Introduction to Set Theory

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Definition of a Set

The field of “set theory” is a branch of mathematics that is concerned with describing collections of objects.

A set is a collection of objects, where an “object” is a generic term that refers to the elements (or members) of the set.

The notation $a \in A$ denotes that the object a belongs to the set A.

- The symbol “$\in$” should be read as “is a member of”
- $a \in A$ denotes the lower case letter a is a member (or element) of the upper case letter A (the set)

Specifying a Set

Sets are defined by listing the elements inside curly braces, such as

$$A = \{a_1, \ldots, a_n\}$$

where a_i is the i-th element of the set A for $i = 1, \ldots, n$.

We could also use a rule or property that specifies the elements, such as

$$A = \{a \mid \text{some rule or property}\}$$

where a denotes an arbitrary member that satisfies the specified rule.

- The vertical bar symbol "$|$" should be read as "such that"
- The notation $\{a \mid \text{rule}\}$ denotes that the set consists of all elements a such that the rule is satisfied

Set Example 1

Define the set C to denote the possible outcomes of a coin toss:

$$C = \{\text{heads, tails}\}$$

which is a list containing all of the elements of C.

If we wanted to use the "rule" notation, we could define the set C as

$$C = \{c \mid c \text{ is either "heads" or "tails"}\}$$

which denotes the same set that was previously defined.

Given that there are only two elements, the list notation is preferable.

Set Example 2

Define the set D to denote the possible outcomes of the roll of a dice:

$$D = \{1, 2, 3, 4, 5, 6\}$$

which is a list containing all of the elements of D.

If we wanted to use the "rule" notation, we could define the set D as

$$D = \{d \mid d \text{ is a positive integer less than or equal to } 6\}$$

which denotes the same set that was previously defined.

Given that there are only six elements, the list notation is preferable.

Set Example 3

Define the set S to denote the states in the United States of America:

$$S = \{\text{Alabama, Alaska, } \ldots, \text{ Wisconsin, Wyoming}\}$$

which is a list suggesting all of the elements of S.

- "…" denotes some additional elements are implied but omitted

Using the rule notation:

$$S = \{s \mid s \text{ is a state in the United States of America}\}$$

which denotes the same set that was previously suggested.

The rule notation is preferred because the list (with …) is vague.

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Definition of a Subset

Given sets A and B, we say that B is a subset of A if every member of set B is also a member of set A.

- The notation $B \subseteq A$ denotes that B is a subset of A.
- "$\subseteq$" includes the possibility that A and B are equivalent sets.

Given sets A and B, we say that A and B are equivalent if the two sets contain the exact same elements.

- The notation $A = B$ denotes that two sets are equivalent.
- Equivalent sets satisfy $A \subseteq B$ and $B \subseteq A$.

Subset Examples

Example. Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 3, 4, 5, 6\}$.

- The sets A and B are equivalent, i.e., $A = B$ (A is equal to B)
- $A \subseteq B$ (A is a subset of B) and $B \subseteq A$ (B is a subset of A)

Example. Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 7\}$.

- Each set contains at least one unique element
- $A \not\subseteq B$ (A is not a subset of B)
- $B \not\subseteq A$ (B is not a subset of A)

Definition of a Proper Subset

Given sets A and B, we say that B is a proper subset of A if (i) every member of B is also a member of A, and (ii) A contains at least one member that is not in B.

- The notation $B \subset A$ denotes that B is a proper subset of A.

Example. Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 3\}$.

- $B \subseteq A$ (B is a subset of A)
- $B \subset A$ (B is a proper subset of A)
- The proper subset notation provides more information

Definition of the Universal Set

The universal set U refers to the set that contains all other objects of interest, such that any other set A is a proper subset of U.

- $A \subset U$ where A is any other set

Example. Suppose that $U = \{1, 2, 3, \ldots\}$ is the set of all natural numbers, i.e., all positive integers.

- U would be considered the universal set for the previous examples
- If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3\}$, then
- $A \subset U$ and $B \subset U$ for each example

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Cardinality of a Set

The cardinality (or size) of a set refers to the number of elements in the set. Note that the cardinality of a set A is typically denoted by $|A|$.

If $|A| \leq |B|$ and $|B| \leq |A|$, then we write that $|A| = |B|$, which is the Schröder-Bernstein theorem.

Note that $|A| = |B|$ does *not* imply that $A = B$.

- If $A = \{\text{cat, dog, fish}\}$ and $B = \{\text{red, white, blue}\}$
- Then $|A| = |B|$ but $A \neq B$

Examples of Set Cardinality

The previously discussed examples all had finite cardinalities.

Example. If $C = \{\text{heads, tails}\}$, then $|C| = 2$.

Example. If $D = \{1, 2, 3, 4, 5, 6\}$, then $|D| = 6$.

Example. If $S = \{s \mid s \text{ is a state in the United States of America}\}$, then $|S| = 50$.

Finite versus Infinite Sets

A set is finite if the number of elements of the set is countable, whereas a set is infinite if the number of elements of the set is uncountable.

A set A is considered "countable" (i) if the set has a finite number of elements, i.e., $|A| < \infty$, or (ii) if the number of elements of the set has a 1-to-1 relation with the set of natural numbers $N = \{1, 2, 3, \ldots\}$.

Example. The set of even numbers is a countably infinite set:
$E = \{e \mid e = 2n \text{ where } n \text{ is a natural number}\}$

Example. The set $A = \{a \mid a \text{ is a point on a circle}\}$ is an infinite set

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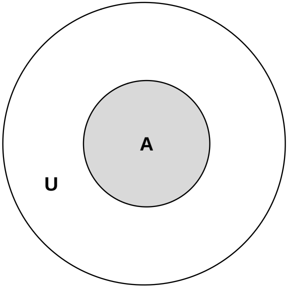
Graphical Depictions of Sets

Relations between sets can be shown using a Venn (or Euler) Diagram.



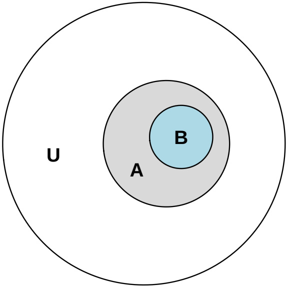


Figure 1: $A \subset U$ (left) and $B \subset A \subset U$ (right). Created with **eulerr** R package.

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Unions and Intersections of Sets

The union of two sets A and B contains all of the objects that are in either set. The union is denoted as $C = A \cup B$, where the $C = \{c \mid c \in A \text{ or } c \in B\}$.

The intersection of two sets A and B contains all of the objects that are in both sets. The intersection is denoted as $C = A \cap B$, where the $C = \{c \mid c \in A \text{ and } c \in B\}$.

Example. If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$, then the union is $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and the intersection is $A \cap B = \{3\}$.

Depiction of Unions and Intersections

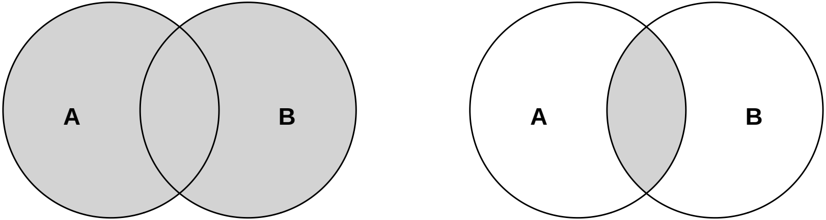

Figure 2: $A \cup B$ (left) and $A \cap B$ (right). Created with **eulerr** R package.

The Empty Set and Disjoint Sets

The empty set is the set that contains no elements, which is denoted by $\emptyset = \{\}$. Two sets are said to be disjoint if they have no elements in common, i.e., if $A \cap B = \emptyset$.

Note that the empty set is considered to be a subset of all sets, i.e., $\emptyset \subseteq A$. As a result, we have that $\emptyset \cup A = A$ for any set A. Note that we also have that $\emptyset \cap A = \emptyset$ for any set A.

Example. If $A = \{\text{cat, dog, fish}\}$ and $B = \{\text{red, white, blue}\}$, then $A \cap B = \emptyset$.

Example. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{1, 4, 7, 8, 9, 10\}$, then $A \cap B \cap C = \emptyset$.

Depiction of Disjoint Sets

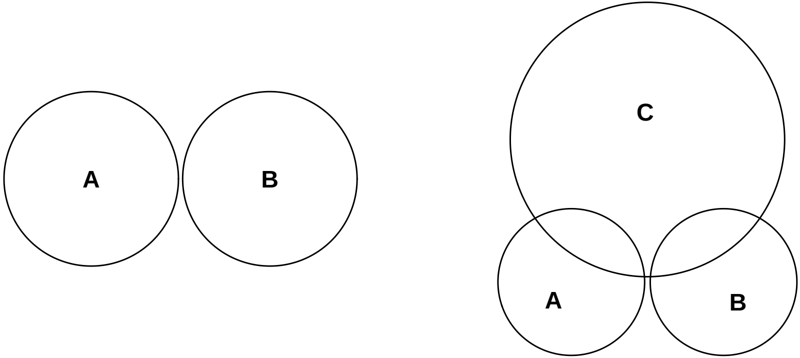

Figure 3: $A \cap B = \emptyset$ (left) and $A \cap B \cap C = \emptyset$ (right). Created with **eulerr** R package.

Order of Operations

Note that the order of operations is important if you are talking about unions and intersections with more than two sets:

- $(A \cup B) \cap C$ is not necessarily equivalent to $A \cup (B \cap C)$

Example. Using the three sets $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{1, 4, 7, 8, 9, 10\}$, we have that $(A \cup B) \cap C = \{1, 4\}$ and $A \cup (B \cap C) = \{1, 2, 3, 4\}$.

Question for the Reader: Does the order of operations matter if we are only talking about unions or interactions?

- Is $(A \cup B) \cup C$ the same as $A \cup (B \cup C)$?
- Is $(A \cap B) \cap C$ the same as $A \cap (B \cap C)$?

Depiction of Order of Operations

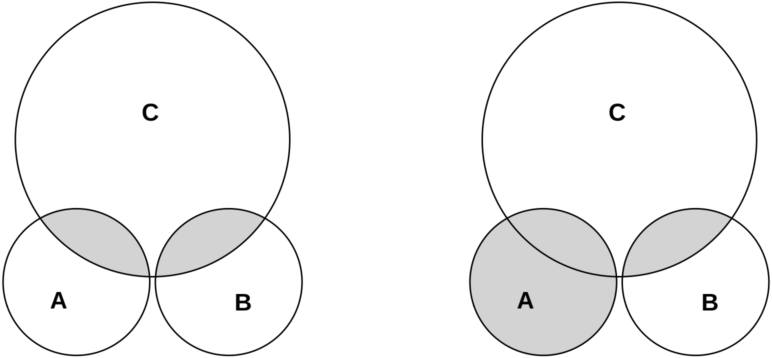

Figure 4: $(A \cup B) \cap C$ (left) and $A \cup (B \cap C)$ (right). Created with **eulerr** R package.

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Complement of a Set

The complement of a set A, denoted by A^c (or sometimes by $\overline{A}$), consists of all elements that are in the universal set U but not in A, i.e., $A^c = \{a \mid a \in U, a \notin A\}$.

The concept of a complement requires the definition of both the set of interest (i.e., A) and the universal set (i.e., U). In other words, the complement of a set is defined with respect to the universal set.

Example. If $A = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the complement of A is defined as $A^c = \{6, 7, 8, 9, 10\}$.

Difference Between Two Sets

The difference of a set A minus a set B, denoted by $A \backslash B$ (or sometimes by $A - B$), consists of all elements that are in A but not in B, i.e., $A \backslash B = \{a \mid a \in A, a \notin B\}$.

Note that the set difference $(A - B)$ is the intersection of A and the complement of B, i.e., $A \backslash B = A \cap B^c$. Also, note that $A \backslash B$ is not the same as $B \backslash A$ (i.e., non-commutative).

Example. If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6\}$, then $A \backslash B = \{1, 2, 3\}$ and $B \backslash A = \emptyset$.

Depiction of Set Complements and Differences

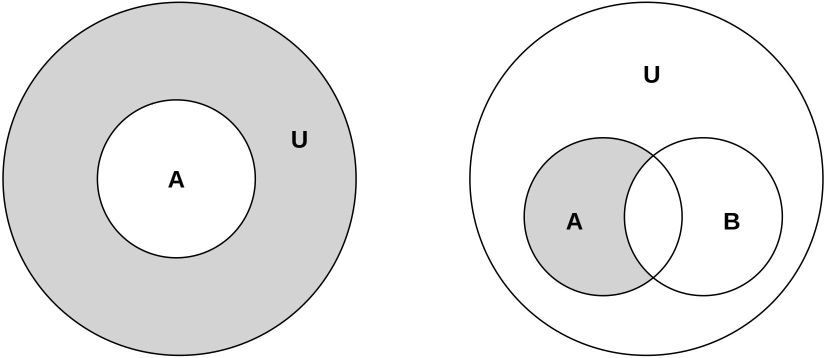

Figure 5: A^c (left) and $A \backslash B$ (right). Created with **eulerr** R package.

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Some Helpful Set Theory Rules

1. $A \cup \emptyset = A$
2. $A \cap \emptyset = \emptyset$
3. $A \cup U = U$
4. $A \cap U = A$
5. $A \cup A^c = U$
6. $A \cap A^c = \emptyset$
7. $A \cup B = B \cup A$
8. $A \cap B = B \cap A$
9. $A \cup (B \cup C) = (A \cup B) \cup C$
10. $A \cap (B \cap C) = (A \cap B) \cap C$
11. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
12. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
13. $(A \cup B)^c = A^c \cap B^c$
14. $(A \cap B)^c = A^c \cup B^c$
15. $A \backslash B = B \backslash A$ if and only if $A = B$