

# One Sample $t$ Test

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## William Sealy Gosset (1876 - 1937)

William Sealy Gosset was a chemist and statistician who was employed as the Head Experimental Brewer at Guinness in the early 1900s.

Through this work, Gosset noticed that the normal distribution was not adequate for testing hypotheses about the mean in small samples of data with an unknown variance.



**Figure 1:** Picture of William Sealy Gosset and Commemorative plaque for Gosset at Guinness. From [https://en.wikipedia.org/wiki/William\\_Sealy\\_Gosset](https://en.wikipedia.org/wiki/William_Sealy_Gosset)

## Student's Important Discovery

As a reminder, if  $X \sim N(\mu, \sigma^2)$ , then  $\bar{x} \sim N(\mu, \sigma^2/n)$  which implies

$$Z = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Gosset discovered that when the sample variance is used in place of the true population variance, the test statistic

$$T = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

where  $t_\nu$  denotes Student's  $t$  distribution (Student, 1908) with  $\nu$  degrees of freedom.

Gosset published this work under the pseudonym “Student” because Guinness did not allow employees to include their name or affiliation on any scientific publications.

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# Overview of Student's $t$ Distribution

Student's  $t$  distribution is a family of distributions that depend on a single parameter: the degrees of freedom parameter  $\nu$ .

Student's  $t$  distribution arises when we standardize a normal variable using the population mean and the sample estimate of the variance.

Student's  $t$  distribution can be a reasonable approximation to a standardized variable formed from a variety of different population distributions (e.g., see Helwig, 2019).

# Properties of Student's $t$ Distribution

Student's  $t$  distribution has the follow properties:

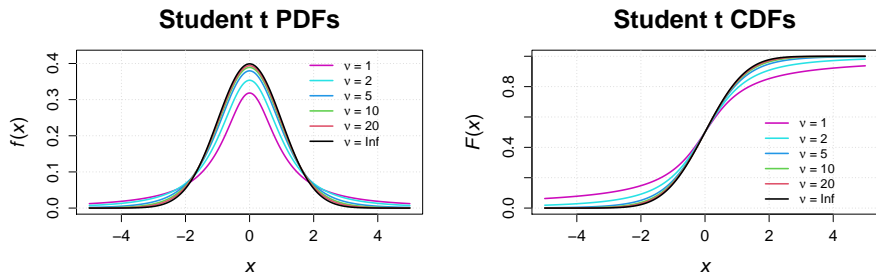
- PDF:  $f(t) = \frac{\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}}{\sqrt{\nu} B\left(\frac{1}{2}, \frac{\nu}{2}\right)}$  where  $B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt$  is the beta function
- CDF:  $F(t) = 1 - \frac{1}{2} I_{\frac{\nu}{t^2 + \nu}}\left(\frac{\nu}{2}, \frac{1}{2}\right)$  where  $I_x(u, v) = \frac{B(x; u, v)}{B(u, v)}$  is the regularized incomplete beta function and  $B(x; u, v) = \int_0^x t^{u-1} (1-t)^{v-1} dt$  is the incomplete beta function
- Mean:  $E(T) = 0$  if  $\nu > 1$ ; otherwise undefined
- Variance:  $\text{Var}(T) = \frac{\nu}{\nu-2}$  if  $\nu > 2$ ,  $\text{Var}(T) = \infty$  if  $1 < \nu \leq 2$ ; otherwise undefined

Relations to other distributions:

- $t_\nu \rightarrow N(0, 1)$  as  $\nu \rightarrow \infty$
- $T = Z \sqrt{\frac{\nu}{V}}$  where  $Z \sim N(0, 1)$  and  $V \sim \chi_\nu^2$  (where  $Z$  and  $V$  are independent)
- $T^2 \sim F(1, \nu)$



# Visualizations of Student's $t$ Distribution



**Figure 2:** Student's  $t$  distribution PDFs and CDFs with various different degrees of freedom.

For small  $\nu$ , the Student's  $t$  PDF has “fatter tails”, which implies that extreme observations are more likely to occur when generating data from a  $t_\nu$  distribution (compared to a  $N(0, 1)$  distribution).

# Degrees of Freedom Detour

In statistics, the degrees of freedom associated with some quantity is the number of values that are free to vary.

Assuming that  $x_i \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ , the vector of sample observations  $(x_1, \dots, x_n)^\top$  has  $n$  degrees of freedom, and note that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \bar{x} \\ \vdots \\ \bar{x} \end{bmatrix} + \begin{bmatrix} x_1 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{bmatrix}$$

where the first vector  $(\bar{x}, \dots, \bar{x})^\top$  has a single degree of freedom (because it is the same number replicated  $n$  times), and the second vector  $(x_1 - \bar{x}, \dots, x_n - \bar{x})^\top$  has  $n - 1$  degrees of freedom (because the deviation scores have one constraint, i.e., that  $\sum_{i=1}^n (x_i - \bar{x}) = 0$ ).

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# Hypothesis Testing Problem

Suppose  $x_1, \dots, x_n$  are an iid sample with  $x_i \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2)$ .

Possible hypotheses to test:

- $H_0 : \mu = \mu_0$  versus  $H_1 : \mu \neq \mu_0$  (exact  $H_0$  with two-sided  $H_1$ )
- $H_0 : \mu \geq \mu_0$  versus  $H_1 : \mu < \mu_0$  (inexact  $H_0$  with less than  $H_1$ )
- $H_0 : \mu \leq \mu_0$  versus  $H_1 : \mu > \mu_0$  (inexact  $H_0$  with greater than  $H_1$ )

where  $\mu_0 \in \mathbb{R}$  is the null hypothesized value of the mean.

In practice, the one-sided alternative hypothesis is often preferred.

# Test Statistic and Null Distribution

As a test statistic, we will use Student's  $t$  statistic

$$T_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where

- $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$  is the sample mean
- $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$  is the sample variance

Assuming that  $H_0$  is true,  $T_0$  follows Student's  $t$  distribution with  $n - 1$  degrees of freedom, i.e.,  $T_0 \sim t_{n-1}$  under  $H_0$ .

## P-Value Calculations

For the directional (one-sided) tests, the p-values are

- $H_1 : \mu < \mu_0$ , the p-value is defined as  $p = P(T < T_0)$
- $H_1 : \mu > \mu_0$ , the p-value is defined as  $p = P(T > T_0)$

where  $T$  is a random variable that follows Student's  $t$  distribution with  $n - 1$  degrees of freedom, i.e.,  $T \sim t_{n-1}$ .

For the two-sided test, the p-value is defined as

$$p = P(T < -|T_0|) + P(T > |T_0|) = 2P(T > |T_0|)$$

where  $T \sim t_{n-1}$ . We can simply double the probability that  $T > |T_0|$  because the  $t_{n-1}$  distribution is symmetric around zero.

## Cereal Example

A store sells “16-ounce” boxes of Captain Crisp cereal. A random sample of 9 boxes was taken and weighed. The results were

15.5, 16.2, 16.1, 15.8, 15.6, 16.0, 15.8, 15.9, 16.2

Suppose that we want to test the null hypothesis  $H_0 : \mu \geq \mu_0$  versus the alternative hypothesis  $H_1 : \mu < \mu_0$ . In this case,  $\mu_0 = 16$ .

The sample mean and variance are

$$\bar{x} = \frac{1}{9}(15.5 + 16.2 + 16.1 + 15.8 + 15.6 + 16.0 + 15.8 + 15.9 + 16.2) = 15.9$$
$$s^2 = \frac{1}{8} \left( \sum_{i=1}^9 x_i^2 - 9\bar{x}^2 \right) = \frac{1}{8} (2275.79 - 9(15.9)^2) = 0.0625$$

## Cereal Example (continued)

The observed  $T$  test statistic and p-value have the form

$$T_0 = \frac{15.9 - 16}{\sqrt{0.0625/9}} = -1.2 \quad \text{and} \quad P(T < -1.2) = 0.1322336$$

where  $T$  follows a  $t$  distribution with 8 degrees of freedom, i.e.,  $T \sim t_8$ .

To confirm this result in R, we can use the `t.test` function:

```
> x <- c(15.5, 16.2, 16.1, 15.8, 15.6, 16.0, 15.8, 15.9, 16.2)
> t.test(x, mu = 16, alternative = "less")
```

One Sample t-test

```
data:  x
t = -1.2, df = 8, p-value = 0.1322
alternative hypothesis: true mean is less than 16
95 percent confidence interval:
 -Inf 16.05496
sample estimates:
mean of x
15.9
```



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# Forming Confidence Intervals

To form a  $100(1 - \alpha)\%$  confidence interval for the mean, note that

$$1 - \alpha = P \left( t_{\alpha/2} < \frac{\bar{x} - \mu}{s/\sqrt{n}} < t_{1-\alpha/2} \right)$$

where  $t_\alpha$  denotes the  $\alpha$ -th quantile of the  $t_{n-1}$  distribution.

Rearranging the terms inside the probability statement reveals that

$$\begin{aligned} 1 - \alpha &= P \left( t_{\alpha/2}s/\sqrt{n} < \bar{x} - \mu < t_{1-\alpha/2}s/\sqrt{n} \right) \\ &= P \left( t_{\alpha/2}s/\sqrt{n} - \bar{x} < -\mu < t_{1-\alpha/2}s/\sqrt{n} - \bar{x} \right) \\ &= P \left( \bar{x} - t_{\alpha/2}s/\sqrt{n} > \mu > \bar{x} - t_{1-\alpha/2}s/\sqrt{n} \right) \end{aligned}$$

## Forming Confidence Intervals (continued)

Implies that the  $100(1 - \alpha)\%$  confidence interval is given by

$$\left[ \bar{x} - t_{1-\alpha/2} \frac{s}{\sqrt{n}}, \quad \bar{x} + t_{\alpha/2} \frac{s}{\sqrt{n}} \right]$$

Given that  $-t_{\alpha/2} = t_{1-\alpha/2}$  the confidence interval can be written as

$$\bar{x} \pm t_{1-\alpha/2} \frac{s}{\sqrt{n}}$$

which is the “normal” confidence interval formula using the Student’s  $t$  distribution critical value  $t_{1-\alpha/2}$  in place of  $z_{1-\alpha/2}$ .

# Forming Confidence Bounds

For directional (one-sided) tests, it would make more sense to use a confidence bound, which places all of the uncertainty in the direction relevant to the alternative:

- $H_1 : \mu < \mu_0$ , use upper confidence bound:  $[-\infty, \bar{x} + t_{1-\alpha}s/\sqrt{n}]$
- $H_1 : \mu > \mu_0$ , use lower confidence bound:  $[\bar{x} - t_{1-\alpha}s/\sqrt{n}, \infty]$

For the previous example, the 95% one-sided upper confidence bound is

$$[-\infty, 16.05496]$$

which was output from the `t.test` function in the previous example.

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# Nonparametric Hypothesis Tests

Suppose that we want to test a null hypothesis about  $\mu$ , but we are not certain that the assumption of normality is reasonable.

Instead of comparing the observed test statistic  $T_0$  to a Student's  $t_{n-1}$  distribution, we could compare the observed  $T_0$  to a null distribution that is obtained in a nonparametric fashion.

Such a test is referred to as a nonparametric randomization test, which is also known as a permutation test (see Helwig, 2019).

- randomly permutes aspects of the observed data under the assumption  $H_0$  is true
- generates a (conditional) null distribution for  $T_0$  without assuming a particular data generating distribution for the data

## Example of Nonparametric Hypothesis Test

Revisiting the previous example, we could use the `np.loc.test` function in the **npctest** R package (Helwig, 2020) to conduct a nonparametric permutation test:

```
> library(npctest)
> np.loc.test(x, mu = 16, alternative = "less")
```

```
Nonparametric Location Test (One Sample t-test)
alternative hypothesis: true mean is less than 16
t = -1.2,  p-value = 0.1602
sample estimate:
mean
15.9
```

This is an “exact” p-value, given that there are only  $2^9 = 512$  elements of the null distribution.

# Nonparametric Confidence Intervals

The `np.boot` function computes a nonparametric confidence interval:

```
> set.seed(0)
> np.boot(x, mean)
```

Nonparametric Bootstrap of Univariate Statistic  
using R = 9999 bootstrap replicates

```
t0: 15.9
SE: 0.0786
Bias: 4e-04
```

BCa Confidence Intervals:

	lower	upper
90%	15.7556	16.0111
95%	15.7333	16.0333
99%	15.6889	16.0778

95% upper bound is given by  $[-\infty, 16.0111]$



# References

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- Student (1908). The probable error of a mean. *Biometrika* 6(1), 1–25.