HW 4

1. Let X be a random variable with two possible probability functions f_0 and f_1 defined just below

X	1	2	3	4	5	6
$f_0(x)$	5/20	4/20	3/20	3/20	3/20	2/20
$f_1(x)$	1/20	2/20	3/20	3/20	5/20	6/20

For testing $H : f_0$ against $H : f_1$ find the MP level α test. Graph the lower boundary of the set

$$S = \{(E_0\phi(X), E_1\phi(X)) : \text{ all tests } \phi\}$$

2. Let X be a random variable which takes on values in the interval (0,1). Consider two possible densities for X given by

$$f_0(x) = 1 \qquad \text{for } 0 < x < 1$$

and

$$f_1(x) = 2x \qquad \text{for } 0 < x < 1$$

For testing $H : f_0$ against $H : f_1$ find the MP level α test. Let $w(\alpha)$ be the power of this test. Find the value of α for which $w(\alpha) - \alpha$ is maximum. Graph the set $S = \{(E_0\phi(X), E_1\phi(X)) : \text{ all tests } \phi\}$.

3, Let X be a random variable whose family of possible probability functions, indexed by $\theta \in \Theta = (0, 1)$ is given below:

X	1	2	3	4	5
$f_\theta(x)$	$\frac{\theta}{8(1+\theta)}$	$\frac{3\theta}{8(1+\theta)}$	$\frac{4\theta}{8(1+\theta)}$	$\frac{3}{8(1+\theta)}$	$\frac{5}{8(1+\theta)}$

Using just X (i.e. a sample of size 1) solve the following problems.

i) Find the MP level $\alpha = 0.2$ test for testing $H : \theta = 0.5$ against $K : \theta = 0.75$. For any test ϕ let $\alpha(\phi)$ and $\beta(\phi)$ be the probability of the type I and type II errors. Graph the set $N = \{\alpha(\phi), \beta(\phi) : \text{for all test } \phi\}$

ii) Find the UMP level $\alpha = 0.2$ of $H : \theta = 0.5$ against $K : \theta > 0.5$.

iii) Let $0 < \theta_1 < \theta_2 < 1$ be given. For testing $H : \theta = \theta_1$ against $K : \theta = \theta_2$ graph the set N. For testing $H : \theta \leq \theta_1$ against $K : \theta > \theta_1$ find a UMP level α test when $0 < \alpha < 1$. (Hint:You many need to consider several cases depending on the size of α.

4. Let $X = (X_1, \ldots, X_n)$ be a sample from the uniform distribution on $(0, \theta)$

i) For testing $H : \theta \leq \theta_0$ against $K : \theta > \theta_0$ show that any test is UMP at level α for which $E_{\theta_0}\phi(X) = \alpha$, $E_\theta\phi(X) \leq \alpha$ for $\theta \leq \theta_0$, and $\phi(x) = 1$ when $\max(x_1, \ldots, x_n) > \theta_0$.

ii) For testing $H : \theta = \theta_0$ against $K : \theta \neq \theta_0$ show that a unique UMP test exists and is given by $\phi(x) = 1$ when $\max(x_1, \ldots, x_n) > \theta_0$ or $\max(x_1, \ldots, x_n) \leq \theta_0\alpha^{1/n}$ and $\phi(x) = 0$ otherwise.

5. Let X be a random variable taking on the values $1, 2, \ldots, n$. Let f and g be two possible probability functions for X and consider testing $H : f$ against $K : g$.

i) Suppose K is known to be true a priori with probability $\pi \in (0, 1)$. Assuming zero-one loss, explain what is meant by "the test ϕ is Bayes against π". Find the form of such a Bayes test.

ii) Let $r(i) = g(i)/f(i)$ for $i = 1, \ldots, n$. Assume that

$$0 < r(1) \leq r(2) \leq \cdots \leq r(n) < \infty$$

and that at least one $r(i) \neq 1$. Let $\pi \in (0, 1)$ be fixed and let ϕ_π be a Bayes test against π.

Suppose now that instead of just observing X we may observe two iid copies of X to use in testing H against K. Although ϕ_π is just a test based on one observation it can obviously be thought of a s a test for the two observation problem as well, where it just ignores the second observation. Now it may happen that ϕ_π is also Bayes against π for the 2 observation problem. We denote this by saying "2=1" for ϕ_π. Finally, a test is nontrivial if it is neither identically 0 nor 1. For a nontrivial test ϕ_π prove that $2 = 1$ for ϕ_π if and only if there exists an i_0 such that $1 < i_0 \leq n$ and

$$r(1) = r(2) = \cdots = r(i_0 - 1) < \frac{1-\pi}{\pi} < r(i_0) = \cdots = r(n) \tag{1}$$

and

$$\frac{1-\pi}{\pi} = r(1)r(n)$$

iii) Let R^1 be the risk set for the one observation problem, i.e. is the collection of all possible type I and type II errors for testing H against K. Let R^2 be the risk set for the two observation problem. Assuming equation (1) holds for some π, graph R^1 and R^2.

6. Let F be a continuous function defined on [0,1] satisfying $F(0) = 0$, $F(1) = 1$, $F'(t)$ and $F''(t)$ exist for $t \in (0, 1)$, $F''(t) < 0$ for $t \in (0, 1)$ and $\lim_{t\to1} F'(t) \geq 0$. Let $N(F)$ be the closed convex set which contains (0,0) and (1,1) and is symmetric with respect to the point (1/2,1/2) whose upper boundary is given by F. For a simple vs simple hypothesis testing problem let $N = \{(\alpha(\phi), \beta(\phi)) : \text{for all tests } \phi\}$. Show that there exist some simple vs simple hypothesis testing problem for which $N(F) = N$.

7. Typically, as α varies the most powerful level-α tests for testing a hypothesis H against a simple alternative are nested in the sense that the associated rejection regions, say R_α. satisfy $R_\alpha \subset R_{\alpha'}$ for any $\alpha < \alpha'$ This relation always holds when H is simple, but the following example shows that it need not be satisfied for composite H. Let X take on the values 1, 2, 3 and 4 with probabilities given by

	1	2	3	4
f_0	2/13	4/13	3/13	4/13
f_1	4/13	2/13	1/13	6/13
q	4/13	3/13	2/13	4/13

Show that for testing $H : f_0$ or f_1 against $K : q$ the MP level $\alpha = 5/13$ test rejects when $x = 1$ or $x = 3$ and the MP level $\alpha = 6/13$ test rejects when $x = 1$ or $x = 2$

8. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent samples from $N(\mu, 1)$ and $N(\nu, 1)$, and consider testing $H : \nu \leq \mu$ against $K : \nu > \mu$. Show that there exists a UMP test and that it rejects H when $\bar{Y} - \bar{X}$ is too large. (If $\mu_1 < \nu_1$ is a particular alternative, the distribution assigning probability 1 to the point $\nu = \mu = (m\mu_1 + n\nu_1)/(m + n)$ is least favorable.)

9. p-values. Consider a family of tests of $H : \theta = \theta_0$ (or $\theta \leq \theta_0$), with level-α rejection regions S_α such that

a) $P_{\alpha_0}\{X \in S_\alpha\} = \alpha$ for all $0 < \alpha < 1$,

b) $S_{\alpha_0} = \cap_{\alpha > \alpha_0} S_\alpha$ for all $0 < \alpha < \alpha_0$.

Note b) implies that $S_\alpha \subset S_{\alpha'}$ for $\alpha < \alpha'$.

Then the p-value $\hat{\alpha}$ is given by

$$\hat{\alpha} = \hat{\alpha}(x) = \inf\{\alpha : x \in S_\alpha\}$$

i) When $\theta = \theta_0$, the distribution of $\hat{\alpha}$ is uniform on (0,1).

ii) If the tests S_α are unbiased, the distribution of $\hat{\alpha}$ under any alternative θ satisfies

$$P_\theta\{\hat{\alpha} \leq \alpha\} \geq P_{\theta_0}\{\hat{\alpha} \leq \alpha\} = \alpha$$

So that it is shifted toward the origin.