1 (Tiefeng Jiang, Feb. 03, 2015)

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of an $n \times n$ GUE with density

$$\frac{1}{(2\pi)^{n/2} \prod_{j=1}^{n} j!} \prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2 \cdot e^{-\sum_{i=1}^{n} \lambda_i^2/2}. \tag{1.1}$$

Then $x_i = \lambda_i / \sqrt{2}$ for $i = 1, 2, \cdots, n$ have density

$$f(x_1, \cdots, x_n) = \frac{2^{n/2}}{(2\pi)^{n/2} \prod_{j=1}^{n} j!} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \cdot e^{-\sum_{i=1}^{n} x_i^2}. \tag{1.2}$$

See Ch 7 from Mehta’s book. Let us rewrite the determinant part:

$$\begin{vmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \\ \vdots & \cdots & \vdots \\ x_1^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}^2 = \begin{vmatrix} p_0(x_1) & \cdots & p_0(x_n) \\ p_1(x_1) & \cdots & p_1(x_n) \\ \vdots & \cdots & \vdots \\ p_{n-1}(x_1) & \cdots & p_{n-1}(x_n) \end{vmatrix}^2 \tag{1.2}$$

where $p_0(x) = 1$ and $p_j(x) = x^j + \sum_{i=0}^{j-1} c_i x^i$ for arbitrary $c_j$’s by adding any linear combination of rows to any other row. Recall the Hermite polynomials $H_0(x) = 1$ and

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$= 2^n x^n + \sum_{i=0}^{n-1} c_i x^i, \ x \in \mathbb{R},$$

for all $n \geq 1$ by Rodrigues’s formula. Easily, $H_n(x)$ is an even function for even $n$, and is odd for odd $n$. They have the orthogonal property

$$\int_{\mathbb{R}} H_i(x) H_j(x) e^{-x^2} dx = \delta_{ij} \sqrt{\pi} 2^i i!$$

for all $i \geq 0$ and $j \geq 0$. Then

$$f(x_1, \cdots, x_n) = \frac{1}{\pi^{n/2} \prod_{j=1}^{n} j!} \begin{vmatrix} H_0(x_1) e^{-x_1^2/2} & \cdots & H_0(x_n) e^{-x_n^2/2} \\ H_1(x_1) e^{-x_1^2/2} & \cdots & H_1(x_n) e^{-x_n^2/2} \\ \vdots & \cdots & \vdots \\ H_{n-1}(x_1) e^{-x_1^2/2} & \cdots & H_{n-1}(x_n) e^{-x_n^2/2} \end{vmatrix}^2.$$

Define the normalized Hermite function

$$h_i(x) = \frac{1}{\pi^{1/4} \sqrt{2^i i!}} H_i(x), \ i = 0, 1, 2, \cdots \tag{1.3}$$
(It is known that \( \{h_i(x); i = 0, 1, \ldots \} \) form a normal basis on \( L^2(\mathbb{R}, \mu) \) where \( \mu(dx) = e^{-x^2} dx \). The polynomials \( \{h_i(x); i \geq 0 \} \) can be also obtained by Gram-Schmidt orthogonalization method on \( \{1, x, x^2, \ldots \} \) with respect to probability measure \((\sqrt{\pi})^{-1/2}e^{-x^2}\). Finally,

\[
f(x_1, \ldots, x_n) = \frac{1}{n!} \begin{vmatrix} h_0(x_1)e^{-x_1^2/2} & \cdots & h_0(x_n)e^{-x_n^2/2} \\ h_1(x_1)e^{-x_1^2/2} & \cdots & h_1(x_n)e^{-x_n^2/2} \\ \vdots & \cdots & \vdots \\ h_{n-1}(x_1)e^{-x_1^2/2} & \cdots & h_{n-1}(x_n)e^{-x_n^2/2} \end{vmatrix}^2 = \frac{1}{n!} \det(K_n(x_i, x_j))_{n \times n}
\]

where

\[
K_n(x, y) = \sum_{i=0}^{n-1} h_i(x)h_i(y)e^{-(x^2+y^2)/2}, \ x \in \mathbb{R} \tag{1.4}
\]

by looking at the inner product of any two columns of the matrix. Reviewing the correlation function \( p_n(x_1, \ldots, x_n) = n!f(x_1, \ldots, x_n) \). This says that the eigenvalues \( \{x_1, \ldots, x_n\} \) form a determinantal point process with kernel \( K_n(x, y) \). By the previous proposition,

\[
P(\lambda_{\max} \leq t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \int_{(t, \infty)^m} \det(K_n(x_i, x_j)) dx_1 \cdots dx_m. \tag{1.5}
\]

Take \( t = 2\sqrt{n} + n^{-1/6}x \) with \( x \in \mathbb{R} \). To prove the largest eigenvalue converging to the Tracy-Widom law, we need to show that the above converges pointwise to

\[
\det(I - K)_{L^2(t, \infty)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(t, \infty)^m} \det(A(x_i, x_j)) dx_1 \cdots dx_m
\]

where the Airy kernel

\[
A(x, y) = \int_0^\infty \text{Ai}(x + t)\text{Ai}(y + t) \, dt,
\]

which is Hermitian and \( \int_t^\infty A(x, x) \, dx < \infty \).

We next collect some basic results on Hermite polynomials. They are Rotach-Plancherel type formula.

**Lemma 1.1.** Review \( h_n(x) \) in (1.3). Define \( f_n(x) = h_n(x)e^{-x^2/2} \). Let \( \delta \in (0, \frac{1}{2}) \). The following holds.

(i) \( \sup_{x \in \mathbb{R}} |f_n(x)| \leq C n^{-1/12} \).

(ii) For \(-1 + \delta \leq x \leq 1 - \delta\),

\[
f_n(\sqrt{2n} x) = \frac{2^{1/4}}{n^{1/4}\sqrt{\pi}} \cdot \frac{1}{(1 - x^2)^{1/4}} \left( \cos \left[ 2nF(x) - \frac{1}{2} \arcsin x \right] + O\left( \frac{1}{n} \right) \right), \quad \text{where}
\]

\[
F(x) = \int_x^1 \sqrt{1 - y^2} \, dy = \frac{1}{2} \left( \arccos x - x\sqrt{1 - x^2} \right).
\]
(iii) There exist constants \( c > 0 \) and \( C > 0 \) such that \( |f_n(\sqrt{2n} x)| \leq C n^{-1/12} \exp(-cn(1-x-1)^{3/2}) \) for all \( x \in (1, 1+\delta) \).

(iv) \( |f_n(\sqrt{2n} x)| \leq C_1 n^{-1/4} e^{-C_2 n^{3/2}} \) for all \( x \geq 1 + \delta \).

**Proof.** (i) is (61), (ii) is (62), (iii) is (66), and (iv) is on p. 99 from Johansson (2007).

**Lemma 1.2.** (Lemma 7.2 from Adler and Moerbeke, 2005) For large \( n > 0 \), we have that

\[
f_{n-k}(\sqrt{2n+1} + \frac{u}{\sqrt{2n^{1/6}}}) = \frac{2^{1/4}}{n^{1/12}} Ai(u + \frac{k}{n^{1/3}}) \left[ 1 + O\left( \frac{\log n}{n^{2/3}} \right) \right]
\]

uniformly for all \( u \) in any given compact set and \( -M_0 n^{1/3} \log n \leq k \leq M_0 n^{1/3} \log n \) with fixed \( M_0 > 0 \).

**Theorem 1.** Let \( \lambda_1, \cdots, \lambda_n \) have the density function as in (1.1). Then

\[
P\left( n^{1/6}(\lambda_{\text{max}} - 2\sqrt{n}) \leq x \right) \to \det(I - K)_{L^2(x, \infty)}
\]

for all \( x \in \mathbb{R} \) as \( n \to \infty \).

**Proof.** Set \( x_i = \lambda_i/\sqrt{2} \) for \( i = 1, 2, \cdots, n \). Then we need to show

\[
P\left( n^{1/6}(x_{\text{max}} - \sqrt{2n}) \leq \frac{x}{\sqrt{2}} \right) \to \det(I - K)_{L^2(x, \infty)} \tag{1.6}
\]

for all \( x \in \mathbb{R} \) as \( n \to \infty \).

We first prove

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2} n^{1/6}} K_n\left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right) = \int_0^\infty Ai(u + t)Ai(v + t) \, dt \tag{1.7}
\]

uniformly for all \( u, v \) in a compact set of \( \mathbb{R} \). Recall (1.4) and \( f_n(x) = h_n(x)e^{-x^2/2} \). Make transform \( k = n - i \) to have

\[
K_n\left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right) = \sum_{j=1}^n f_{n-k}\left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_{n-k}\left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right) = \left[ \sum_{k=1}^{n^{1/3} \log n} + \sum_{k=n^{1/3} \log n}^n \right] f_{n-k}\left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_{n-k}\left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right)
\]

Since \( \sqrt{2n} = \sqrt{2n+1} - \frac{1}{\sqrt{8n}} + O(\frac{1}{n^{3/2}}) \), we can write

\[
\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} = \sqrt{2n+1} + \frac{1}{\sqrt{2} n^{1/6}} \left( u - \frac{1}{2n^{1/3}} + O\left( \frac{1}{n^{3/2}} \right) \right)
\]
as $n \to \infty$. By Lemma 1.2, as $n \to \infty$,

$$f_{n-k} \left( \sqrt{2n + \frac{u}{\sqrt{2} n^{1/6}}} \right) = \frac{2^{1/4}}{n^{1/12}} \text{Ai} \left( u + \frac{k - (1/2)}{n^{1/3}} + O \left( \frac{1}{n^{1/3}} \right) \right) \left[ 1 + O \left( \frac{\log n}{n^{2/3}} \right) \right]$$

uniformly for $1 \leq k \leq n^{1/3} \log n$ and $u$ in any given compact interval. Therefore,

$$W_n = \frac{1}{\sqrt{2} n^{1/6}} \sum_{k=1}^{n^{1/3} \log n} f_{n-k} \left( \sqrt{2n + \frac{u}{\sqrt{2} n^{1/6}}} \right) f_{n-k} \left( \sqrt{2n + \frac{v}{\sqrt{2} n^{1/6}}} \right) \left[ 1 + O \left( \frac{\log n}{n^{2/3}} \right) \right] \sum_{k=1}^{n^{1/3} \log n} \text{Ai} \left( u + \frac{k - (1/2)}{n^{1/3}} + O \left( \frac{1}{n^{1/3}} \right) \right) \cdot \text{Ai} \left( v + \frac{k - (1/2)}{n^{1/3}} + O \left( \frac{1}{n^{1/3}} \right) \right).$$

(1.8)

For $\text{Ai}(x)$ is entire on the complex plane and $\lim_{x \to +\infty} |\text{Ai}(x)| + |\text{Ai}'(x)| = 0$, we know $C := \sup_{x \geq 0} |\text{Ai}(x)| \cdot \sup_{x \geq 0} |\text{Ai}'(x)| < \infty$. Use the identity $\int_{k}^{k+1} q(x) \, dx = q(k) + \int_{k}^{k+1} (q(x) - q(k)) \, dx$ and the error term $|\int_{k}^{k+1} (q(x) - q(k)) \, dx| \leq \sup_{k \leq x \leq k+1} |q'(x)|$ to get that the sum in (1.8) is equal to

$$\int_{1}^{n^{1/3} \log n + 1} \text{Ai} \left( u + \frac{x - (1/2)}{n^{1/3}} + O \left( \frac{1}{n^{1/3}} \right) \right) \text{Ai} \left( v + \frac{x - (1/2)}{n^{1/3}} + O \left( \frac{1}{n^{1/3}} \right) \right) \, dx + \epsilon_n$$

(1.9)

with $|\epsilon_n| \leq 3C \log n$ as $n \to \infty$. Therefore, the whole term in (1.8) equals

$$W_n = \left[ 1 + O \left( \frac{\log n}{n^{2/3}} \right) \right] \int_{n^{-1/3}/2}^{\log n + o(1)} \text{Ai} \left( u + t + O \left( \frac{1}{n^{1/3}} \right) \right) \text{Ai} \left( v + t + O \left( \frac{1}{n^{1/3}} \right) \right) \, dt$$

$$+ O \left( \frac{\log n}{n^{1/3}} \right)$$

$$= \left[ 1 + O \left( \frac{\log n}{n^{2/3}} \right) \right] \int_{n^{-1/3}(1+o(1))}^{\log n + o(1)} \text{Ai}(u + t) \text{Ai}(v + t) \, dt + O \left( \frac{\log n}{n^{1/3}} \right)$$

(1.10)

$$\to \int_{0}^{\infty} \text{Ai}(u + t) \text{Ai}(v + t) \, dt$$

(1.11)

since the two “$O \left( \frac{\log n}{n^{1/3}} \right)$” in the integral are the same. Further, the above implies that

$$|W_n| \leq 2 \int_{0}^{\infty} |\text{Ai}(u + t) \text{Ai}(v + t)| \, dt$$

(1.12)

as $n$ is sufficiently large. The integral above is finite thanks to the fact

$$\text{Ai}(y) \sim \frac{1}{2 \sqrt{\pi} y^{1/4}} e^{- (2/3) y^{3/2}}$$

as $y \to +\infty$. 

4
Now we estimate
\[
\frac{1}{n^{1/6}} \sum_{k=n^{1/3} \log n}^n \frac{n-k}{2n^{1/6}} f_{n-k} \left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_{n-k} \left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right)
\]
\[
= \frac{1}{n^{1/6}} \sum_{j=0}^{n-n^{1/3} \log n} f_j \left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_j \left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right).
\]

We break the above sum into two terms: \(\sum_{j=0}^{n/9}\) and \(\sum_{j=n/9}^{n-n^{1/3} \log n}\) and estimate them respectively. Write
\[
\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} = \sqrt{2j} \cdot \frac{\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}}{\sqrt{2j}}.
\]

**Estimate 1:** \(\sum_{j=0}^{n/9}\). Notice
\[
\frac{\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}}{\sqrt{2j}} - 1 > 1
\]
for \(j \geq 1\). Taking \(\delta = 1\) in (iv) of Lemma 1.1, we see
\[
\frac{1}{n^{1/6}} \sum_{j=0}^{n/9} f_j \left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_j \left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right)
\]
\[
\leq \frac{1}{n^{1/6}} \left( \|f_0\|_\infty + C_1 \sum_{j=1}^{n/9} j^{-1/4} e^{-C_2j} \right) = \frac{C_3}{n^{1/6}}
\]
for all \(n \geq 1\).

**Estimate 2:** \(\sum_{j=n/9}^{n-n^{1/3} \log n}\). Observe
\[
U_n : = \frac{\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}}{\sqrt{2j}} - 1
\]
\[
= \frac{1}{\sqrt{2j}} \left( \sqrt{2n} - \sqrt{2j} + \frac{u}{\sqrt{2} n^{1/6}} \right)
\]
\[
= \frac{1}{\sqrt{2j}} \left( \sqrt{2} \frac{n-j}{\sqrt{n} + \sqrt{j}} + \frac{u}{\sqrt{2} n^{1/6}} \right)
\]
\[
\leq 4
\]
as \(n\) is sufficiently large since \(\sqrt{n} + \sqrt{j} \geq 2\sqrt{n}\) and \(j \leq n - n^{1/3} \log n\). Also, \(U_n \geq \frac{n-j}{3\sqrt{2n}}\) as
n is sufficiently large. Taking $\delta = 6$ in (iii) of Lemma 1.1,

\[
\frac{1}{n^{1/6}} \sum_{j=n/9}^{n-n^{1/3} \log n} f_j \left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_j \left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right) \leq \frac{C}{n^{1/3}} \sum_{j=n/9}^{n-n^{1/3} \log n} \exp \left[ - \frac{C'}{n} \left( \frac{n-j}{\sqrt{n}} \right)^{3/2} \right] \leq \frac{C}{n^{1/3}} \sum_{j=n/9}^{n-n^{1/3} \log n} \exp \left[ - \frac{C'}{n} \left( 1 - \frac{j}{n} \right)^{3/2} \right].
\]

Since the $j$-th term in the sum is increasing in $j$, then the last quantity is dominated by

\[
\frac{C}{n^{1/3}} \int_{n/9}^{n-n^{1/3} \log n} \exp \left[ - \frac{C'}{n} \left( 1 - \frac{x}{n} \right)^{3/2} \right] dx.
\]

Let $s = n^{2/3} \left( 1 - \frac{x}{n} \right)$. Then the above integral is equal to

\[
C \int_{\log n}^{\frac{8}{5} n^{1/3}} e^{-c' s^{3/2}} ds \leq C \int_{\log n}^{\infty} e^{-c' s^{3/2}} ds.
\]

Therefore the above assertions imply

\[
\frac{1}{n^{1/6}} \sum_{j=n/9}^{n-n^{1/3} \log n} f_j \left( \sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}} \right) f_j \left( \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}} \right) \leq C \int_{\log n}^{\infty} e^{-c' s^{3/2}} ds
\]

as $n$ is sufficiently large. This together with (1.11) and (1.14) concludes that

\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}^{1/6}} K_n(\sqrt{2n} + \frac{u}{\sqrt{2} n^{1/6}}, \sqrt{2n} + \frac{v}{\sqrt{2} n^{1/6}}) = \int_{0}^{\infty} \text{Ai}(u + t) \text{Ai}(v + t) dt
\]

uniformly for all $u, v$ in a compact set of $\mathbb{R}$. Also, from (1.12), (1.14) and (1.15) and the Hadamard inequality, we get

\[
\left( \frac{1}{\sqrt{2} n^{1/6}} \right)^m \det(K_n(\sqrt{2n} + \frac{u_i}{\sqrt{2} n^{1/6}}, \sqrt{2n} + \frac{u_j}{\sqrt{2} n^{1/6}}))_{m \times m} \leq \prod_{i=1}^{m} \frac{1}{\sqrt{2} n^{1/6}} \cdot K_n(\sqrt{2n} + \frac{u_i}{\sqrt{2} n^{1/6}}, \sqrt{2n} + \frac{u_j}{\sqrt{2} n^{1/6}})^m \leq C^m
\]

uniformly for all $u_1, \cdots, u_m$ in any compact set $A$ as $n$ is sufficiently large, where $C$ is a finite constant depending on the set $A$. 

6
Review (1.18). By an earlier proposition,
\[
P\left(\text{no eigenvalues } \in \left(\sqrt{2n + \frac{x}{\sqrt{2n^{1/6}}}}, \sqrt{2n + \frac{y}{\sqrt{2n^{1/6}}}}\right)\right)
= \sum_{m=0}^{n} \frac{(-1)^m}{m!} \left(\frac{1}{\sqrt{2n^{1/6}}}\right)^m \int_{[x,y]^m} \det(K_n(\sqrt{2n + \frac{u_i}{\sqrt{2n^{1/6}}}}, \sqrt{2n + \frac{u_j}{\sqrt{2n^{1/6}}}})) \, du_1 \cdots du_m
\]
as \(n \to \infty\). From (1.7) and (1.17), we conclude from the dominated convergence theorem that
\[
P\left(\text{no eigenvalues } \in \left(\sqrt{2n + \frac{x}{\sqrt{2n^{1/6}}}}, \sqrt{2n + \frac{y}{\sqrt{2n^{1/6}}}}\right)\right)
\to \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{[x,y]^m} \det(K(u_i, u_j)) \, du_1 \cdots du_m
= \det(I - K)_{L^2(x,y)}
\]
as \(n \to \infty\).

**Fact 1.** Notice
\[
\det(I - K)_{L^2(x,y)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(x,y)^m} \det(A(x_i, x_j)) \, dx_1 \cdots dx_m.
\]
By the Hadamard inequality, the integral above is dominated by \(\left(\int_x^\infty Ai(t, t) \, dt\right)^m\) and
\[
\int_x^\infty A(t, t) \, dt = \int_x^\infty \int_0^\infty Ai(t + u)^2 \, du \, dt
= \int_x^\infty \int_t^\infty Ai(y)^2 \, dy \, dt
= \int_x^\infty (y - x)Ai(y)^2 \, dy < \infty
\]
by the Fubini theorem and then the tail of the Airy function: \(0 < Ai(y) \leq e^{-\frac{2}{3}y^{3/2}}\) as \(y\) is sufficiently large. By the bounded convergence theorem,
\[
\lim_{y \to +\infty} \det(I - K)_{L^2(x,y)} = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \int_{(x,\infty)^m} \det(A(x_i, x_j)) \, dx_1 \cdots dx_m.
\]

**Fact 2.**

**Lemma 1.3.** (Ledoux) There exist constant \(c > 0\) and \(C > 0\) such that
\[
P(n^{1/6}(\lambda_{\max} - 2\sqrt{n}) \geq t) \leq Ce^{-ct^{3/2}}
\]
for all \(0 < t \leq n^{2/3}\).
Now let’s make summarization. Set

\[ S = \left\{ \text{no eigenvalues } \lambda \in \left( \sqrt{2n + \frac{x}{\sqrt{2n^{1/6}}}}, \sqrt{2n + \frac{y}{\sqrt{2n^{1/6}}}} \right) \right\} \]

Rewrite the left hand side of (1.18) as follows:

\[ P(\lambda_{\max} \leq \sqrt{2n + \frac{x}{\sqrt{2n^{1/6}}}}) + P(\left\{ \lambda_{\max} \geq \sqrt{2n + \frac{y}{\sqrt{2n^{1/6}}}} \right\} \cap S). \]

The last probability is bounded by

\[ P(\lambda_{\max} \geq \sqrt{2n + \frac{y}{\sqrt{2n^{1/6}}}}) \leq e^{-Cy^{3/2}} \]

by Lemma 1.3, which goes to zero as \( y \to +\infty \). This combines with (1.19) prove the theorem.

\[ \blacksquare \]

References

