Problem 1. Let \( \{X, X_n; n = 1, 2, \cdots\} \) be a sequence of random variables. Show that \( X_n \to X \ a.s. \) if and only if

\[
P(\{|X_n - X| \geq \epsilon\} \ i.o.) = 0
\]

for any \( \epsilon > 0. \)

Problem 2. Let \( X_n; n \geq 1 \) be a sequence of random variables. Using Problem 1 to show that “\( X_n \to X \ a.s. \)” implies “\( X_n \to X \) in probability”. Show also that “\( X_n \to X \) in probability” implies “\( X_n \to X \) in distribution”, that is, \( \lim_{n \to \infty} Ef(X_n) = Ef(X) \) for every bounded continuous function \( f(x) \).

Problem 3. Let \( \{X_i; i = 1, 2, \cdots\} \) be a sequence of i.i.d. random variables. Define \( S_n = \sum_{i=1}^{n} X_i \). There exists a random variable \( Y \) such that \( S_n/n \to Y \) a.s. as \( n \to \infty \). Prove that \( E|X_1| < \infty. \) Hint: First show \( X_n/n = (S_n - S_{n-1})/n \to 0 \) a.s. then use the Second Borel Cantelli lemma.

Problem 4. Let \( \{X_n; n \geq 1\} \) be a sequence of random variables. Prove that there exists a sequence of positive constants \( \{a_n; n \geq 1\} \) such that \( X_n/a_n \to 0 \) a.s.

Problem 5. Let \( \{X, X_n; n \geq 1\} \) be a sequence of random variables. Construct a counterexample such that \( X_n \to X \) in probability but \( X_n \) does not converge to \( X \) almost surely.

Problem 6. Let \( X_1, \cdots, X_n \) be a random sample from \( N(\theta, \sigma^2) \). Consider testing \( H_0 : \theta \leq \theta_0 \) vs \( H_1 : \theta > \theta_0 \).

(a) If \( \sigma^2 \) is known, show that the test that rejects \( H_0 \) when

\[
\bar{X} > \theta_0 + z_\alpha \sqrt{\sigma^2/n}
\]

is a test of size \( \alpha. \) Show that the test can be derived as an LRT.

(b) Show that the test in part (a) is a UMP test.

(c) If \( \sigma^2 \) is unknown, show that the test that rejects \( H_0 \) when

\[
\bar{X} > \theta_0 + t_{n-1,\alpha} \sqrt{S^2/n}
\]
is a test of size $\alpha$. Show that the test can be derived as an LRT.

**Problem 7.** Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with the exponential distribution of mean $\lambda$. Set $W_n = \max_{1 \leq i \leq n} X_i$. Find a sequence of constants $\{b_n\}$ and prove that

$$\lim_{n \to \infty} \frac{W_n}{b_n} = 1 \quad a.s.$$