1. Since $E(X^k)$ has the same value for every positive integer k, we might try to find a random variable X such that X, X^2, X^3, X^4,... all have the same distribution. If X can take only the values 0 and 1, then $X^k = X$ for every positive integer k since $0^k = 0$ and $1^k = 1$. If $\Pr(X = 1) = p = 1 - \Pr(X = 0)$, then in order for $E(X^k) = 1/3$, as required, we must have $p = 1/3$. Therefore, a random variable X such that $\Pr(X = 1) = 1/3$ and $\Pr(X = 0) = 2/3$ satisfies the required conditions.

8. The number X of components that fail will have a binomial distribution with parameters $n = 10$ and $p = 0.2$. Therefore,

$$\begin{aligned}\Pr(X \geq 2 \mid X \geq 1) &= \frac{\Pr(X \geq 2)}{\Pr(X \geq 1)} = \frac{1 - \Pr(X = 0) - \Pr(X = 1)}{1 - \Pr(X = 0)} \\ &= \frac{1 - .1074 - .2684}{1 - .1074} = \frac{.6242}{.8926} = .6993.\end{aligned}$$

9. $$\Pr\left(X_1 = 1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right) = \frac{\Pr\left(X_1 = 1 \text{ and } \sum_{i=1}^{n} X_i = k\right)}{\Pr\left(\sum_{i=1}^{n} X_i = k\right)} = \frac{\Pr\left(X_1 = 1 \text{ and } \sum_{i=2}^{n} X_i = k - 1\right)}{\Pr\left(\sum_{i=1}^{n} X_i = k\right)}.$$

Since the random variables $X_1, \ldots, X_n$ are independent, it follows that X_1 and $\sum_{i=2}^{n} X_i$ are independent. Therefore, the final expression can be rewritten as

$$\frac{\Pr(X_1 = 1)\Pr\left(\sum_{i=2}^{n} X_i = k - 1\right)}{\Pr\left(\sum_{i=1}^{n} X_i = k\right)}.$$

The sum $\sum_{i=2}^{n} X_i$ has a binomial distribution with parameters $n - 1$ and p, and the sum $\sum_{i=1}^{n} X_i$ has a binomial distribution with parameters n and p. Therefore,

$$\Pr\left(\sum_{i=2}^{n} X_i = k - 1\right) = \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)} = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k},$$

and

$$\Pr\left(\sum_{i=1}^{n} X_i = k\right) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Also, $\Pr(X_1 = 1) = p$. It now follows that

$$\Pr\left(X_1 = 1 \,\middle|\, \sum_{i=1}^{n} X_i = k\right) = \frac{\binom{n-1}{k-1} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{k}{n}.$$

2. Let X denote the number of red balls that are obtained. Then X has a hypergeometric distribution with parameters $A = 5, B = 10$, and $n = 7$. The maximum value of X is $\min\{n, A\} = 5$, hence,

$$\Pr(X \geq 3) = \sum_{x=3}^{5} \frac{\binom{5}{x}\binom{10}{7-x}}{\binom{15}{7}} = \frac{2745}{6435} \approx 0.4266.$$

4. By Eq. (5.3.7),

$$\text{Var}(X) = \frac{(8)(20)}{(28)^2(27)}\, n(28-n).$$

The quadratic function $n(28-n)$ is a maximum when $n = 14$.

2. From the table of the Poisson distribution in the back of the book it is found that

$$\Pr(X \geq 3) = .0284 + .0050 + .0007 + .0001 + .0000 = .0342.$$

8. For $x = 0, 1, \ldots, k$,

$$\Pr(X_1 = x \,|\, X_1 + X_2 = k) = \frac{\Pr(X_1 = x \text{ and } X_1 + X_2 = k)}{\Pr(X_1 + X_2 = k)} = \frac{\Pr(X_1 = x \text{ and } X_2 = k - x)}{\Pr(X_1 + X_2 = k)}.$$

Since X_1 and X_2 are independent,

$$\Pr(X_1 = x \text{ and } X_2 = k - x) = \Pr(X_1 = x)\Pr(X_2 = k - x).$$

Also, by Theorem 5.4.1, the sum $X_1 + X_2$ will have a Poisson distribution with mean $\lambda_1 + \lambda_2$. Hence,

$$\begin{aligned}
\Pr(X_1 = x) &= \frac{\exp(-\lambda_1)\lambda_1^x}{x!} \\
\Pr(X_2 = k - x) &= \frac{\exp(-\lambda_2)\lambda_2^{k-x}}{(k-x)!} \\
\Pr(X_1 + X_2 = k) &= \frac{\exp(-(\lambda_1 + \lambda_2))(\lambda_1 + \lambda_2)^k}{k!}
\end{aligned}$$

It now follows that

$$\Pr(X_1 = x \,|\, X_1 + X_2 = k) = \frac{k!}{x!(k-x)!}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{k-x} = \binom{k}{x} p^x (1-p)^{k-x},$$

where $p = \lambda_1/(\lambda_1 + \lambda_2)$. It can now be seen that this conditional distribution is a binomial distribution with parameters k and $p = \lambda_1/(\lambda_1 + \lambda_2)$.

13. It can be assumed that the exact number of sets of triplets in this hospital is a binomial distribution with parameters $n = 700$ and $p = 0.001$. Therefore, this distribution can be approximated by a Poisson distribution with mean $700(0.001) = 0.7$. It is found from the table of the Poisson distribution that

$$\Pr(X = 1) = 0.3476.$$

2. (a) The number of tails will have a negative binomial distribution with parameters $r = 5$ and $p = 1/30$. By Eq. (5.5.9),

$$E(X) = \frac{rq}{p} = 5(29) = 145.$$

(b) By Eq. (5.5.9), $\text{Var}(X) = \frac{rq}{p^2} = 4350.$

4. (a) The number of failures X_A obtained by player A before he obtains r successes will have a negative binomial distribution with parameters r and p. The total number of throws required by player A will be $Y_A = X_A + r$. Therefore,

$$E(Y_A) = E(X_A) + r = r\frac{q}{p} + r = \frac{r}{p}.$$

The number of failures X_B obtained by player B before he obtains mr successes will have a negative binomial distribution with parameters mr and mp. The total number of throws required by player B will be $Y_B = X_B + mr$. Therefore,

$$E(Y_B) = E(X_B) + mr = (mr)\frac{(1-mp)}{mp} + mr = \frac{r}{p}.$$

(b)

$$\begin{aligned} \text{Var}(Y_A) &= \text{Var}(X_A) = \frac{rq}{p^2} = \frac{r}{p^2}(1-p) \text{ and} \\ \text{Var}(Y_B) &= \text{Var}(X_B) = \frac{(mr)(1-mp)}{(mp)^2} = \frac{r}{p^2}\left(\frac{1}{m} - p\right). \end{aligned}$$

Therefore, $\text{Var}(Y_B) < \text{Var}(Y_A)$.

5. By Eq. (5.5.6), the m.g.f. of X_i is

$$\psi_i(t) = \left(\frac{p}{1 - q\exp(t)}\right)^{r_i} \quad \text{for} \quad t < \log\left(\frac{1}{q}\right).$$

Therefore, the m.g.f. of $X_1 + \cdots + X_k$ is

$$\psi(t) = \prod_{i=1}^{k} \psi_i(t) = \left(\frac{p}{1 - q\exp(t)}\right)^{r_1 + \cdots + r_k} \quad \text{for} \quad t < \log\left(\frac{1}{q}\right).$$

Since $\psi(t)$ is the m.g.f. of a negative binomial distribution with parameters $r_1 + \cdots + r_k$ and p, that must be the distribution of $X_1 + \cdots + X_k$.

2. Let $Z = (X - 1)/2$. Then Z has a standard normal distribution.

(a) $\Pr(X \le 3) = \Pr(Z \le 1) = \Phi(1) = 0.8413$

(b) $\Pr(X > 1.5) = \Pr(Z > 0.25) = 1 - \Phi(0.25) = 0.4013$.

(c) $\Pr(X = 1) = 0$, because X has a continuous distribution.

(d) $\Pr(2 < X < 5) = \Pr(0.5 < Z < 2) = \Phi(2) - \Phi(0.5) = 0.2858$.

(e) $\Pr(X \ge 0) = \Pr(Z \ge -0.5) = \Pr(Z \le 0.5) = \Phi(0.5) = 0.6915$.

(f) $\Pr(-1 < X < 0.5) = \Pr(-1 < Z < -0.25) = \Pr(0.25 < Z < 1) = \Phi(1) - \Phi(0.25) = 0.2426$.

10. We know that $E(\overline{X}_n) = \mu$ and $\mathrm{Var}(\overline{X}_n) = \sigma^2/n = 4/25$. Hence, if we let $Z = (\overline{X}_n - \mu)/(2/5) = (5/2)(\overline{X}_n - \mu)$, then Z will have a standard normal distribution. Hence,

$$\Pr(|\overline{X}_n - \mu| \le 1) = \Pr(|Z| \le 2.5) = 2\Phi(2.5) - 1 = 0.9876.$$

15. Let $f_1(x)$ denote the p.d.f. of X if the person has glaucoma and let $f_2(x)$ denote the p.d.f. of X if the person does not have glaucoma. Furthermore, let A_1 denote the event that the person has glaucoma and let $A_2 = A_1^C$ denote the event that the person does not have glaucoma. Then

$$\begin{aligned}
\Pr(A_1) &= 0.1, \quad \Pr(A_2) = 0.9, \\
f_1(x) &= \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(x-25)^2\right\} \quad \text{for} \quad -\infty < x < \infty, \\
f_2(x) &= \frac{1}{(2\pi)^{1/2}} \exp\left\{-\frac{1}{2}(x-20)^2\right\} \quad \text{for} \quad -\infty < x < \infty.
\end{aligned}$$

(a) $$\Pr(A_1 \mid X = x) = \frac{\Pr(A_1) f_1(x)}{\Pr(A_1) f_1(x) + \Pr(A_2) f_2(x)}$$

(b) The value found in part (a) will be greater than 1/2 if and only if

$$\Pr(A_1) f_1(x) > \Pr(A_2) f_2(x).$$

All of the following inequalities are equivalent to this one:

(i) $\exp\{-(x-25)^2/2\} > 9\exp\{-(x-20)^2/2\}$

(ii) $-(x-25)^2/2 > \log 9 - (x-20)^2/2$

(iii) $(x-20)^2 - (x-25)^2 > 2\log 9$

(iv) $10x - 225 > 2\log 9$

(v) $x > 22.5 + \log(9)/5$.