

# Fuzzy Confidence Intervals

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## Normal Theory Intervals

$X_1, X_2, \dots, X_n$  iid Normal( $\theta, \sigma^2$ )

$$\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \sim \text{Normal}(\theta, 1)$$

Hence a  $(1 - \alpha)\%$  confidence interval for  $\theta$  is

$$\bar{X} \mp z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

When  $\sigma$  is unknown we replace it with the sample standard deviation and use a  $t$  distribution.

## Tests and Confidence Intervals

- For fixed  $\alpha$  and  $\theta_0$ ,

$$\bar{x} \mapsto \phi(\bar{x}, \alpha, \theta_0)$$

is the size  $\alpha$  *test* of  $H_0 : \theta = \theta_0$ .

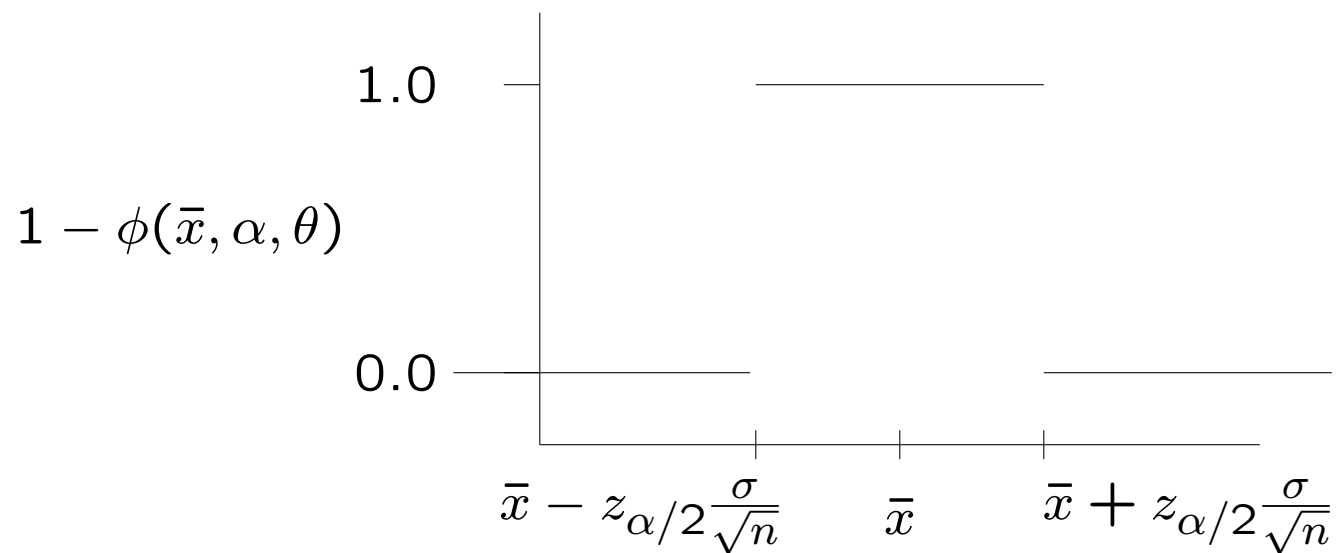
- For fixed  $\bar{x}$  and  $\alpha$ ,

$$\theta \mapsto 1 - \phi(\bar{x}, \alpha, \theta)$$

is (the indicator function of) the *confidence interval* with coverage  $1 - \alpha$ .

## A C.I. as an Indicator Function

For fixed values of  $\bar{x}$  and  $\alpha$  the *Indicator function* of the  $(1 - \alpha)\%$  confidence interval as a function of  $\theta$ .



## Ordinary Confidence Intervals

OK for continuous data, but a **really bad idea** for **discrete** data.

Why?

Coverage Probability

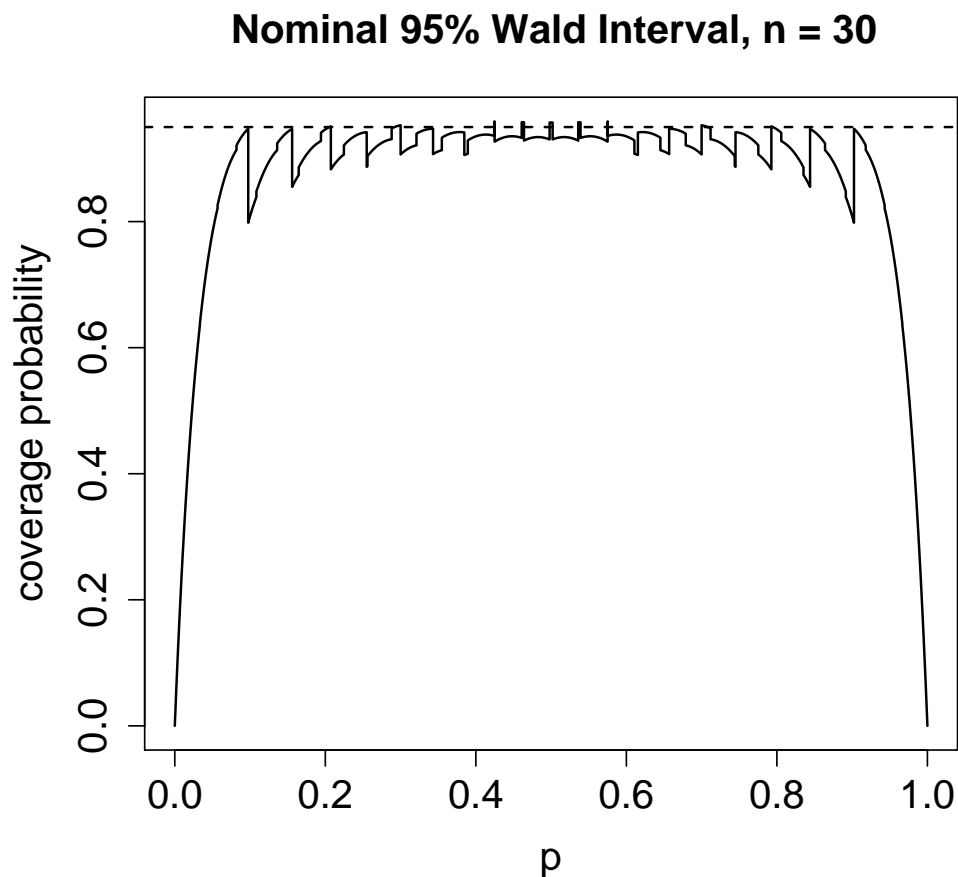
$$\begin{aligned}\gamma(\theta) &= \text{pr}_{\theta}\{l(X) < \theta < u(X)\} \\ &= \sum_{x \in S} I_{(l(x), u(x))}(\theta) \cdot f_{\theta}(x)\end{aligned}$$

As  $\theta$  moves across the boundary of a possible confidence interval  $(l(x), u(x))$ , the coverage probability jumps by  $f_{\theta}(x)$ .

Ideally,  $\gamma$  is a **constant function** equal to the nominal confidence coefficient.

But that's **not possible**.

## Usual Binomial Interval

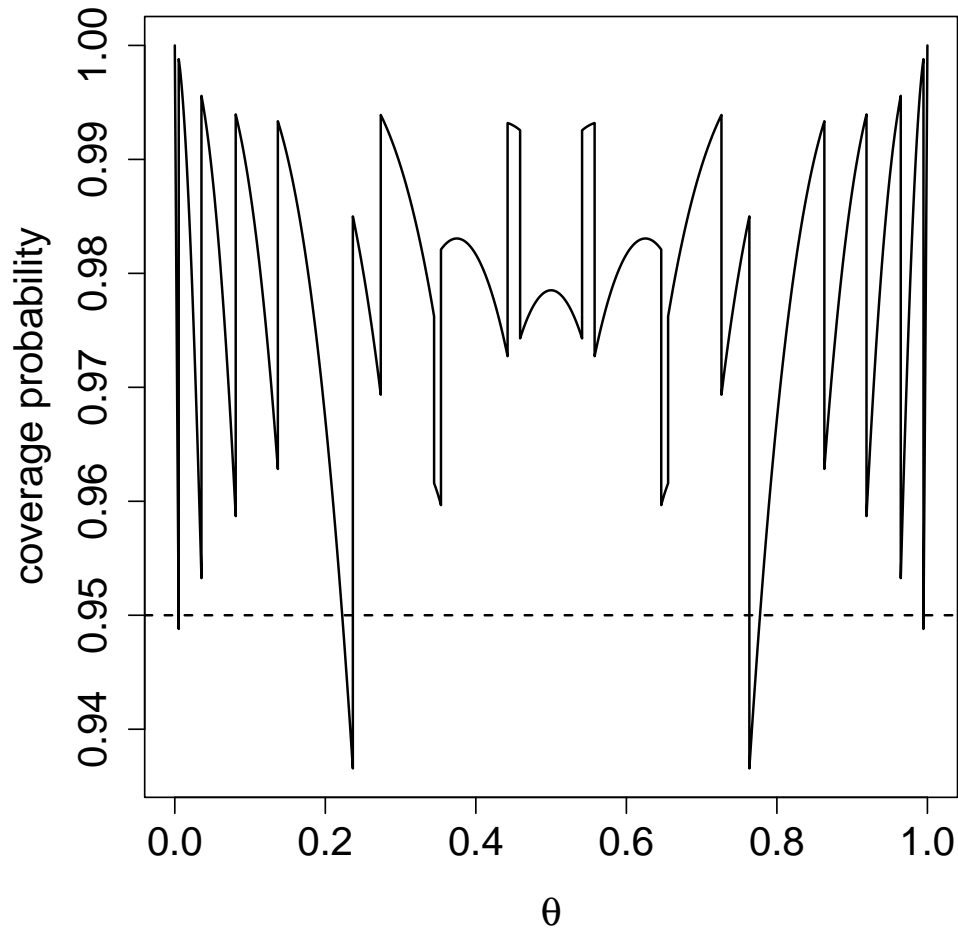


Performance of usual (Wald) interval

$$\hat{p} \pm 1.96 \sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

for Binomial(30,  $p$ ). Dotted line is nominal level (0.95).

## Another Binomial Example



Binomial data, sample size  $n = 10$ , confidence interval calculated by R function `prop.test`

## Recent Literature

Agresti and Coull (*Amer. Statist.*, 1998)

Approximate is better than “exact” for interval estimation of binomial proportions.

Brown, Cai, and DasGupta (*Statist. Sci.*, 2001)

Interval estimation for a binomial proportion (with discussion).

Casella (*Statist. Sci.*, 2001)

Comment on Brown, et al.

All recommend different intervals. All recommended intervals are bad, just slightly less bad than other possibilities.

Ordinary confidence intervals for discrete data are irreparably bad.

Geyer and Meeden (*Statist. Sci.*, 2005)

## Randomized Tests

Randomized test defined by *critical function*  $\phi(x, \alpha, \theta)$ .

- observed data  $x$ .
- significance level  $\alpha$
- null hypothesis  $H_0 : \theta = \theta_0$

Decision is randomized: reject  $H_0$  with probability  $\phi(x, \alpha, \theta_0)$ .

Since probabilities are between zero and one, so is  $\phi(x, \alpha, \theta)$ .

Classical uniformly most powerful (UMP) and UMP unbiased (UMPU) tests are randomized when data are discrete.

## Randomized Test Example

Observe  $X \sim \text{Binomial}(20, \theta)$ . Test

$$H_0 : \theta = 0.5$$

$$H_1 : \theta < 0.5$$

Distribution of  $X$  under  $H_0$ .

$x$	$f(x)$	$F(x)$
4	0.0046	0.0059
5	0.0148	0.0207
6	0.0370	0.0577
7	0.0739	0.1316

Nonrandomized test can have  $\alpha = 0.0207$  or  $\alpha = 0.0577$ , but nothing in between.

Randomized test that rejects with probability one when  $X \leq 5$  and with probability 0.7928 when  $X = 6$  has  $\alpha = 0.05$ .

$$\begin{aligned} & \Pr(X \leq 5) + 0.7928 \cdot \Pr(X = 6) \\ &= 0.0207 + 0.7928 \cdot 0.0370 \\ &= 0.0500 \end{aligned}$$

For fixed  $\alpha$  and  $\theta_0$ ,

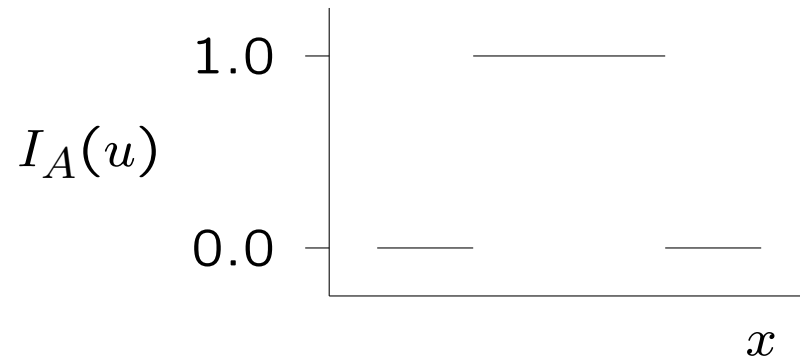
$$x \mapsto \phi(x, \alpha, \theta_0)$$

is the *abstract randomized decision* for the size  $\alpha$  test of  $H_0 : \theta = \theta_0$ .

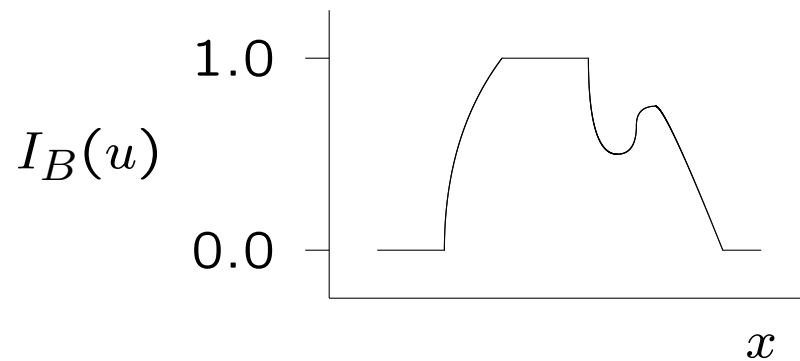
Just report this value, not the final randomized decision.

## Fuzzy Sets

*Indicator function*  $I_A$  of ordinary set  $A$



*Membership function*  $I_B$  of fuzzy set  $B$



The value  $I_B(u)$  is the **degree of membership** of the point  $u$  in the fuzzy set  $B$ .

Ordinary sets are special case of fuzzy sets called ***crisp sets***.

A non-probabilistic measure of uncertainty.

Think **partial credit**!

## Fuzzy Confidence Intervals

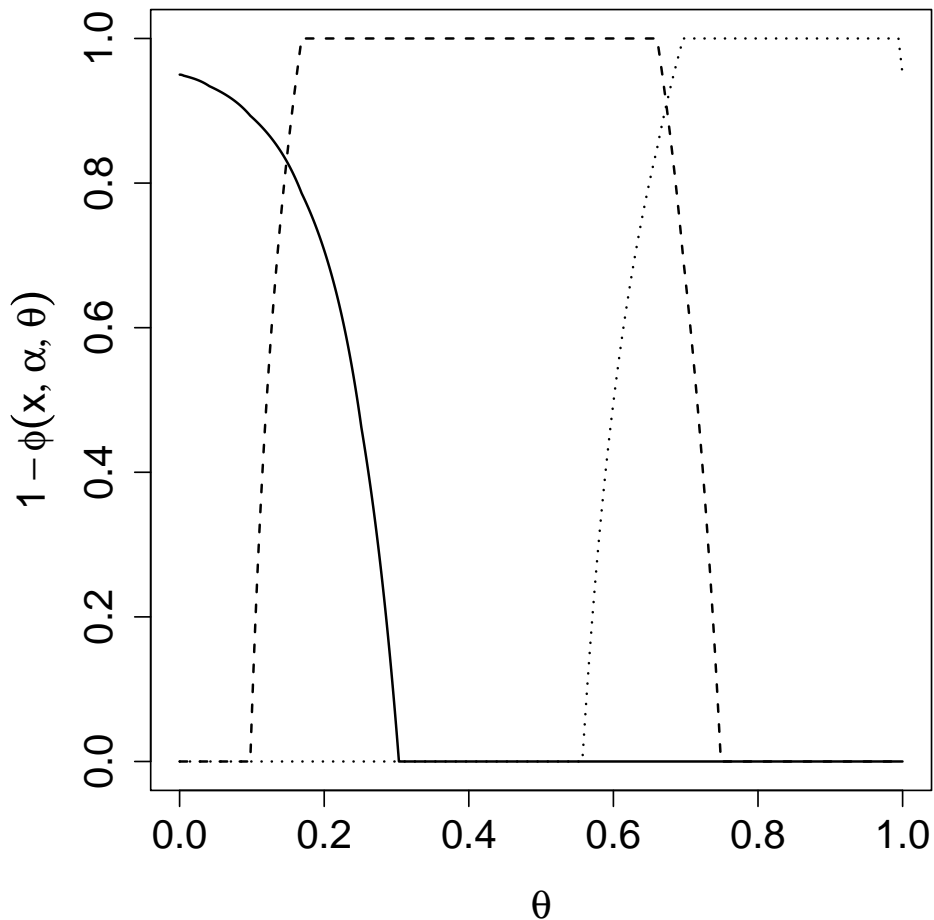
For fixed  $x$  and  $\alpha$ ,

$$\theta \mapsto 1 - \phi(x, \alpha, \theta)$$

is (the membership function of) the *fuzzy confidence interval* with coverage  $1 - \alpha$ .

For the binomial problem we have an R package that computes the above.

## Binomial Example



Sample size  $n = 10$ , fuzzy confidence interval associated with UMPU test, confidence level  $1 - \alpha = 0.95$ .

### Data

$x = 0$  (solid curve)

$x = 4$  (dashed curve)

$x = 9$  (dotted curve)

## Optimal property of fuzzy CI's

Let

$$\begin{aligned}c(\theta, \theta') &= E_{\theta}\{1 - \phi(X, \alpha, \theta')\} \\ &= \text{coverage at } \theta' \text{ when true is } \theta \\ &= \text{average height at } \theta' \text{ when true is } \theta\end{aligned}$$

Subject to

- i)*  $c(\theta, \theta) = 1 - \alpha$  for all  $\theta$  (exactness)
- ii)*  $c(\theta, \theta) \geq c(\theta, \theta')$  for all  $\theta, \theta'$  (unbiasedness)

the UMPU fuzzy interval,  $c^*$  satisfies i) and ii) and for any other fuzzy interval,  $\tilde{c}$ , satisfying i) and ii)

$$c^*(\theta, \theta') \leq \tilde{c}(\theta, \theta') \text{ whenever } \theta' \neq \theta$$

## Fuzzy and Randomized Concepts

### $\gamma$ -Cuts

If  $I_B$  is the membership function of a fuzzy set  $B$ , the  $\gamma$ -cut of  $B$  is the crisp set

$$\gamma I_B = \{x : I_B(x) \geq \gamma\}.$$

For the *fuzzy confidence interval*  $B$  with membership function

$$I_B(\theta) = 1 - \phi(x, \alpha, \theta).$$

A *randomized confidence interval* is the crisp set  $^U I_B$ , where  $U$  is uniform  $(0, 1)$  random variate.

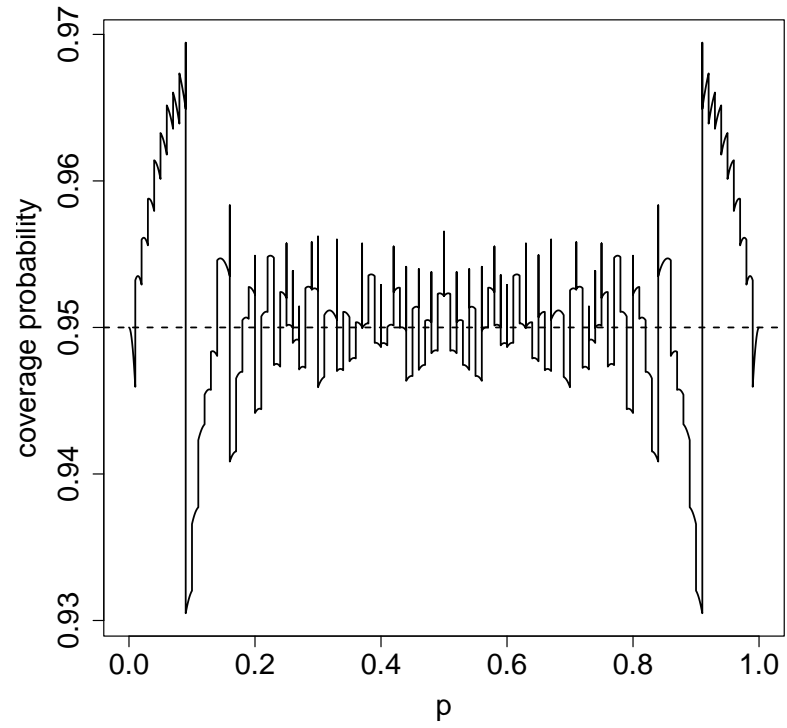
Other ways to randomize.

## An Early Randomized Interval

Blyth-Hutchinson (*Biometrika*, 1960) proposed another randomized interval for Binomial problem.

It would be exact except for rounding error. Randomization variate  $U$  rounded to one significant figure. Interval endpoints rounded to two significant figures.

95% Blythe–Hutchinson Randomized Interval,  $n = 10$



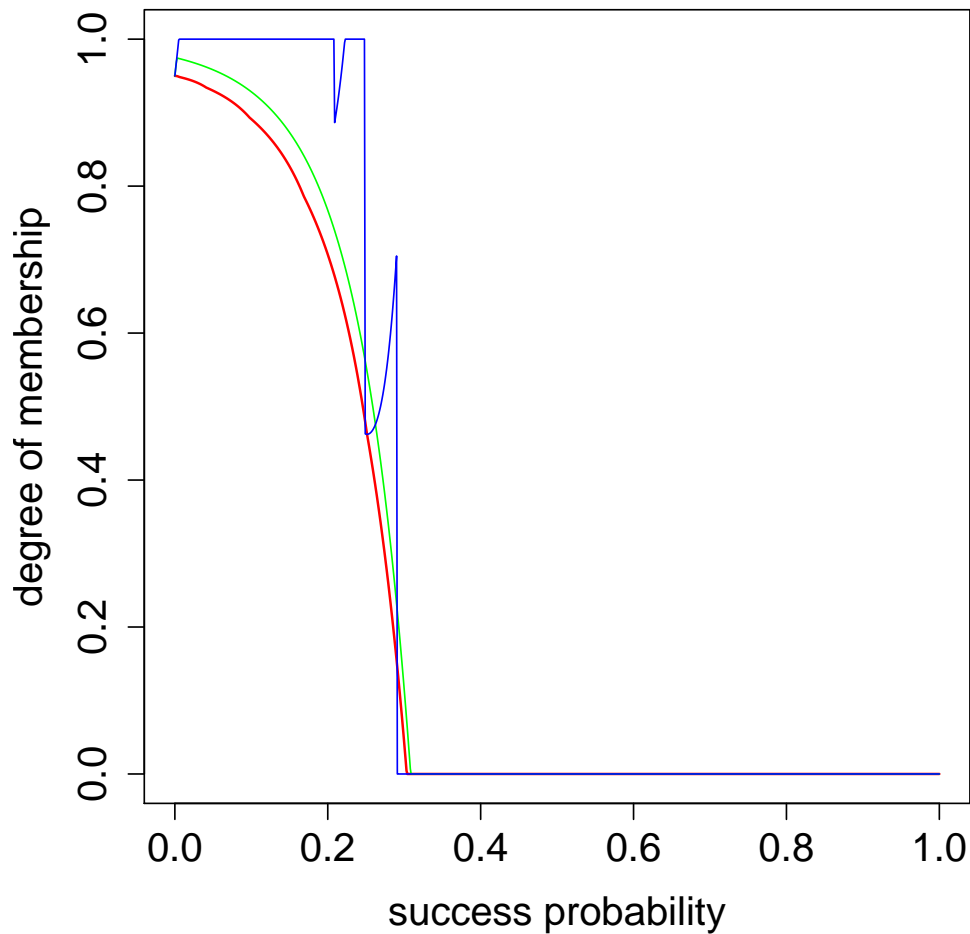
## Pratt (JASA 1961)

Showed that the confidence procedure which minimizes average expected length is the inversion of the MP tests for

$$H_0 : \theta = \theta_0 \text{ versus } H_a : \theta \text{ is unifrom}(0,1)$$

Resulting fuzzy intervals are not unimodal.

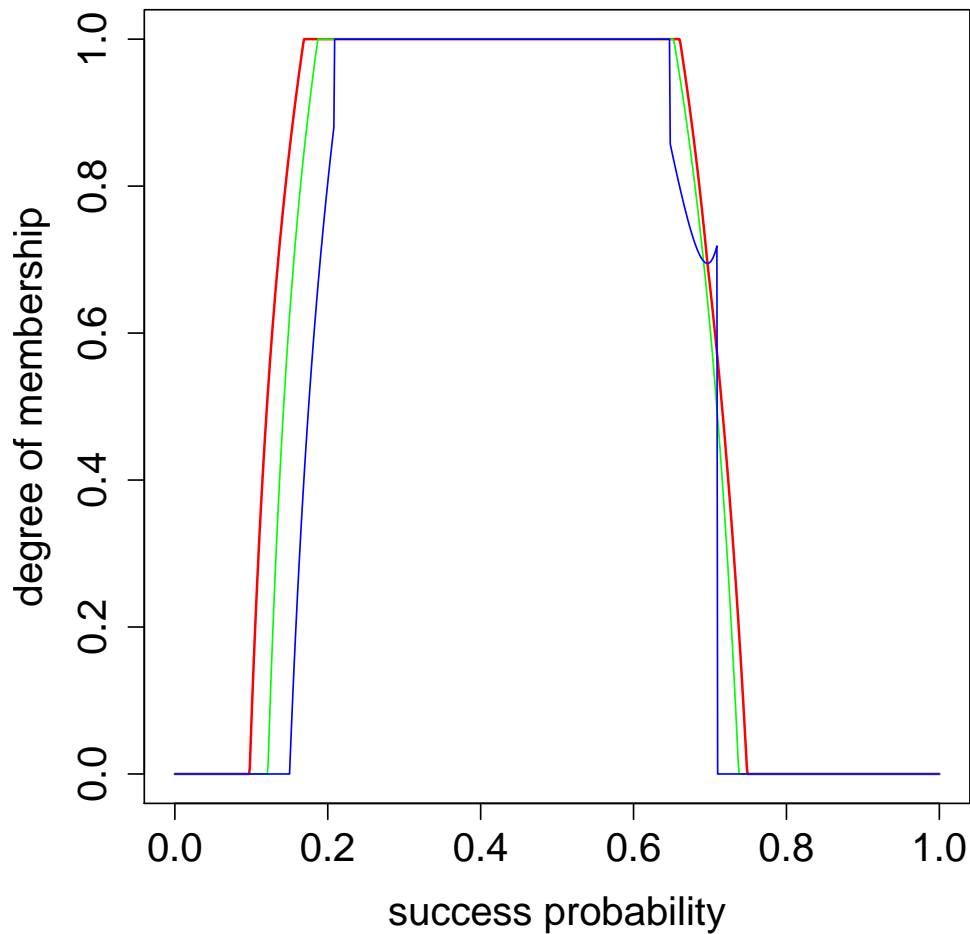
## Three different fuzzy intervals



95% fuzzy intervals for  
 $n = 10$  and  $x = 0$ .

Blue curve (Pratt)  
Green curve (Equal tail)  
Red curve (UMPU)

## Three different fuzzy intervals



95% fuzzy intervals for  
 $n = 10$  and  $x = 4$ .

Blue curve (Pratt)  
Green curve (Equal tail)  
Red curve (UMPU)

## Fuzzy $P$ -values

For fixed  $x$  and  $\theta_0$

$$\alpha \mapsto \phi(x, \alpha, \theta)$$

is the membership function of the *fuzzy  $P$ -value* or the distribution function of the *abstract randomized  $P$ -value* for the test of  $H_0 : \theta = \theta_0$ .

## Situations with UMP and UMPU Tests

- Binomial, Poisson and Negative Binomial
- Two Binomials  $p_1(1 - p_2)/p_2(1 - p_1)$
- Two Poissons  $\mu_1/\mu_2$
- Two Negative Binomials  $(1 - p_2)/(1 - p_1)$
- McNemar  $p_{12}/(p_{12} + p_{21})$
- Fisher  $p_{11}p_{22}/p_{12}p_{21}$

## Summary

- Confidence intervals, based on UMP and UMPU tests, are the **Right Thing** (exact and uniformly most powerful).
- Crisp CI's are the **Wrong Thing** (for discrete data).
- Fuzzy is to hard?
- R code available for binomial problem.
- Fuzzy outside classical UMP and UMPU?
- Fuzzy or Randomized?