Problem 1. Let $X$ be a positive random variable. Show that $(EX) \cdot (EX^{-1}) \geq 1$. Give one example such that “$>$” holds.

Problem 2. Let $X$ and $Y$ be two random variables such that

$$E(X^{2k}) = E(Y^{2k}) = 1 \cdot 3 \cdots (2k - 1) = (2k - 1)!! \text{ for any } k = 1, 2, \cdots ,$$

and

$$E(X^{2k+1}) = E(Y^{2k+1}) = 0 \text{ for any } k = 0, 1, 2, \cdots .$$

(i) Show that $L(X) = L(Y)$.

(ii) What is the characteristic function of $X$? Can you identify the distribution of $X$?

Problem 3. Which of the following are characteristic functions? Justify your answers.

(a) $\phi(t) = (e^{it} + e^{-it})/2$;

(b) $\phi(t) = \exp(it^{2003})$.

Problem 4. Let $\{X_n; n \geq 1\}$ be a sequence of identically distributed random variables with $X_1 \sim N(0, 1)$. Prove that

$$E \left( 1 + \frac{X_n^2}{n} \right)^{-n} \to \frac{1}{\sqrt{3}}$$

as $n \to \infty$. 

Problem 5. Suppose $X_n$ has density function $f_n(x)$ for each $n = 0, 1, 2, \cdots$. If $|X_n| \leq 1$ for each $n \geq 0$ and $f_n(x) \to f_0(x)$ as $n \to \infty$ for each $x \in [-1, 1]$. Prove that

$$\int_{-\infty}^{\infty} |f_n(x) - f_0(x)| \, dx \to 0$$

as $n \to \infty$.

Problem 6. Suppose $P(|X| \geq x) \leq e^{-x}$ for $x > 0$. Show that $E|X|^k \leq k!$ for $k = 1, 2, \cdots$.

Problem 7. Let $X_1, X_2, \cdots, X_n$ be a sequence of i.i.d. random variables with $X_1 \sim U[0, 1]$. Let $S_n = X_1 + \cdots + X_n$. Show that

$$P \left( \frac{1}{5} \leq \frac{S_n}{n} \leq \frac{1}{4} \right) \leq e^{-n\lambda}$$

for some $\lambda > 0$. Calculate this $\lambda$. 