Nonparametric Location Tests: *k*-Sample

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 - Overview
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 - Estimating treatment
 - Confidence intervals

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 - Overview
 - Test statistic
 - Relation to Wilcoxon's test

- 3) Kruskal-Wallis ANOVA:
 - Overview
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 - Hypothesis testing
 - Example

Wilcoxon's Rank Sum Test

Problem of Interest

For the two-sample location problem, we have N = m + n observations

- X_1, \ldots, X_m are iid random sample from population 1
- Y_1, \ldots, Y_n are iid random sample from population 2

We want to make inferences about difference in distributions

- Let F_1 and F_2 denote distributions of populations 1 and 2
- Null hypothesis is same distribution $\Leftrightarrow H_0: F_1(z) = F_2(z)$ for all z

Using the location-shift model, we have

- $F_1(z) = F_2(z \delta)$ where $\delta = E(Y) E(X)$ is treatment effect
- Null hypothesis is no treatment effect $\Leftrightarrow H_0: \delta = 0$

Typical Assumptions

Within sample independence assumption

- X_1, \ldots, X_m are iid random sample from population 1
- Y_1, \ldots, Y_n are iid random sample from population 2

Between sample independence assumption

• Samples $\{X_i\}_{i=1}^m$ and $\{Y_i\}_{i=1}^n$ are mutually independent

Continuity assumption: both F_1 and F_2 are continuous distributions

Assumptions and Hypothesis

Assumes independence (within and between sample) and continuity.

The null hypothesis about δ (treatment effect) is

$$H_0: \delta = 0$$

and we could have one of three alternative hypotheses:

- One-Sided Upper-Tail: $H_1: \delta > 0$
- One-Sided Lower-Tail: $H_1: \delta < 0$
- Two-Sided: $H_1: \delta \neq 0$

Test Statistic

 R_k denotes the rank of the combined sample $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$ for $k = 1, \ldots, N$, where N = m + n

Defining the indicator variable

$$\psi_{\mathbf{k}} = \left\{ \begin{array}{ll} \mathbf{1} & \text{if from 2nd population} \\ \mathbf{0} & \text{otherwise} \end{array} \right.$$

the Wilcoxon rank sum test statistic W is defined as

$$W = \sum_{k=1}^{N} R_k \psi_k = \sum_{j=1}^{n} S_j$$

where S_i is the (combined) rank associated with Y_i for j = 1, ..., n

Distribution of Test Statistic under H_0

Under H_0 all $\binom{N}{n}$ arrangements of Y-ranks occur with equal probability

- Given (N, n), calculate W for all $\binom{N}{n}$ possible outcomes
- Each outcome has probability $1/\binom{N}{n}$ under H_0

Example null distribution with m=3 and n=2:

_	Probability under H ₀	W	Y-ranks
_	1/10	3	1,2
	1/10	4	1,3
	1/10	5	1,4
	1/10	6	1,5
Note: there are $\binom{5}{2} = 10$ possibilities	1/10	5	2,3
	1/10	6	2,4
	1/10	7	2,5
	1/10	7	3,4
	1/10	8	3,5
	1/10	9	4.5

Hypothesis Testing

One-Sided Upper Tail Test:

- $H_0: \delta = 0 \text{ versus } H_1: \delta > 0$
- Reject H_0 if $W \ge w_\alpha$ where $P(W > w_\alpha) = \alpha$

One-Sided Lower Tail Test:

- $H_0: \delta = 0$ versus $H_1: \delta < 0$
- Reject H_0 if $W \leq n(m+n+1) w_{\alpha}$

Two-Sided Test:

- $H_0: \delta = 0$ versus $H_1: \delta \neq 0$
- Reject H_0 if $W \ge w_{\alpha/2}$ or $W \le n(m+n+1) w_{\alpha/2}$

Large Sample Approximation

Under H_0 , the expected value and variance of W are

- $E(W) = \frac{n(m+n+1)}{2}$
- $V(W) = \frac{mn(m+n+1)}{12}$

We can create a standardized test statistic W^* of the form

$$W^* = \frac{W - E(W)}{\sqrt{V(W)}}$$

which asymptotically follows a N(0,1) distribution.

Derivation of Large Sample Approximation

Note that we have $W = \sum_{i=1}^{n} S_{i}$, which implies that

- W/n is the average of the (combined) Y-ranks
- W/n has same distribution as sample mean of size n drawn without replacement from finite population $\{1, \ldots, N\}$

Using some basic results of finite population theory, we have

- $E(W/n) = \mu$, where $\mu = \frac{1}{N} \sum_{i=1}^{N} i = \frac{N+1}{2}$
- $V(W/n) = \sigma^2 \frac{N-n}{n(N-1)}$, where $\sigma^2 = (\frac{1}{N} \sum_{i=1}^{N} i^2) \mu^2 = \frac{(N-1)(N+1)}{12}$

Putting things together, we have that

- $E(W) = n\mu = \frac{n(N+1)}{2}$
- $V(W) = n^2 \sigma^2 \frac{N-n}{n(N-1)} = \frac{mn(N+1)}{12}$

Handling Ties

If $Z_i = Z_j$ for any two observations from combined sample $(X_1, \ldots, X_m, Y_1, \ldots, Y_n)$, then use the average ranking procedure.

- W is calculated in same fashion (using average ranks)
- Average ranks with null distribution is approximate level α test
- ullet Can still obtain an exact level α test via conditional distribution
- Need to adjust variance term for large sample approximation

Example 4.2: Description

SST = Social Skills Training program for alcoholics

Supplement to traditional treatment program (Control)

N = 23 total patients (m = 12 Control and n = 11 SST).

Table 4.2 gives post-treatment alcohol intake for each patient group, as well as the overall rank of the combined sample (R_k)

Want to test if the SST program reduced alcohol intake

- $H_0: \delta = 0$ versus $H_1: \delta < 0$.
- δ is treatment effect (location difference)

Example 4.2: Data

Nonparametric Statistical Methods, 3rd Ed. (Hollander et al., 2014)

Table 4.2 Alcohol Intake for 1 Year (Centiliter of Pure Alcohol)

			-
Control	R_k	SST	R_k
1042	(13)	874	(9)
1617	(23)	389	(2)
1180	(18)	612	(4)
973	(12)	798	(7)
1552	(22)	1152	(17)
1251	(19)	893	(10)
1151	(16)	541	(3)
1511	(21)	741	(6)
728	(5)	1064	(14)
1079	(15)	862	(8)
951	(11)	213	(1)
1319	(20)		

Source: L. Eriksen, S. Björnstad, and K. G. Götestam (1986).

Example 4.2: By Hand

Control	R_k	SST	R_k
1042	(13)	874	(9)
1617	(23)	389	(2)
1180	(18)	612	(4)
973	(12)	798	(7)
1552	(22)	1152	(17)
1251	(19)	893	(10)
1151	(16)	541	(3)
1511	(21)	741	(6)
728	(5)	1064	(14)
1079	(15)	862	(8)
951	(11)	213	(1)
1319	(20)		
\sum	195	\sum	81

$$W = \sum_{i=1}^{11} S_i = 81$$

Example 4.2: Using R (Hard Way)

```
> library(NSM3)
> data(alcohol.intake)
> alcohol.intake
Śχ
 [1] 1042 1617 1180 973 1552 1251 1151 1511 728 1079 951 1319
$v
     874 389 612 798 1152 893 541 741 1064 862 213
> r = rank(c(alcohol.intake$x,alcohol.intake$y))
> sum(r[1:12])
> sum(r[13:23])
[11 81
```

Example 4.2: Using R (Easy Way)

```
> control = alcohol.intake$x
> sst = alcohol.intake$y
> wilcox.test(control,sst,alternative="greater")

Wilcoxon rank sum test

data: control and sst
W = 117, p-value = 0.0004904
alternative hypothesis: true location shift is greater than 0
```

We reject H_0 : $\delta = 0$ and conclude that SST program results in reduced alcohol intake in recovering alcoholic patients.

Note: \mathbb{W} value output by wilcox.test is NOT $W = \sum_{j=1}^{n} S_j = 81$

•
$$W = mn - W + n(n+1)/2 = 12 * 11 - 81 + 11 * 12/2 = 117$$

An Estimator of δ

To estimate the treatment effect δ , first form the *mn* differences

$$D_{ij}=Y_j-X_i$$

for i = 1, ..., m and j = 1, ..., n.

The estimate of δ corresponding to Wilcoxon's rank sum test is

$$\hat{\delta} = \mathsf{median}(D_{ij}; i = 1, \dots, m; j = 1, \dots, n)$$

which is the median of the differences.

Motivation: make mean of $(X_1, \ldots, X_m, Y_1 - \hat{\delta}, \ldots, Y_n - \hat{\delta})$ as close as possible to E(W) = n(m+n+1)/2.

Symmetric Two-Sided Confidence Interval for δ

Define the following terms

- Let $U^{(1)} < U^{(2)} < \cdots \le U^{(mn)}$ denote the ordered D_{ii} scores
- $w_{\alpha/2}$ is the critical value such that $P(W > w_{\alpha/2}) = \alpha/2$ under H_0
- $C_{\alpha} = \frac{n(2m+n+1)}{2} + 1 w_{\alpha/2}$ is the transformed critical value

A symmetric $(1-\alpha)100\%$ confidence interval for δ is given by

$$\delta_L = U^{(C_{\alpha})}$$
 $\delta_{II} = U^{(mn+1-C_{\alpha})}$

One-Sided Confidence Intervals for δ

Define the following additional terms

- w_{α} is the critical value such that $P(W > w_{\alpha}) = \alpha$ under H_0
- $C_{\alpha}^* = \frac{n(2m+n+1)}{2} + 1 w_{\alpha}$ transformed critical value

An asymmetric $(1 - \alpha)100\%$ upper confidence bound for δ is

$$\delta_L = -\infty$$

$$\delta_U = U^{(mn+1-C_{\alpha}^*)}$$

An asymmetric $(1 - \alpha)100\%$ lower confidence bound for δ is

$$\delta_L = U^{(C_\alpha^*)}$$
$$\delta_U = \infty$$

Example 4.2: Estimate δ

Get $U^{(1)} < U^{(2)} < \cdots < U^{(M)}$ and $\hat{\theta}$ for previous example:

```
> d = as.vector(outer(control,sst,"-"))
> sort(d)
  [1] -424 -336 -201 -179 -165 -146 -134 -113 -110 -91
                                                            -73
                -1 15
                                             80
                                                  87
                                                      89
                                                            99
                                                                 99
 [131 -22 -13]
                           2.8
                                        77
 [25] 111 116
                116
                      149
                            153
                                 167
                                            175
                                                 180
                                                       186
                                                            187
                                                                 187
                217
                      232
                            2.44
                                                 2.81
                                                       2.87
 [37] 205
            210
                                 255
                                       258
                                            2.77
 [49]
       306
            318
                 338
                      339
                            339
                                 353
                                       358
                                            359
                                                 361
                                                       377
                                                            382
                                                                 389
 [61]
       400
            410
                 410
                      426
                            430
                                 432
                                       439
                                            445
                                                 447
                                                       453
                                                            457
                                                                 465
 [73]
       467
            488
                 501
                      510
                            515
                                 52.1
                                       538
                                            539
                                                 553
                                                       562
                                                                 578
                                                       678
 [85]
       584
            610
                 618
                      637
                            639
                                 639
                                       649
                                            653
                                                 659
                                                                 690
 [971
                 713
                      724
                            738
                                 743
                                       754
                                            755
                                                      762
[109]
                       829
                            862
                                 866
                                       876
                                            899
                                                 930
                                                       938
                                                            940
                                                                 967
       791
            811
                  819
[121] 970 1005 1011 1038 1076 1106 1122 1163 1228 1298 1339 1404
> median(d)
```

[11 435.5

Example 4.2: Confidence Interval for δ

Efficiency of Wilcoxon Rank Sum Test

Efficiency of W relative to two-sample t test:

			<u> </u>			
F	Normal	Uniform	Logistic	Double Exp	Cauchy	Exp
E(W, t)	0.955	1.000	1.097	1.500	∞	3.000

Interpreting the table:

- If F is normal, W is almost as efficient as t (4.5% efficiency loss)
- If F is non-normal, W is more efficient than t

Mann-Whitney *U*-Test

Assumptions and Hypothesis

Same assumptions and hypotheses as Wilcoxon Rank Sum Test.

Assumes independence (within and between sample) and continuity.

The null hypothesis about δ (treatment effect) is

$$H_0: \delta = 0$$

and we could have one of three alternative hypotheses:

- One-Sided Upper-Tail: $H_1: \delta > 0$
- One-Sided Lower-Tail: $H_1: \delta < 0$
- Two-Sided: $H_1: \delta \neq 0$

Test Statistic

Defining the indicator function

$$\phi(X_i, Y_j) = \begin{cases} 1 & \text{if } X_i < Y_j \\ 0 & \text{otherwise} \end{cases}$$

the Mann-Whitney test statistic *U* is defined as

$$U = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi(X_i, Y_j)$$

which counts the number of times X is before Y in combined sample.

Relation to Wilcoxon's Rank Sum Test Statistic

It was shown by Mann and Whitney that

$$W=U+\frac{n(n+1)}{2}$$

where W and U are the Wilcoxon and Mann-Whitney test statistics.

For a fixed sample size N = m + n, this implies that tests based on the W and U test statistics are equivalent.

- For a fixed N = m + n, we have W = U +constant
- Adding constant only changes location of distribution

Kruskal-Wallis ANOVA

Problem of Interest

For the *k*-sample location problem, we have $N = \sum_{i=1}^{k} n_i$

- X_{1i}, \ldots, X_{ni} are iid random sample from population j
- k > 2 is the number of sampled populations

We want to make inferences about difference in locations

- Let F_i denote distribution of population j
- Assume $F_i(z) = F(z \tau_i)$ where τ_i is *j*-th treatment effect

Using the location-shift model, we have

- $X_{ii} = \theta + \tau_i + e_{ii}$ where θ is median and e_{ii} is error (0 median)
- Null hypothesis is no treatment difference $\Leftrightarrow H_0: \tau_1 = \cdots = \tau_k$

Assumptions and Hypothesis

Within sample independence assumption

• X_{1j}, \ldots, X_{njj} are iid random sample from population j

Between sample independence assumption

• Samples $\{X_{ij}\}_{i=1}^{n_j}$ and $\{X_{ij'}\}_{i=1}^{n_{j'}}$ are mutually independent $\forall j \neq j'$

Continuity and form assumption

• F_j is continuous and has the form $F_j(z) = F(z - \tau_j)$ for all j, z

The null and alternative hypotheses are

• $H_0: \tau_1 = \cdots = \tau_k$ versus $H_1: \tau_j \neq \tau_{j'}$ for some j, j'

Test Statistic

Let r_{ii} denote the rank of X_{ii} in the combined sample of size $N = \sum_{i=1}^{k} n_i$ observations

Defining the sum and average of the joint ranks for each group

$$R_j = \sum_{i=1}^{n_j} r_{ij}$$
 and $R_{\cdot j} = R_j/n_j$

the Kruskall-Wallis test statistic H is defined as

$$H = \frac{12}{N(N+1)} \sum_{j=1}^{k} n_j \left(R_{\cdot j} - \frac{N+1}{2} \right)^2$$
$$= \left(\frac{12}{N(N+1)} \sum_{j=1}^{k} \frac{R_j^2}{n_j} \right) - 3(N+1)$$

where $\frac{N+1}{2} = (\sum_{i=1}^{k} \sum_{i=1}^{n_j} r_{ij}/N)$ is the average of the joint rankings

Hypothesis Testing & Large Sample Approximation

One-Sided Upper Tail Test:

- $H_0: \tau_1 = \cdots = \tau_k$ versus $H_1: \tau_i \neq \tau_{i'}$ for some j, j'
- Reject H_0 if $H \ge h_\alpha$ where $P(H > h_\alpha) = \alpha$

This is the only appropriate test here...

- As $(R_{i} \frac{N+1}{2})^2$ increases, we have more evidence against H_0
- We only reject H_0 if test statistic H is too large

Under H_0 and as $\min_i(n_i) \to \infty$, we have that $H \sim \chi^2_{(k-1)}$

- $\chi^2_{(k-1)}$ denotes a chi-squared distribution with k-1 df
- Reject H_0 if $H \ge \chi^2_{(k-1):\alpha}$ where $P(\chi^2_{(k-1)} > \chi^2_{(k-1):\alpha}) = \alpha$

Handling Ties

When there are ties, we need to replace *H* with

$$H^* = \frac{H}{1 - \frac{1}{N^3 - N} \sum_{j=1}^{g} (t_j^3 - t_j)}$$

where

- H is computed using averaged ranks
- g is the number of tied groups
- t_i is the size of the tied group

Example: Description

Visual and auditory cues example from Hays (1994) *Statistics*.

Does lack of visual/auditory synchrony affect memory?

Total of n = 30 college students participate in memory experiment.

- Watch video of person reciting 50 words
- Try to remember the 50 words (record number correct)

Randomly assign $n_i = 10$ subjects to one of g = 3 video conditions:

- fast: sound precedes lip movements in video
- normal: sound synced with lip movements in video
- slow: lip movements in video precede sound

Example: Data

From *Statistics* (Hays, 1994)

Number of correctly remembered words:

	Fast $(j = 1)$		Fast $(j = 1)$ Normal $(j = 2)$		Slov	v (<i>j</i> = 3)
Subject (i)	X_{ij}	(r_{ij})	X_{ij}	(r_{ij})	X_{ij}	(r_{ij})
1	23	(15.5)	27	(23.0)	23	(15.5)
2	22	(12.5)	28	(24.0)	24	(18.5)
3	18	(5.0)	33	(29.0)	21	(10.5)
4	15	(1.0)	19	(7.0)	25	(20.5)
5	29	(25.5)	25	(20.5)	19	(7.0)
6	30	(27.5)	29	(25.5)	24	(18.5)
7	23	(15.5)	36	(30.0)	22	(12.5)
8	16	(2.0)	30	(27.5)	17	(3.5)
9	19	(7.0)	26	(22.0)	20	(9.0)
10	17	(3.5)	21	(10.5)	23	(15.5)
$R_j = \sum_{i=1}^{10} r_{ij}$		115		219		131

Example: By Hand

There are N = 30 subjects, so Kruskal-Wallis test statistic is

$$H = \left(\frac{12}{30(31)} \left[115^2 + 219^2 + 131^2\right] / 10\right) - 3(31)$$
= 8.092903

but this needs to be corrected for the ties.

There are q = 9 groups of ties with group sizes

$$(t_1,\ldots,t_9)=(2,3,2,2,4,2,2,2,2)$$

so the corrected test statistic value is

$$H^* = \frac{H}{1 - \frac{1}{30^3 - 30} \sum_{j=1}^{9} (t_j^3 - t_j)}$$
$$= \frac{8.092903}{0.9953281} = 8.13089$$

Example: Using R (Hard Way)

```
> sync = c(23,27,23,22,28,24,18,33,21,15,
           19, 25, 29, 25, 19, 30, 29, 24, 23, 36,
+
           22, 16, 30, 17, 19, 26, 20, 17, 21, 23)
+
> cond = factor(rep(c("fast", "normal", "slow"), 10))
> N = 30
> Rj = tapply(rank(sync),cond,sum)
> H = (12/(N*(N+1)))*sum(Rj^2)/10 - 3*(N+1)
> H
[11 8.092903
> tj = tapply(sync,sync,length)
> tj = tj[tj>1]
> t.i
17 19 21 22 23 24 25 29 30
2 3 2 2 4 2 2 2 2
> Hstar = H/(1-sum(tj^3-tj)/(N^3-N))
> Hstar
[1] 8.13089
> 1 - pchisq(Hstar, 2)
[1] 0.01715536
```

Example: Using R (Easy Way)

Friedman Test

Problem of Interest

For two-way layout, we have N = nk observations

- X_{i1}, \ldots, X_{ik} is *i*-th block of data
- k > 2 is the number of treatments

We want to make inferences about difference in locations

- Let F_{ii} denote distribution of i-th block and j-th treatment
- Assume $F_{ij}(z) = F(z \beta_i \tau_j)$ where β_i is the *i*-th block effect and τ_j is the *j*-th treatment effect

Using the location-shift model, we have

- $X_{ij} = \theta + \beta_i + \tau_j + e_{ij}$ where θ is median and e_{ij} is error (0 median)
- Null hypothesis is no treatment difference $\Leftrightarrow H_0: \tau_1 = \cdots = \tau_k$

Assumptions and Hypothesis

Within block independence assumption

• X_{i1}, \ldots, X_{ik} are administered in random order

Between block independence assumption

• Blocks $\{X_{ij}\}_{j=1}^k$ and $\{X_{i'j}\}_{j=1}^k$ are mutually independent $\forall i \neq i'$

Continuity and form assumption

• F_{ij} is continuous and has form $F_{ij}(z) = F(z - \beta_i - \tau_j)$ for all i, j, z

The null and alternative hypotheses are

• $H_0: \tau_1 = \cdots = \tau_k$ versus $H_1: \tau_j \neq \tau_{j'}$ for some j, j'

Test Statistic

 $r_{ii} \in \{1, \dots, k\}$ denotes the rank of X_{i1}, \dots, X_{ik} within the *i*-th block

Defining the sum and average of the joint ranks for each group

$$R_j = \sum_{i=1}^n r_{ij}$$
 and $R_{\cdot j} = R_j/n$

the Friedman test statistic S is defined as

$$S = \frac{12n}{k(k+1)} \sum_{j=1}^{k} \left(R_{\cdot j} - \frac{k+1}{2} \right)^{2}$$
$$= \left(\frac{12}{nk(k+1)} \sum_{j=1}^{k} R_{j}^{2} \right) - 3n(k+1)$$

where $\frac{k+1}{2} = (\sum_{i=1}^{n} \sum_{j=1}^{k} r_{ij}/N)$ is average of within-block rankings

Hypothesis Testing & Large Sample Approximation

One-Sided Upper Tail Test:

- $H_0: \tau_1 = \cdots = \tau_k$ versus $H_1: \tau_i \neq \tau_{i'}$ for some j, j'
- Reject H_0 if $S \geq s_{\alpha}$ where $P(S > s_{\alpha}) = \alpha$

This is the only appropriate test here...

- As $(R_{i} \frac{k+1}{2})^2$ increases, we have more evidence against H_0
- We only reject H_0 if test statistic S is too large

Under H_0 and as $n \to \infty$, we have that $S \sim \chi^2_{(k-1)}$

- $\chi^2_{(k-1)}$ denotes a chi-squared distribution with k-1 df
- Reject H_0 if $S \ge \chi^2_{(k-1):\alpha}$ where $P(\chi^2_{(k-1)} > \chi^2_{(k-1):\alpha}) = \alpha$

Handling Ties

When there are ties, we need to replace S with

$$S^* = \frac{12\sum_{j=1}^{k} \left(R_j - \frac{n(k+1)}{2}\right)^2}{nk(k+1) - \frac{1}{k-1}\sum_{j=1}^{n} \left[\left(\sum_{j=1}^{g_i} t_{ij}^3\right) - k\right]}$$

where

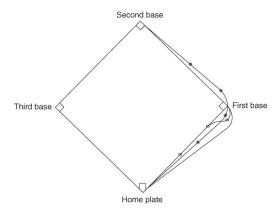
- g_i is the number of tied groups in i-th block
- t_{ii} is the size of the j-th tied group in i-th block

Example 7.1: Description

n=22 baseball players participated in a baserunning study

Compared k = 3 methods to round first base:

- Round out (diamond)
- Narrow angle (asterisk)
- Wide angle (solid)



From Nonparametric Statistical Methods. 3rd Ed. (Hollander et al., 2014)

Response variable is time to run to 2nd base

Example 7.1: Data

Nonparametric Statistical Methods, 3rd Ed. (Hollander et al., 2014)

Table 7.1 Rounding-First-Base Times

	Round Out $(j = 1)$		Narrow Angle $(j = 2)$		Wide Angle (j = 3)	
Player (i)	X_{ij}	r_{ij}	X _{ij}	r _{ij}	X_{ij}	r_{ij}
1	5.40	1.0	5.50	2.0	5.55	3
2	5.85	3.0	5.70	1.0	5.75	2
3	5.20	1.0	5.60	3.0	5.50	2
4	5.55	3.0	5.50	2.0	5.40	1
5	5.90	3.0	5.85	2.0	5.70	1
6 7	5.45	1.0	5.55	2.0	5.60	3
7	5.40	2.5	5.40	2.5	5.35	1
8	5.45	2.0	5.50	3.0	5.35	1
9	5.25	3.0	5.15	2.0	5.00	1
10	5.85	3.0	5.80	2.0	5.70	1
11	5.25	3.0	5.20	2.0	5.10	1
12	5.65	3.0	5.55	2.0	5.45	1
13	5.60	3.0	5.35	1.0	5.45	2
14	5.05	3.0	5.00	2.0	4.95	1
15	5.50	2.5	5.50	2.5	5.40	1
16	5.45	1.0	5.55	3.0	5.50	2
17	5.55	2.5	5.55	2.5	5.35	1
18	5.45	1.0	5.50	2.0	5.55	3
19	5.50	3.0	5.45	2.0	5.25	1
20	5.65	3.0	5.60	2.0	5.40	1
21	5.70	3.0	5.65	2.0	5.55	1
22	6.30	2.5	6.30	2.5	6.25	1
$R_j = \sum_{i=1}^n r_{ij}$		53		47		32

Source: W. F. Woodward (1970).

Example: By Hand

There are ties in blocks $i \in \{7, 15, 17, 22\}$ such that

$$t_{i1} = 2$$
 and $t_{i2} = 1$

because there is one tied group of size 2 and one tied group of size 1 for each block $\Longrightarrow (\sum_{i=1}^{g_i} t_{ii}^3) - k = (2^3 + 1^3) - 3 = 6$ for each block.

The corrected test statistic value is

$$S^* = \frac{12 \left[(53 - 44)^2 + (47 - 44)^2 + (32 - 44)^2 \right]}{22 * 3 * 4 - 0.5 * (6 * 4)}$$
$$= 11.14286$$

Example: Using R (Enter Data)

```
rounding.times = matrix(c(5.40, 5.50, 5.55,
                           5.85, 5.70, 5.75,
                           5.20, 5.60, 5.50,
                           5.55, 5.50, 5.40,
                           5.90, 5.85, 5.70,
                           5.45. 5.55. 5.60.
                           5.40, 5.40, 5.35,
                           5.45. 5.50. 5.35.
                           5.25. 5.15. 5.00.
                           5.85, 5.80, 5.70,
                           5.25, 5.20, 5.10,
                           5.65, 5.55, 5.45,
                           5.60, 5.35, 5.45,
                           5.05, 5.00, 4.95,
                           5.50, 5.50, 5.40,
                           5.45. 5.55. 5.50.
                           5.55. 5.55. 5.35.
                           5.45, 5.50, 5.55,
                           5.50, 5.45, 5.25,
                           5.65, 5.60, 5.40,
                           5.70, 5.65, 5.55,
                           6.30, 6.30, 6.25), ncol=3, byrow=TRUE)
```

Example: Using R (Hard Way)

```
> rtrank = t(apply(rounding.times,1,rank))
> n = 22
> k = 3
> vrt = as.vector(rtrank)
> tj = tapply(vrt, list(rep(1:n,k), vrt), length)
> cval = 0
> for(i in 1:n){
+ tidx = which(is.na(tj[i,]) == FALSE)
+ tij = tj[i,tidx]
+
     if (length(tij) < k) {cval=cval+sum(tij^3) - k}
+ }
> top = 12*sum((colSums(rtrank)-n*(k+1)/2)^2)
> bot = n*k*(k+1)-(1/(k-1))*cval
> Sc = top/bot
> Sc
[1] 11.14286
> 1 - pchisq(Sc, 2)
[1] 0.003805041
```

Example: Using R (Easy Way)

> friedman.test(rounding.times)

```
Friedman rank sum test
```

```
data: rounding.times
Friedman chi-squared = 11.1429, df = 2, p-value = 0.003805
```