HW 3

1. Let X_1 and X_2 be independent random variables each Bernoulli(θ) where $\theta \in \{1/4, 3/4\}$. Let $D = \Theta$ and assume squared error loss. Let Δ be all non-randomized decision rules base on (X_1, X_2). Let Δ_T be all non-randomized decision rules base on $T = X_1 + X_2$. Show that there is a member of $\Delta - \Delta_T$ which is not dominated by any member of Δ_T.

2. Let $X_1, \ldots, X_n$ be iid each Poisson(λ) where $\lambda \in (0, +\infty) = \Theta$. Let $D = [0, +\infty)$ and assume convex loss. Show that if $\gamma(\lambda)$ has an unbiased estimator then it can be expressed as a power series in λ. Find the best unbiased estimator of $\gamma(\lambda) = \lambda^i$ where i is some poiitive integer. Show that the best unbiased esitmator of

$$\gamma(\lambda) = P_\lambda(X_1 = k) = \exp(-\lambda)\lambda^k/k!$$

is

$$\begin{aligned} \delta(t) &= \frac{1}{n^k}\binom{t}{k}(1 - 1/n)^{t-k} \quad \text{for} \quad 0 \le k \le t \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{1}$$

where $T = \sum_{i=1}^n X_i$

3. Let X be Poisson(λ) where $\lambda \in \{0, 1, \ldots\}$. Show that X is not complete.

4. Let $X_1, \ldots, X_n$ be iid each uniform on $\{1, 2, \ldots, N\}$ where $N \in \{1, 2, \ldots\} = \Theta$. Show that $T(X_1, \ldots, X_n) = \max X_i$ is complete and sufficient. Let the decision space D be the set of real numbers. Let Γ be the class of all functions defined on Θ having unbiased estimators. Show that Γ is the class of all real-valued functions defined on Θ. Given a $\gamma \in \Gamma$ find an explicit representation for its unbiased estimator. Apply this to the case where $\gamma(N) = N$.

5. Let X be a random variable with

$$\begin{aligned} f_\theta &= \binom{\theta}{x}(1/2)^\theta \qquad \text{for} \quad x = 0, 1, \ldots, \theta \\ &= 0 \qquad \text{otherwise} \end{aligned} \tag{2}$$

We wish to estimate $\gamma(\theta) = \theta$ with squared error loss and with the decision space D equal to the set of real numbers.

i) If $\Theta = \{0, 1, 2, \ldots\}$ show that X is complete and sufficient and find the best unbiased estimator of θ.

ii) If $\Theta = \{1, 2, \ldots\}$ show that X is no longer complete and that there does not exist a best unbiased estimator for θ.

iii) If $\Theta = \{1, 2, \ldots\}$ and $D = \{1, 2, \ldots\}$ show that a best unbiased estimator of θ exists. Find this estimator.

6. Consider a coin with unknown probability $\theta \in [0, 1]$ of coming up heads. Let a and n be known integers satisfying $1 < a < n$. Consider the experiment which consists of the tossing the coin independently until a heads are observed or n tosses are completed (i.e. the coin is tossed at most n times).

The outcomes of such an experiment can be represented as a random walk in the plane. The walk starts at $(0,0)$ and moves one unit to the right or up according to whether the first toss is a head or a tail. From the resulting point, $(1,0)$ or $(0,1)$ it again moves a unit to the right or up and continues in this way until it reaches a stopping point.

i) Give the probability of an outcome which contained h heads and t tails in some specified order.

ii) Consider the statistic T which is the total number of tails observed when the the random walk stops. Find the probability distribution of T (as a function of θ) and show that T is sufficient for this model.

iii) Prove that T is complete for this model.

iv) Find the best unbiased estimator of θ.

7. Consider the following idealized situation. An urn contains an infinite number of coins. A fraction θ of the coins is of Type I where each has probability λ of coming up heads on a given toss while the remaining fraction of the coins is of a second type, say Type II, where each has probability 1 of coming up heads when tossed. The two types of coins are indistinguishable however. Suppose N coins are selected at random from the urn and let N_1 be the number of coins that are of Type I. Then suppose that each of the selected coins are tossed and it is observed that x_1 of the tosses resulted in a tail where $(0 < x_1 < N)$. Assume that the prior distributions for θ and λ are independent beta distributions where $\theta \sim beta(a, b)$ and $\lambda \sim beta(\alpha, \beta)$

i) Find the posterior distribution (up to the normalizing constant) of $p(N_1 \mid x_1)$.

ii) Now suppose that all of the coins that turned up heads on the first set of tosses are all tossed again and this time there were x_2 tails observed where $0 < x_2 < N - x_1$. Now find $p(N_1 \mid x_1, x_2)$ up to the normalizing constant.

8. Suppose events are happening in time over the unit interval. With probability θ, assumed to be known, the distribution of the events follows a Poisson process with intensity parameter λ or with probability $1-\theta$ there is some change point $x \in (0,1)$ such that on $[0,x)$ the events follow a Poisson process with parameter λ_1 and on $[x,1]$ the events follow a Poisson process with parameter λ_2.

When no change point exist we take as our prior for λ the exponential distribution (i.e., $g(\lambda) = \exp(-\lambda)$ for $\lambda > 0$). When a change point does exist our prior for λ_1 and λ_2 are independent exponentials. Given that a change point does exist our prior distribution for where it occurs is uniform(0,1) independent of the priors for λ_1 and λ_2.

Suppose we have have observed the process over the unit interval and the *data* consists of k events occurring at times $0 < t_1 < t_2 < \cdots < t_k < 1$. For $0 < x < 1$ let $k(x)$ be the number of events that occurred in $[0, x)$.

i) Show that under this model the probability of observing these *data* is

$$\theta/2^{k+1} + (1-\theta)\phi(data)$$

where

$$\phi(data) = \int_0^1 \frac{x^{k(x)}}{(1+x)^{k(x)+1}} \frac{(1-x)^{k-k(x)}}{(2-x)^{k-k(x)+1}}\, dx$$

ii) Find the posterior probability that a change point exists.

9. Consider a decision problem where both the parameter space Θ and the decision space D are the set of real numbers. We will consider two possible loss functions

$$L_1(\theta, d) = \exp[a(d-\theta)] - a(d-\theta) - 1$$

where $a \neq 0$ is a fixed, known real number and

$$L_2(\theta, d) = (\theta - d)^2 + C^2\phi(d)$$

where $C > 0$ is a fixed known real number and

$$\phi(d) = \begin{cases} 0 & \text{if } d = 0 \\ 1 & \text{if } d \neq 0 \end{cases}$$

i) Let g be a prior density for θ with mean μ, variance σ^2 and moment generating function ψ. For each of the loss functions find the Bayes estimator for θ for the no data problem for the prior g.

ii)Now consider the data problem where the distribution of X given θ is Normal(θ,1) and the prior g is Normal(μ, σ^2). Find the Bayes estimator for the data problem for each of the two loss functions.

iii) Briefly describe scenarios where the two loss functions would be more reasonable than squared error loss.

10. Let X be a real valued random variable with $\{f_\theta : \theta \in \Theta\}$ a family of everywhere positive densities. Assume Θ is an open interval of real numbers. Assume that this family of densities has the monotone likelihood ratio property in X.

i) Show that if ψ is a non-decreasing function of x then $E_\theta\psi(X)$ is a non-decreasing function of θ.

Hint: let $\theta_1 < \theta_2$ and consider the sets

$$A = \{x : f_{\theta_2}(x) < f_{\theta_1}(x)\} \quad \text{and} \quad B = \{x : f_{\theta_2}(x) > f_{\theta_1}(x)\}$$

ii) Now let ψ be a function with a single change of sign. That is, there exists a value x_0 such that $\psi(x) \leq 0$ for $x < x_0$ and $\psi(x) \geq 0$ for $x \geq x_0$. Show that there exist θ_0 such that

$$E_\theta\psi(X) \leq 0 \text{ for } \theta < \theta_0 \quad \text{and} \quad E_\theta\psi(X) > 0 \text{ for } \theta > \theta_0$$

unless $E_\theta\psi(X)$ is either positive for all θ or negative for all θ.

Hint: let $\theta_1 < \theta_2$ and show that if $E_{\theta_1}(X) > 0$ then $E_{\theta_2}(X) > 0$.

11. Let X be a normal random variable with mean θ and variance equal to one. Let $g(\theta) = \exp -\alpha(\theta)$ be a prior distribution for $-\infty < \theta < \infty$. We are interested in studying the behavior of the posterior distribution of θ given x, $p(\theta \mid x)$, for large values of x.

Let $\lambda(\theta) = \theta^2/2 + \alpha(\theta)$. We suppose that λ', the derivative of λ, exists and that there exist constants K and M such that for $\theta \leq K$, $\lambda'(\theta) \leq M$ and for $\theta > K$, λ' is strictly increasing and everywhere greater than M.

i) Show that for x sufficiently large $p(\theta \mid x)$, as a function of θ is maximized at $\theta = (\lambda')^{-1}(x)$ where $(\lambda')^{-1}$ is the inverse of λ' on the set $\theta > K$.

ii) For $x > K$ let $y = (\lambda')^{-1}(x)$. For such a y find the posterior distribution of θ given y.

iii) Suppose that $\alpha''(\cdot)$ exists for all θ and that $inf_\theta \, \alpha''(\theta) \geq a > -1$ and $lim_{\theta \to \infty} \alpha''(\theta) = c$ where $-1 < c < \infty$.

Let f_y be the posterior density of $\theta - y$ given y. Show that for every real number z

$$\lim_{y \to \infty} f_y(z) = \sqrt{\frac{1+c}{2\pi}} \exp\{-\frac{z^2}{2}(1+c)\}$$

In addition show that there exist constants c_1, c_2 and c_3 such that for $y > c_3$

$$f_y(z) \leq c_1 \exp(-c_2 z^2) \quad \text{for } z \in (-\infty, \infty)$$

Hint: It may be useful to expand $\alpha(y+z)$ in a Taylor series about y.

iv) Let $\delta_g(x)$ be the Bayes estimate of θ against the prior g when the loss is squared error. What do the above results imply about the behavior of $\delta_g(x)$ for large values of x.