrices without independence structures in columns. *Statist. Sinica* **18** 425–442. MR2411613

[5] BAI, Z. D. and SILVERSTEIN, J. (2009). *Spectral Analysis of Large Dimensional Random Matrices*, 2nd ed. Springer.

[6] BAI, Z. D. (1999). Methodologies in spectral analysis of large-dimensional random matrices, a review. *Statist. Sinica* **9** 611–677. MR1711663

[7] BAUER, M. and GOLINELLI, O. (2001). Random incidence matrices: Moments of the spectral density. *J. Stat. Phys.* **103** 301–337. MR1828732

[8] BEN AROUS, G., DEMBO, A. and GUIONNET, A. (2001). Aging of spherical spin glasses. *Probab. Theory Related Fields* **120** 1–67. MR1856194

[9] BIGGS, N. L., LLOYD, E. K. and WILSON, R. J. (1976). *Graph Theory*: 1736–1936. Clarendon, Oxford. MR0444418

[10] BOLLOBÁS, B. (1985). *Random Graphs*. Academic Press, London. MR809996

[11] BOLLOBÁS, B. (1979). *Graph Theory*: *An Introductory Course*. *Graduate Texts in Mathematics* **63**. Springer, New York. MR536131

[12] BRYC, W., DEMBO, A. and JIANG, T. (2006). Spectral measure of large random Hankel, Markov and Toeplitz matrices. *Ann. Probab.* **34** 1–38. MR2206341

[13] CHOW, Y. S. and TEICHER, H. (1988). *Probability Theory*, *Independence*, *Interchangeability*, *Martingales*, 2nd ed. Springer, New York. MR953964

[14] CHUNG, F. R. K. (1997). *Spectral Graph Theory*. *CBMS Regional Conference Series in Mathematics* **92**. Conf. Board Math. Sci., Washington, DC. MR1421568

[15] CHUNG, F. and LU, L. (2006). *Complex Graphs and Networks*. *CBMS Regional Conference Series in Mathematics* **107**. Conf. Board Math. Sci., Washington, DC. MR2248695

[16] COLIN DE VERDIÈRE, Y. (1998). *Spectres de Graphes*. *Cours Spécialisés* [*Specialized Courses*] **4**. Société Mathématique de France, Paris. MR1652692

[17] DUDLEY, R. M. (2002). *Real Analysis and Probability*. *Cambridge Studies in Advanced Mathematics* **74**. Cambridge Univ. Press, Cambridge. MR1932358

[18] DURRETT, R. (2007). *Random Graph Dynamics*. Cambridge Univ. Press, Cambridge. MR2271734

[19] ERDÖS, P. and RÉNYI, A. (1959). On random graphs. I. *Publ. Math. Debrecen* **6** 290–297. MR0120167

[20] ERDÖS, P. and RÉNYI, A. (1960). On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **5** 17–61. MR0125031

[21] ERDÖS, P. and RÉNYI, A. (1961). On the evolution of random graphs. *Bull. Inst. Internat. Statist.* **38** 343–347. MR0148055

[22] ERDÖS, P. and RÉNYI, A. (1961). On the strength of connectedness of a random graph. *Acta Math. Acad. Sci. Hungar.* **12** 261–267. MR0130187

[23] ERDŐS, P. and SPENCER, J. (1974). *Probabilistic Methods in Combinatorics*. Academic Press, New York. MR0382007

[24] EVANGELOU, S. N. (1992). A numerical study of sparse random matrices. *J. Stat. Phys.* **69** 361–383. MR1184780

[25] EVANGELOU, S. N. and ECONOMOU, E. N. (1992). Spectral density singularities, level statistics, and localization in sparse random matrices. *Phys. Rev. Lett.* **68** 361–364.

[26] EVANGELOU, S. N. (1983). Quantum percolation and the Anderson transition in dilute systems. *Phys. Rev. B* **27** 1397–1400.

[27] FEY, A., HOFSTAD, R. and KLOK, M. (2008). Large deviations for eigenvalues of sample covariance matrices, with applications to mobile communication systems. *Adv. in Appl. Probab.* **40** 1048–1071.

[28] FÜREDI, Z. and KOMLÓS, J. (1981). The eigenvalues of random symmetric matrices. *Combinatorica* **1** 233–241. MR637828

[29] HALL, P. and HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press [Harcourt Brace Jovanovich Publishers], New York. MR624435

[30] JANSON, S., ŁUCZAK, T. and RUCINSKI, A. (2000). *Random Graphs*. Wiley, New York. MR1782847

[31] JUHÁSZ, F. (1981). On the spectrum of a random graph. In *Algebraic Methods in Graph Theory*, *Vol. I*, *II* (*Szeged*, 1978). *Colloquia Mathematica Societatis János Bolyai* **25** 313–316. North-Holland, Amsterdam. MR642050

[32] KHORUNZHY, O., SHCHERBINA, M. and VENGEROVSKY, V. (2004). Eigenvalue distribution of large weighted random graphs. *J. Math. Phys.* **45** 1648–1672. MR2043849

[33] KHORUNZHY, A., KHORUZHENKO, B., PASTUR, L. and SHCHERBINA, M. (1992). *The Large n-Limit in Statistical Mechanics and the Spectral Theory of Disordered Systems*. *Phase Transition and Critical Phenomenon* **15** 73. Academic Press, New York.

[34] KOLCHIN, V. F. (1999). *Random Graphs*. *Encyclopedia of Mathematics and Its Applications* **53**. Cambridge Univ. Press, Cambridge. MR1728076

[35] KRIVELEVICH, M. and SUDAKOV, B. (2003). The largest eigenvalue of sparse random graphs. *Combin*. *Probab*. *Comput*. **12** 61–72. MR1967486

[36] LEDOUX, M. (2007). Deviation inequalities on largest eigenvalues. In *Geometric Aspects of Functional Analysis*. *Lecture Notes in Math*. **1910** 167–219. Springer, Berlin. MR2349607

[37] MCKAY, B. D. (1981). The expected eigenvalue distribution of a large regular graph. *Linear Algebra Appl*. **40** 203–216. MR629617

[38] MIRLIN, A. D. and FYODOROV, Y. V. (1991). Universality of level correlation function of sparse random matrices. *J. Phys*. *A* **24** 2273-2286. MR1118532

[39] NOVIKOV, S. P. (1998). Schrödinger operators on graphs and symplectic geometry. In *The Arnoldfest* **2**. Field Institute, Toronto. MR1733586

[40] PALMER, E. M. (1985). *Graphical Evolution*: *An Introduction to the Theory of Random Graphs*. Wiley, Chichester. MR795795

[41] PAN, G.-M., GUO, M.-H. and ZHOU, W. (2007). Asymptotic distributions of the signal-to-interference ratios of LMMSE detection in multiuser communications. *Ann*. *Appl*. *Probab*. **17** 181–206. MR2292584

[42] PUPPE, T. (2008). *Spectral Graph Drawing*: *A Survey*. VDM, Verlag.

[43] RODGERS, G. J. and BRAY, A. J. (1988). Density of states of a sparse random matrix. *Phys*. *Rev. B* (3) **37** 3557–3562. MR932406

[44] RODGERS, G. J. and DE DOMINICIS, C. (1990). Density of states of sparse random matrices. *J. Phys*. *A* **23** 1567–1573. MR1048785

[45] VIVO, P., MAJUMDAR, S. N. and BOHIGAS, O. (2007). Large deviations of the maximum eigenvalue in Wishart random matrices. *J. Phys*. *A* **40** 4317–4337. MR2316708

[46] WATTS, D. J. and STROGATZ, S. H. (1998). Collective dynamics of "small-world" networks. *Nature* **393** 440–442.

[47] WIGNER, E. P. (1958). On the distribution of the roots of certain symmetric matrices. *Ann*. *of Math*. (2) **67** 325–327. MR0095527

SCHOOL OF MATHEMATICS
JILIN UNIVERSITY
2699 QIANJIN STREET
CHANGCHUN
CHINA
AND
SCHOOL OF STATISTICS
UNIVERSITY OF MINNESOTA
224 CHURCH STREET
MINNEAPOLIS, MINNESOTA 55455
USA
E-MAIL: dingxue@stat.umn.edu

SCHOOL OF STATISTICS
UNIVERSITY OF MINNESOTA
224 CHURCH STREET
MINNEAPOLIS, MINNESOTA 55455
USA
E-MAIL: tjiang@stat.umn.edu