Introduction to Set Theory

Nathaniel E. Helwig

University of Minnesota

1 What is a Set?

The field of "set theory" is a branch of mathematics that is concerned with describing collections of objects. Set theory is fundamental to probability theory, which is the cornerstone of the field of Statistics. Thus, we need to understand some basic set theory as a prerequisite to understanding probability and statistics.

Definition. A set is a collection of objects, where an "object" is a generic term that refers to the elements (or members) of the set.

The notation $a \in A$ denotes that the object a belongs to the set A. Note that the symbol "$\in$" should be read as "is a member of", so that notation $a \in A$ denotes that the lower case letter a is a member (or element) of the upper case letter A (the set). Sets are defined by listing the elements inside curly braces, such as

$$A = \{a_1, \ldots, a_n\}$$

where a_i is the i-th element of the set A for $i = 1, \ldots, n$. We could also define a set using a rule that specifies the elements of the set, such as

$$A = \{a \mid \text{some rule or property}\}$$

where a denotes an arbitrary member that satisfies the specified rule or property, which is listed to the right of the vertical bar. Note that the vertical bar symbol "|" should be read as "such that", so that the notation $\{a \mid \text{rule}\}$ denotes that the set consists of all elements a such that the rule is satisfied.

***Example* 1.** Define the set C to denote the possible outcomes of a coin toss:

$$C = \{\text{heads, tails}\}$$

which is a list containing all of the elements of C. If we wanted to use the "rule" notation, we could define the set C as

$$C = \{c \mid c \text{ is either "heads" or "tails"}\}$$

which denotes the same set that was previously defined with the curly braces. Given that there are only two elements of the set C, the curly braces notation should be preferred.

***Example* 2.** Define the set D to denote the possible outcomes of the roll of a dice:

$$D = \{1, 2, 3, 4, 5, 6\}$$

which is a list containing all of the elements of D. If we wanted to use the "rule" notation, we could define the set D as

$$D = \{d \mid d \text{ is a positive integer less than or equal to } 6\}$$

which denotes the same set that was previously defined with the curly braces.

***Example* 3.** Define the set S to denote the states in the United States of America:

$$S = \{\text{Alabama, Alaska}, \ldots, \text{Wisconsin, Wyoming}\}$$

which is a list suggesting all of the elements of S. Note that the "..." notation denotes that some additional elements are suggested but omitted. Using the rule notation:

$$S = \{s \mid s \text{ is a state in the United States of America}\}$$

which denotes the same set that was previously suggested with the curly braces. Note that, in this case, the rule notation is preferred because the curly braces notations (with the ...) is vague. For example, without the rule notation, it would be unclear if the set S is referring to all states in the United States of America, only the "red states", or some other subset.

2 What is a Subset?

When working with sets, it is necessary to define some language that can be used to talk about how sets are similar to (or different from) one another. One basic relation that we can talk about is whether one set is a subset of another set.

Definition. Given sets A and B, we say that B is a subset of A if every member of set B is also a member of set A. The notation $B \subseteq A$ denotes that B is a subset of A. The notation "$\subseteq$" includes the possibility that A and B are equivalent sets.

Definition. Given sets A and B, we say that A and B are equivalent if the two sets contain the exact same elements. The notation $A = B$ denotes that two sets are equivalent to one another. Note that equivalent sets satisfy $A \subseteq B$ and $B \subseteq A$.

***Example* 4.** Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 3, 4, 5, 6\}$. Then the sets A and B are equivalent, so we could write $A = B$ (A is equal to B), $A \subseteq B$ (A is a subset of B), and $B \subseteq A$ (B is a subset of A).

***Example* 5.** Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 7\}$. Then we could write $A \nsubseteq B$ (A is not a subset of B) and $B \nsubseteq A$ (B is not a subset of A), which implies that each set contains at least one unique element.

Definition. Given sets A and B, we say that B is a proper subset of A if (i) every member of B is also a member of A, and (ii) A contains at least one member that is not in B. The notation $B \subset A$ denotes that B is a proper subset of A.

***Example* 6.** Suppose that A is all possible outcomes of the roll of a standard (six-sided) dice and $B = \{1, 2, 3\}$. Then we could write $B \subseteq A$ (B is a subset of A) or $B \subset A$ (B is a proper subset of A). Note that the proper subset notation is preferred because it provides more information, i.e., that A contains some elements that are not elements of B.

Definition. The universal set U refers to the set that contains all other objects of interest, such that any other set A is a subset of the universal set, i.e., $A \subset U$.

***Example* 7.** Suppose that $U = \{1, 2, 3, \ldots\}$ is the set of all natural numbers, i.e., all positive integers. The set U would be considered the universal set for Examples 4–6, such that $A \subset U$ and $B \subset U$ for each example.

3 The Size of a Set

Another basic concept that we can use to describe and compare sets is their size, which is a rather intuitive concept. As you might guess, defining the size of a set makes it possible to talk about which sets are smaller or bigger than others.

Definition. The cardinality (or size) of a set refers to the number of elements in the set. Note that the cardinality of a set A is typically denoted by $|A|$.

If $|A| \leq |B|$ and $|B| \leq |A|$, then we write that $|A| = |B|$, which is the Schröder-Bernstein theorem. Note that $|A| = |B|$ does *not* imply that $A = B$.

***Example* 8.** If $A = \{\text{cat, dog, fish}\}$ and $B = \{\text{red, white, blue}\}$, then sets A and B have the same cardinality (i.e., $|A| = |B|$) but are not equivalent sets (i.e., $A \neq B$).

The previously discussed examples all had finite cardinalities:

- The cardinality of the set in Example 1 is $|C| = 2$.
- The cardinality of the set in Example 2 is $|D| = 6$.
- The cardinality of the set in Example 3 is $|S| = 50$.

Definition. A set is finite if the number of elements of the set is countable, whereas a set is infinite if the number of elements of the set is uncountable.

Note that a set A is considered "countable" (i) if the set has a finite number of elements, i.e., $|A| < \infty$, or (ii) if the number of elements of the set has a 1-to-1 correspondence with the set of natural numbers $N = \{1, 2, 3, \ldots\}$.

***Example* 9.** The set of natural numbers $N = \{1, 2, 3, \ldots\}$ is a countably infinite set, given that we could count the infinite number of elements (with an infinite amount of time).

***Example* 10.** The set of even numbers $E = \{e \mid e = 2n \text{ where } n \text{ is a natural number}\}$ is a countably infinite set, given that it has a 1-to-1 mapping with the set of natural numbers.

***Example* 11.** The set $A = \{a \mid a \text{ is a point on a circle}\}$ is an infinite set, given that there are an uncountable number of points on any given circle.

***Example* 12.** The set $A = \{a \mid a \text{ is a real number between 0 and 1}\}$ is an infinite set, given that there are an uncountable number of real numbers between 0 and 1.

4 Venn Diagrams (or Euler Diagrams)

Relations between sets can be graphically depicted using a Venn (or Euler) Diagram.

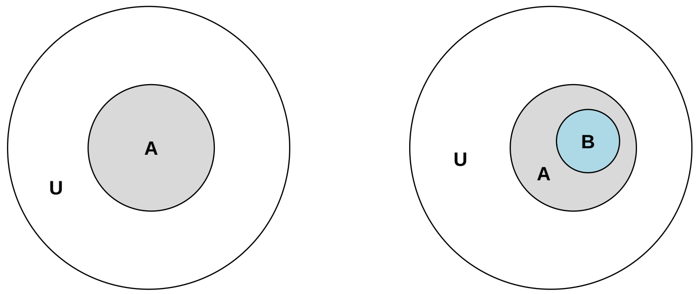

Figure 1: $A \subset U$ (left) and $B \subset A \subset U$ (right). Created with the **eulerr** R package.

5 Unions and Intersections

Definition. The union of two sets A and B contains all of the objects that are in either set. The union is denoted as $C = A \cup B$, where the $C = \{c \mid c \in A \text{ or } c \in B\}$.

Definition. The intersection of two sets A and B contains all of the objects that are in both sets. The intersection is denoted as $C = A \cap B$, where the $C = \{c \mid c \in A \text{ and } c \in B\}$.

***Example* 13.** If $A = \{1, 2, 3\}$ and $B = \{3, 4, 5, 6\}$, then the union is $A \cup B = \{1, 2, 3, 4, 5, 6\}$ and the intersection is $A \cap B = \{3\}$.

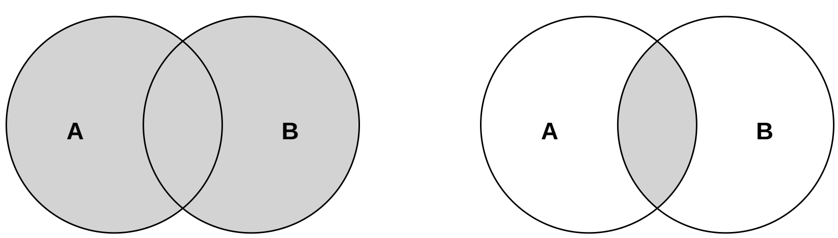

Figure 2: $A \cup B$ (left) and $A \cap B$ (right). Created with the **eulerr** R package.

Definition. The empty set is the set that contains no elements, which is denoted by $\emptyset = \{\}$. Two sets are said to be disjoint if they have no elements in common, i.e., if $A \cap B = \emptyset$.

Note that the empty set is considered to be a subset of all sets, i.e., $\emptyset \subseteq A$. As a result, we have that $\emptyset \cup A = A$ for any set A. Note that we also have that $\emptyset \cap A = \emptyset$ for any set A.

***Example* 14.** If $A = \{\text{cat, dog, fish}\}$ and $B = \{\text{red, white, blue}\}$, then $A \cap B = \emptyset$.

***Example* 15.** If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{1, 4, 7, 8, 9, 10\}$, then $A \cap B \cap C = \emptyset$.

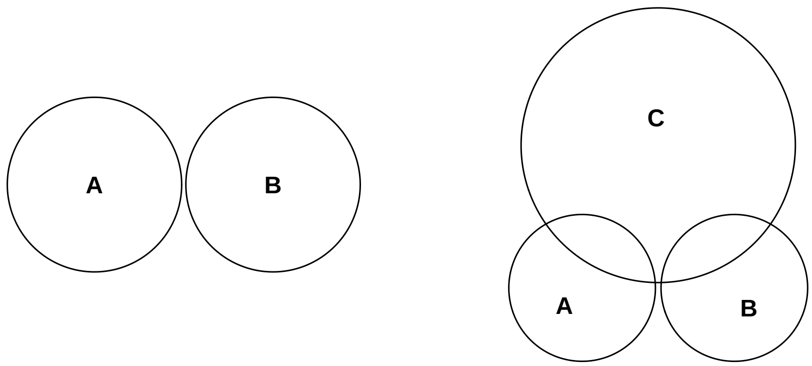

Figure 3: $A \cap B = \emptyset$ (left) and $A \cap B \cap C = \emptyset$ (right). Created with the **eulerr** R package.

Note that the order of operations is important if you are talking about unions and intersections with more than two sets: $(A \cup B) \cap C$ is not necessarily the same set as $A \cup (B \cap C)$.

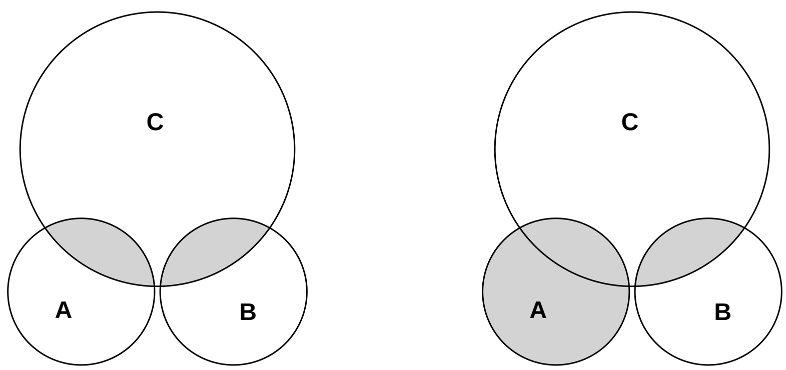

Figure 4: $(A \cup B) \cap C$ (left) and $A \cup (B \cap C)$ (right). Created with the **eulerr** R package.

***Example* 16.** Using the three sets $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, and $C = \{1, 4, 7, 8, 9, 10\}$, we have that $(A \cup B) \cap C = \{1, 4\}$ and $A \cup (B \cap C) = \{1, 2, 3, 4\}$.

6 Complements and Differences

Definition. The complement of a set A, denoted by A^c (or sometimes by $\overline{A}$), consists of all elements that are in the universal set U but not in A, i.e., $A^c = \{a \mid a \in U, a \notin A\}$.

Note that the concept of a complement requires the definition of both the set of interest (i.e., A) and the universal set (i.e., U). In other words, the complement of a set is defined with respect to the universal set.

***Example* 17.** If $A = \{1, 2, 3, 4, 5\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the complement of A is defined as $A^c = \{6, 7, 8, 9, 10\}$.

***Example* 18.** If $A = \{a \mid a = 2n$ where n is a natural number$\}$ is the set of even numbers, and the universal set is the set of natural numbers, then the complement of A is the set of odd numbers, i.e., $A^c = \{a \mid a = 2n - 1$ where n is a natural number$\}$.

Definition. The difference of a set A minus a set B, denoted by $A \backslash B$ (or sometimes by $A - B$), consists of all elements that are in A but not in B, i.e., $A \backslash B = \{a \mid a \in A, a \notin B\}$.

Note that the set difference $(A - B)$ is the intersection of A and the complement of B, i.e., $A \backslash B = A \cap B^c$. Also, note that $A \backslash B$ is not the same as $B \backslash A$ (i.e., non-commutative).

***Example* 19.** If $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$, then $A \backslash B = A$ and $B \backslash A = B$.

***Example* 20.** If $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{4, 5, 6\}$, then $A \backslash B = \{1, 2, 3\}$ and $B \backslash A = \emptyset$.

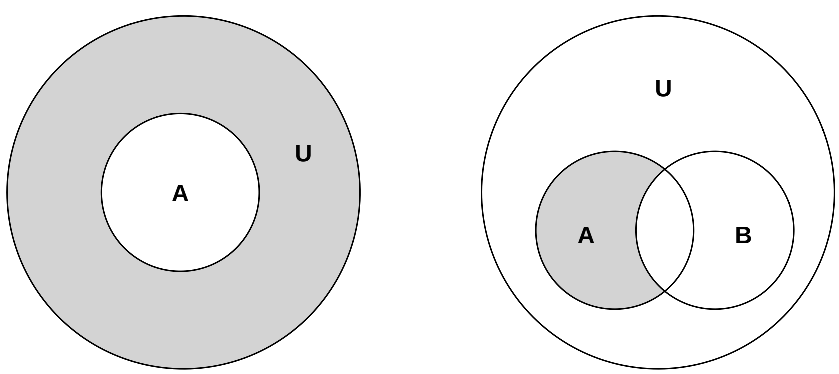

Figure 5: A^c (left) and $A \backslash B$ (right). Created with the **eulerr** R package.

7 Basic Set Properties

Some general rules of set theory:

1. $A \cup \emptyset = A$
2. $A \cap \emptyset = \emptyset$
3. $A \cup U = U$
4. $A \cap U = A$
5. $A \cup A^c = U$
6. $A \cap A^c = \emptyset$
7. $A \cup B = B \cup A$ [commutative law for union]
8. $A \cap B = B \cap A$ [commutative law for intersection]
9. $A \cup (B \cup C) = (A \cup B) \cup C$ [associative law for union]
10. $A \cap (B \cap C) = (A \cap B) \cap C$ [associative law for intersection]
11. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ [distributive law for union]
12. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ [distributive law for intersection]
13. $(A \cup B)^c = A^c \cap B^c$ [De Morgan's Law for union complement]
14. $(A \cap B)^c = A^c \cup B^c$ [De Morgan's Law for intersection complement]
15. $A \backslash B = B \backslash A$ if and only if $A = B$