An Improved Skewness Measure

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ABSTRACT

Skewness is an important property of a population. Although many measures have been proposed there is no consensus on what should be used in practice. We propose a new measure which is appropriate when the direction of the skewness is known a priori. It is simple to interpret and easy to estimate.

Keywords: Skewness measure; influence function; Bayesian bootstrap

1 Introduction

The skewness of a distribution can play an important role in many situations. Many measures of skewness have been proposed and although they are intuitively plausible they can be difficult to interpret. As a result, there seems to be no widely agreed upon practical measure. In many problems where skewness is of interest, it is known a priori whether the distribution is either skewed to the right or to the left. For such problems we define a new family of skewness measures and identify one which is easy to interpret, and for which it is easy to find good point and interval estimates. For definiteness we will assume that it is known that the distribution of interest is skewed to the right.

Let X be a random variable with a distribution function F which has a strictly positive differentiable density function on an interval $I(a,b)$ where a and b may be $-\infty$ or $+\infty$. In Groeneveld and Meeden (1984) a class of skewness measures given by

$$\gamma_p(F) = \frac{F^{-1}(1-p) + F^{-1}(p) - 2m_x}{F^{-1}(1-p) - F^{-1}(p)}, \quad 0 < p < 1/2, \tag{1}$$

where m_x is the median of X, was discussed.

Let G be another distribution function which has a strictly positive differentiable density function on the interval $I(a,b)$. van Zwet (1964) in considering the idea of ordering two distributions with regard to skewness, called G at least as skewed to the right as F if $R(x) = G^{-1}(F(x))$ was convex on $I(a,b)$. In this case $R''(x) \geq 0$ on $I(a,b)$, one writes $F <_c G$ and says "F c-precedes G". It was shown in Groeneveld and Meeden (1984) that γ_p satisfies the following conditions:

$$\gamma_p(cX+d) = \gamma_p(X) \quad \text{for constants } c>0 \text{ and } d, \tag{2}$$

$$\gamma_p(F) = 0 \quad \text{for symmetric } F, \tag{3}$$

$$\gamma_p(-F) = -\gamma_p(F), \tag{4}$$

$$\text{if } F <_c G \text{ then } \gamma_p(F) \leq \gamma_p(G). \tag{5}$$

Oja (1981) called these four conditions appropriate properties for a skewness measure.

In this note we consider replacing the denominator of γ_p by $m_x - F^{-1}(p)$, obtaining

$$\lambda_p(F) = \frac{F^{-1}(1-p) + F^{-1}(p) - 2m_x}{m_x - F^{-1}(p)}, \tag{6}$$

a new skewness measure for distributions which are skewed to the right. We shall see that for such distributions this new measure is more sensitive to right skewness than the standard measure. This new measure was suggested by a similar substitution in the denominator of a family of measures of kurtosis made in Kotz and Seier (2009).

In section 2 we consider some of the basic properties of λ_p In section 3 we recall the methods of Ruppert (1987) and use influence functions to compare the sensitivity to *right* skewness of λ_p with that of γ_p, and conclude that λ_p is more sensitive to right skewness than γ_p. In section 4 we consider four common families of distributions and compare the two measures. In section 5 we discuss how to estimate the measures given a random sample from an unknown distribution. We conclude with some final remarks in section 6.

2 Properties of λ_p

It is easy to see that λ_p may be rewritten as

$$\lambda_p(F) = \frac{F^{-1}(1-p) - m_x}{m_x - F^{-1}(p)} - 1. \tag{7}$$

In this expression it is straightforward to verify condition (2) for λ_p since for a $c > 0$ each term in the fraction is multiplied by c and hence it cancels. Similarly the addition of d to X will cancel in both the numerator and denominator. For symmetric X the distances $F^{-1}(1-p) - m_x$ and $m_x - F^{-1}(p)$ will be equal and $\lambda_p = 0$ proving condition (3).

Next we consider condition (5). If $F <_c G$ then $\lambda_p(F) \leq \lambda_p(G)$ or

$$\frac{F^{-1}(1-p) - m_x}{m_x - F^{-1}(p)} \leq \frac{G^{-1}(1-p) - m_y}{m_y - G^{-1}(p)} \quad \text{for} \quad 0 < p < 1/2.$$

Because translations do not affect λ_p, m_x and m_y may be taken to be 0 and subject to $G^{-1}(F(x)) = R(x)$ being convex ($R''(x) \geq 0$), one must demonstrate

$$F^{-1}(p)/G^{-1}(p) \geq F^{-1}(1-p)/G^{-1}(1-p) \quad \text{for} \quad 0 < p < 1/2. \tag{8}$$

As shown in Groeneveld and Meeden (1984) if $R''(x) \geq 0$ then $R(x)/x$ is nondecreasing where we define at $x = 0$

$$\lim_{x\to 0} \frac{R(x)}{x} = \lim_{x\to 0} \frac{G^{-1}(F(x))}{x} = \frac{f(0)}{g(0)} > 0.$$

But then

$$\frac{G^{-1}(F(F^{-1}(x)))}{F^{-1}(x)} = \frac{G^{-1}(x)}{F^{-1}(x)}$$

is nondecreasing in x and $F^{-1}(x)/G^{-1}(x)$ is nonincreasing in x and as $p < 1/2 < 1-p$; equation (8) follows directly.

Writing $Q_1 = F^{-1}(p)$, $Q_2 = m_x$ and $Q_3 = F^{-1}(1-p)$, it is easy to check that

$$\lambda_p = K(p)\gamma_p \quad \text{where} \quad K(p) = \frac{Q_3 - Q_1}{Q_2 - Q_1}. \tag{9}$$

Note that $K(p) \geq 1$ and it will be large if the random variable X has a long tail to the right and $Q_2 - Q_1$ is small, suggesting a "hump" of probability immediately to the left of the median. Taking $Q_2 = 0$ without loss of generality $K(p) = -Q_3/Q_1 + 1$. If X is replaced by $-X$, $K(p) = -Q_1/Q_3 + 1$ and we see that if X is not symmetric; then $\lambda_p(-X) \neq -\lambda_p(X)$, for at least one p and so λ_p does not satisfy condition (4) because $\gamma_p(-X) \neq -\gamma_p(X)$. It will be shown in the next section that generally λ_p is more sensitive to *right* skewness than γ_p.

3 Influence functions for γ_p and λ_p

An influence function of a functional T, which was introduced by Hampel (1974), is a "directional derivative" of the form

$$T(F;G) = \lim_{\epsilon \downarrow 0} \frac{T(F_\epsilon) - T(F)}{\epsilon}, \tag{10}$$

where $F_\epsilon = (1-\epsilon)F + \epsilon G$. $T(F;G)$ is the "derivative" of T at F in the direction of G. If $G = \delta_x$, where δ_x is a point mass at x, F_ϵ is F contaminated by a proportion ϵ of contamination at x. Groeneveld (1991) showed that by using $G = \delta_x$ and $T = \gamma_p$ the resulting influence function was sensitive to the skewness of F. The function $T(F;\delta_x)$ is called the IF of T at x, i. e.

$$IF(x;F,T) = T(F;\delta_x).$$

He then found, using the notation above:

$$\begin{aligned}(Q_3 - Q_1)^2 IF(x;F,\gamma_p) &= 2(Q_3 - Q_2)\left\{\frac{p - I(x < Q_1)}{f(Q_1)} - \frac{1/2 - I(x < Q_2)}{f(Q_2)}\right\} \\ &\quad + 2(Q_2 - Q_1)\left\{\frac{-[1/2 - I(x < Q_2)]}{f(Q_2)}\right. \\ &\quad \left. + \frac{(1-p) - I(x < Q_3)}{f(Q_3)}\right\} \\ &= \frac{2(Q_2 - Q_1)}{f(Q_3)}\left[(1-p) - I(x < Q_3)\right] + \frac{2(Q_3 - Q_2)}{f(Q_1)}\left[\right. \\ &\quad \left. p - I(x < Q_1)\right] - \frac{2(Q_3 - Q_1)}{f(Q_2)}\left[1/2 - I(x < Q_2)\right].\end{aligned} \tag{11}$$

Here $I(\cdot)$ is the indicator function. It can readily be shown that

$$\lambda_p = \frac{2\gamma_p}{1 - \gamma_p}.$$

For the ratio of two functionals of F at x, say N/D, we can use the following result to calculate $IF(x;F,\lambda_p)$:

$$IF(N/D) = \frac{D[IF(x;F,N)] - N[IF(x;F,D)]}{D^2}.$$

Hence,

$$\begin{aligned} IF(x;F,\lambda_p) &= \frac{2(1-\gamma_p)IF(x;F,\gamma_p) - 2\gamma_p IF(x;F,1-\gamma_p)}{(1-\gamma_p)^2} \\ &= \frac{2IF(x;F,\gamma_p)}{(1-\gamma_p)^2}. \end{aligned} \tag{12}$$

One may also show that

$$\begin{aligned} IF(x;F,\lambda_p) &= \frac{1}{2}\{\frac{Q_3-Q_1}{Q_2-Q_1}\}^2 IF(x;F,\gamma_p) \\ &= \frac{K(p)^2}{2} IF(x;F,\gamma_p). \end{aligned} \tag{13}$$

The expression on the right side of equation (11) for a fixed value of p is just a step function in x with jumps at Q_1, Q_2 and Q_3. It increases at Q_1, decreases at Q_2 and increases again at Q_3. Hence from equations (11) and (12) we see that $IF(x;F,\gamma_p)$ and $IF(x;F,\lambda_p)$ exhibit the same pattern, but not the same magnitude of jumps. This is significant as neither influence function will respond any more to an extreme right observation than an observation slightly to the right of Q_3.

Recall γ_p ranges from -1 to 1 and for distributions skewed to the right, we would expect it to be positive for most values of p. We see from equation (12) that in such cases $IF(x;F,\lambda_p) > IF(x;F,\gamma_p)$, which suggests that the IF function for λ_p is more sensitive than the IF function for γ_p for distributions skewed to the right.

Similarly, note that in light of equation (13) that $IF(x;F,\lambda_p) > IF(x;F,\gamma_p)$ if $K(p) > \sqrt{2}$. But we see from equation (9) that for most distributions which appear skewed to the right we would expect $K(p) > 2$. This again suggests λ_p should be more sensitive to right skewness than γ_p.

4 Examples

We consider four families of distributions all of which are skewed to the right, as we are focusing on distributions for which the direction of skewness is known a priori. These families are a subclass of the beta family, the Pareto family, the gamma family and a subclass of the Weibull distributions.

For $b > 0$ we consider the family of densities of the form $f_b(x) = (b+1)(1-x)^b$ for $0 \le x \le 1$. This is a subset of the beta densities which are skewed to the right, and decrease from $(0, b+1)$ to $(1, 0)$. The corresponding distribution functions are $F_b(x) = 1-(1-x)^{b+1}$, $0 \leq x \leq 1$. The inverses of the distribution functions are $F_b^{-1}(y) = 1-(1-y)^{1/(b+1)}$, $0 \leq y \leq 1$. From equation (1) one finds

$$\gamma_p = \frac{2^{b/(b+1)} - p^{1/(b+1)} - (1-p)^{1/(b+1)}}{(1-p)^{1/(b+1)} - p^{1/(b+1)}}. \tag{14}$$

It is easy to check that if $0 < b_1 < b_2$ then $F_{b_1} <_c F_{b_2}$. That is, right skewness increases as b increases. In Table 1 we report values of γ_p and λ_p, for various values of b and for $p = 0.05$, 0.10 and 0.15. To get λ_p we just need to change the denominator of γ_p.

For $a > 0$ the Pareto distributions have density functions of the form $f_a(x) = a/x^{a+1}$ for $x \geq 1$. These densities are skewed to the right and decrease from $(1, a)$ for $x \geq 1$. The corresponding distribution functions are $F_a(x) = 1 - x^{-a}$ for $x \geq 1$ and the inverses of the distribution functions are $F_a^{-1}(y) = (1-y)^{-1/a}$, $0 \leq y \leq 1$. Using equation (1) one finds

$$\gamma_p = \frac{p^{-1/a} + (1-p)^{-1/a} - 2^{1+1/a}}{p^{-1/a} - (1-p)^{-1/a}}. \tag{15}$$

As before, to get λ_p we change the denominator of γ_p.

It is easy to check that right skewness decreases as a increases. In Table 2 we show the values of γ_p and λ_p for various values of b and for $p = 0.05$, 0.10 and 0.15.

Next we consider the gamma family of distributions with shape parameter $\alpha > 0$ and scale parameter β. Since both γ_p and λ_p do not depend on β we assume that its value is one. For these distributions it is well known that the skewness is to the right. For $\alpha > 1$ the gamma densities are unimodal on $0 \leq x < \infty$. For $0 < \alpha < 1$ the densities decrease from ∞ to 0 on $0 < x < \infty$. For$\alpha = 1$ the distribution is the standard exponential distribution. It has been shown by van Zwet (1964) that the skewness decreases as α increases. For values of $\alpha \neq 1$ there are no closed form expressions for the inverses of its distribution functions and hence no closed form expressions for γ_p and λ_p exist. However many statistical packages can find their quantiles, so in practice this is not a problem. In Table 3, for various choices of α, and for $p = 0.05$, 0.10 and 0.15 we find the values of γ_p and λ_p.

Consider the Weibull distribution with shape parameter $c > 0$ and scale parameter equal to one. Its density is of the form $f_c(x) = cx^{c-1}\exp(-x^c)$ for $x \geq 0$. For $c > 1$ the Weibull densities are unimodal on $0 \leq x < \infty$. For $0 < c < 1$ the densities decrease from ∞ to 0 on $0 < x < \infty$. For $c = 1$ the distribution is the standard exponential distribution, as in the gamma case for $\alpha = 1$. The corresponding distribution functions are $F_c(x) = 1 - \exp(-x^c)$ for $x \geq 0$. Thus the inverses of the distribution functions are given by $F_c^{-1}(y) = (-\ln(1-y))^{1/c}$, $0 \leq y \leq 1$. Hence

$$\gamma_p = \frac{(-\ln(1-p))^{1/c} + (-\ln(p))^{1/c} - 2(\ln(2))^{1/c}}{(-\ln(p))^{1/c} - (-\ln(1-p))^{1/c}}. \tag{16}$$

It was shown in Groeneveld (1986) that if $c_1 > c_2 > 0$ then $F_{c_1} <_c F_{c_2}$ and so skewness decreases as c increases. It is also shown there that the Weibull distribution changes from right to left skewness for c approximately equal to 3.5 (depending on which skewness measure is used). For values of $c \in [3, 3.5]$ the distributions are close to being symmetric. In Table 4, for various choices of c, and for $p = 0.05$, 0.10 and 0.15 we find the values of γ_p and λ_p.

We see from Tables 1-4 that in all cases $K(p) > \sqrt{2}$. Recall that in the light of equation (13) that this implies $IF(x; F, \lambda_p) > IF(x; F, \gamma_p)$ for the distributions covered in the these tables. Also note from these tables, that for all the distributions considered there, that $K(p)$ and $K(p)^2/2$ increase as p decreases from 0.15 to 0.05. This suggests that $IF(x; F, \lambda_p)$ is increasingly more sensitive to right skewness than $IF(F; , x, \gamma_p)$ as p decreases. Also, if $K(p) \geq \sqrt{8} = 2.828$, $IF(F; x, \lambda_p)/IF(x; F, \gamma_p) \geq 4$, so the IF for λ_p is at least four times as sensitive to right skewness as the IF for γ_p. For $p = 0.05$ this is true for all but the two least skewed distributions considered in Table 1 and for all those in Table 2, and for the four most right skewed gamma and Weibull distributions in Tables 3 and 4."

We believe that one advantage of λ_p is that it is easy to interpret. This is clear from the expression in equation (7). For values of p not too much larger than zero its meaning is very clear. Clearly, using a value of p that is too far from zero is not a good idea, since such a choice ignores too much of the tail behavior of the distribution.

For the reasons given above we recommend using λ_p with $p = 0.05$. This choice will include most of the distribution while at the same time giving protection against extreme behavior in the tails of the distribution. We believe that the results in the tables indicate that this is a sensible choice. In particular it will be difficult to estimate λ_p from a small sample when p is very close to zero.

5 Estimation of λ_p

So far we have just considered $\lambda_{0.05}$ as a population parameter. But to be useful in practice one must be able to estimate it given a sample from an underlying distribution. One natural point estimate is just to "plug in" the appropriate sample quantities. Then one can generate bootstrap samples to get an approximate confidence interval. We will compare this to a slightly different approach based on the Bayesian bootstrap (Bbst) of Rubin (1981).

Let $y = (y_1, \ldots, y_n)$ be the data from a random sample of size n drawn from some continuous distribution with distribution function F. Let $\theta = (\theta_1, \ldots, \theta_n)$ be a probability distribution defined on y. The usual bootstrap is a way of simulating different values of θ. Let $\tilde{y} = (\tilde{y}_1, \ldots, \tilde{y}_n)$ be a bootstrap resample drawn from y. For $i = 1, \ldots, n$ let $\tilde{\theta}_i$ be the number of $\tilde{y}_j$'s which take on the value y_i. Then $\tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_n)$ is one simulated value for θ.

Rather than resampling from the observed sample to get different values for θ the Bbst draws values of θ from the uniform distribution over the $n-1$ dimensional simplex. Given the sample it can be interpreted as an objective posterior distribution for the unknown F, where it is assumed that F concentrates all its mass on the observed y. Clearly, it is based on some peculiar assumptions but Rubin showed that, asymptotically, it is equivalent to the usual bootstrap.

Since the Bbst is a "posterior" distribution for the the unknown F it generates a posterior distribution for any population parameter which is a function of F. For example, for $0 < q < 1$ let $q = q(F)$ denote the qth quantile of F. The 'posterior' expectation of q under the Bbst can be thought of as an objective Bayesian estimate of the qth quantile of the unknown continuous distribution that generated the sample. This is just the Bbst applied to the problem of estimating a quantile. For estimating a quantile it was shown in Meeden (1993) that for many problems the Bbst estimator of a quantile was better than the usual "plug in" estimator in the sense that it had smaller average absolute error. He considered six different populations, three of which were the gamma distribution with $\alpha = 1$, the gamma distribution with $\alpha = 20$ and a lognormal distribution. These four are all distributions which are skewed to the right. For sample sizes of 25 and 50 and for estimating various population quantiles the average absolute error of the Bbst was about 5% smaller than the average absolute error of the "plug in" estimators.

This suggests that for estimating γ_p and λ_p we can use their 'posterior' expectations under the Bbst as point estimators. We will now describe a simulation study which compared the two methods.

To find a Bbst estimator approximately, R independent observations are taken from the uniform distribution over the simplex. For each observation we compute γ_p, say, and take the average of all these values as our point estimate. To get an approximate 0.95 credible interval we just find the 0.025 and 0.975 quantiles of the set of computed values. In many standard computing packages this is easy to do.

To compare the two methods we consider three different Weibull distributions. In each case the scale parameter is 1 and the three shape parameters are 1, 2 and 3. We used three different choices for p, 0.05, 0.10 and 0.15. For each population we took 400 random samples of size 50. For each sample we simulated $R = 1,500$ values of γ_p and λ_p. We compared this to the usual frequentist plug-in approach and an interval estimate based on 1,500 bootstrap samples. The results when the shape parameter equals 1 are given in Table 5. We see in all three cases of estimating λ_p that the Bbst point estimator does slightly better than the usual "plug in estimator"in terms of average absolute error. This is true despite the fact that the Bbst estimator appears to be slightly more biased than the " plug in" estimator. It is not surprising that both estimation methods can be biased since both skewness measures involve ratios of other population quantities. Note that the error in estimating γ_p, will in general be less than in estimating λ_p as $|\gamma_p| \leq 1$, while this is not the case for λ_p. For $p = 0.05$ the Bbst 0.95 credible intervals tend to over-cover a bit, but they are less conservative and shorter than the the usual bootstrap intervals. For larger values of p both methods give intervals that are too long and over-cover.

We repeated the simulations for a sample size of 100 and the results were similar. We then repeated the simulations for a sample size of 50 for all the values of c in Table 4 for $p = 0.05$. The Bbst estimator was usually on the average between 5% and 10% closer to the truth and its intervals were 5% to 10% shorter even though they did tend to over-cover a bit.

6 Final remarks

Although the Bayesian bootstrap is not Bayes it has a stepwise Bayes interpretation. This fact can be used to prove the admissibility of many standard nonparametric estimators. More details can be found in Meeden et al. (1985). It is well-documented that for large sample sizes Bayesian bootstrap 0.95 credible intervals for estimating a mean or a quantile behave like standard frequentist 95% confidence intervals. In the context of finite population sampling this was demonstrated in Nelson and Meeden (2006).

For a measure of skewness for distributions which are skewed to the left the denominator in γ_p may be replaced by $F^{-1}(1-p) - m_x$. Let η_p be γ_p with this replaced denominator. Then we have $\eta_p = L(p)\gamma_p$ where

$$L(p) = 1 + (Q_2 - Q_1)/(Q_3 - Q_2). \tag{17}$$

Using the methods of Sections 2 and 3 it is straightforward to show that η_p satisfies (2), (3) and (5). However, it will satisfy (4) if and only if F is symmetric. Also one finds

$$IF(x; F, \eta_p) = \frac{L(p)^2}{2} F(x; F, \gamma_p).$$

Note from (17) that $L(p)$ will be large if F has a long tail to the left and a "hump" of probability immediately to the right of m_x.

In many, if not most, cases involving skewed populations we will know a priori whether we are dealing with right or left skewness. For such problems requiring a skewness measure to satisfy the condition in equation (4) seems too strong. We have seen that dropping this condition admits more sensitive measures of skewness.

For right skewness we believe that $\lambda_{0.05}$ is a good practical measure of skewness which calibrates nicely with most users intuitive notions. The choice of $p = 0.05$ captures most of the distribution but still gives some robust protection against a few outliers. It is easy to interpret and the Bayesian bootstrap is an easy method to find point and interval estimators with good frequentist properties.

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Table 1: The values of γ_p and λ_p for some members of the beta family

	$p=0.05$			$p=0.10$			$p=0.15$		
b	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$
0.5	0.19	0.47	2.47	0.16	0.37	2.37	0.13	0.30	2.30
1.0	0.29	0.81	2.81	0.24	0.62	2.62	0.20	0.49	2.49
2.0	0.38	1.25	3.25	0.31	0.92	2.92	0.26	0.71	2.71
3.0	0.43	1.51	3.51	0.35	1.09	3.09	0.29	0.83	2.83
4.0	0.46	1.69	3.69	0.38	1.21	3.21	0.31	0.91	2.91
5.0	0.48	1.82	3.82	0.39	1.29	3.29	0.33	0.97	2.97
10.0	0.52	2.14	4.14	0.43	1.48	3.48	0.35	1.10	3.10
50.0	0.55	2.49	4.35	0.46	1.68	3.68	0.38	1.23	3.18
99.0	0.56	2.54	4.54	0.46	1.71	3.71	0.38	1.25	3.25

Table 2: The values of γ_p and λ_p for some members of the Pareto family

	$p=0.05$			$p=0.10$			$p=0.15$		
a	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$
0.5	0.99	135.93	137.93	0.94	33.71	35.71	0.88	14.46	16.46
1.0	0.90	18.00	20.00	0.80	8.00	10.00	0.70	4.67	6.67
2.0	0.77	6.88	8.88	0.66	3.85	5.85	0.56	2.54	4.54
3.0	0.71	4.99	6.99	0.60	2.99	4.99	0.51	2.05	4.05
4.0	0.68	4.25	6.25	0.57	2.62	4.62	0.48	1.83	3.83
5.0	0.66	3.85	5.85	0.55	2.42	4.42	0.46	1.70	3.70
15.0	0.60	2.96	4.96	0.51	2.06	4.06	0.41	1.40	3.40
50.0	0.57	2.69	4.69	0.47	1.80	3.80	0.40	1.31	3.51
99.0	0.57	2.64	4.64	0.47	1.77	3.77	0.39	1.29	3.29

Table 3: The values of γ_p and λ_p for some members of the gamma family

	$p = 0.05$			$p = 0.10$			$p = 0.15$		
α	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$
0.5	0.76	6.51	8.51	0.67	4.12	6.12	0.59	2.86	4.86
1.0	0.56	2.59	4.59	0.46	1.74	3.74	0.39	1.27	3.27
2.0	0.40	1.32	3.32	0.32	0.93	2.93	0.26	0.70	2.70
3.0	0.32	0.95	2.95	0.26	0.68	2.68	0.21	0.53	2.53
4.0	0.28	0.77	2.77	0.22	0.56	2.56	0.18	0.43	2.43
5.0	0.25	0.66	2.66	0.19	0.48	2.48	0.16	0.38	2.38
6.0	0.23	0.58	2.58	0.18	0.43	2.43	0.14	0.34	2.34
12.0	0.16	0.38	2.38	0.12	0.28	2.28	0.10	0.22	2.22
20.0	0.12	0.28	2.28	0.10	0.21	2.21	0.08	0.17	2.17

Table 4: The values of γ_p and λ_p for some members of the Weibull family

	$p = 0.05$			$p = 0.10$			$p = 0.15$		
c	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$	γ_p	λ_p	$K(p)$
0.7	0.76	6.27	8.27	0.66	3.89	5.89	0.57	2.68	4.68
1.0	0.56	2.59	4.59	0.46	1.74	3.74	0.39	1.27	3.27
1.3	0.41	1.41	3.41	0.33	0.98	2.98	0.27	0.74	2.74
1.6	0.30	0.86	2.86	0.24	0.62	2.62	0.19	0.47	2.47
1.9	0.22	0.56	2.56	0.17	0.40	2.40	0.13	0.31	2.31
2.2	0.15	0.36	2.36	0.12	0.26	2.26	0.09	0.20	2.20
2.5	0.10	0.23	2.23	0.08	0.16	2.16	0.06	0.13	2.13
2.8	0.06	0.13	2.13	0.04	0.09	2.09	0.03	0.07	2.07
3.1	0.03	0.06	2.06	0.02	0.04	2.04	0.01	0.03	2.03

Table 5: Simulation results from 400 random samples of size 50 from a Weibull distribution with shape parameter 1 and scale parameter 1. For each sample we found the usual "plug in" estimators of the skewness measures and the corresponding 95% bootstrap confidence intervals. We also found the Bayesian bootstrap point estimators and the 0.95 Bayesian credible intervals. For each point estimator we calculated its average absolute error and for each interval estimator we found its average lower bound and length. We also checked to see if it contained the true value. This was done for three choices of p equal to 0.05, 0.10 and 0.15.

True value	Method	Ave. value	Ave. error	Ave. lowbd	Ave. len	Freq of coverage
			Results for γ_p			
0.564	Usual	0.523	0.09	0.228	0.497	0.970
($p = 0.05$)	Bbst	0.49	0.096	0.224	0.473	0.950
0.465	Usual	0.408	0.100	0.107	0.525	0.978
($p = 0.10$)	Bbst	0.383	0.104	0.090	0.534	0.970
0.388	Usual	0.307	0.115	-0.006	0.564	0.975
($p = 0.15$)	Bbst	0.294	0.113	-0.014	0.565	0.965
			Results for λ_p			
2.587	Usual	2.667	0.917	0.726	6.088	0.985
($p = 0.05$)	Bbst	2.614	0.809	0.707	5.032	0.980
1.738	Usual	1.847	0.713	0.347	4.582	0.982
($p = 0.10$)	Bbst	1.907	0.633	0.295	4.412	0.982
1.269	Usual	1.349	0.625	0.037	4.179	0.992
($p = 0.15$)	Bbst	1.480	0.569	0.016	4.044	0.988