HW 2

1. Consider a decision problem with $D = \{d_1, d_2\}$, $\Theta = \{0, 1\}$ and sample space $\mathcal{X} = \{1, 2\}$. Suppose $f_0(1) = 1/2$ and $f_1(1) = 1/3$ and the loss function is give by the table

	$\theta = 0$	$\theta = 1$
d_1	1	3
d_2	4	2

Suppose the non-randomized decision function

$$\delta = (\delta(1) = d_i, \delta(2) = d_j)$$

is represented by (i, j) then the four non-randomized decision functions are (1,1), (1,2), (2,1) and (2,2).

i) Show that every behavorial decision function is a randomized decision function.

ii) Evaluate the risk vector for each of the four non-randomized decision functions and graph the risk set.

iii) For any prior $\pi = P(\theta = 0)$ find a Bayes rule against π.

2. Consider a decision problem with $D = [0, 1]$, $\Theta = \{0, 1\}$ and sample space $\mathcal{X} = \{1, 2\}$. Let $f_\theta(1) = (1+\theta)/3$, and $L(0, d) = d^2$ and $L(1, d) = 1-d$. For any prior $\pi = P(\theta = 0)$ find a Bayes rule, graph the lower boundary of the risk set and find a minimax rule.

3. Let X be binomial(n, θ) where n is known and $\theta \in \Theta = (0, 1)$ is unknown. Let $D = [0, 1]$ and $L(\theta, d) = (\theta - d)^2$.

i) Show that $\delta(X) = X/n$ is not a Bayes estimator of θ.

ii) Show that δ is the unique Bayes estimator of θ when the loss function is $L(\theta, d) = (\theta - d)^2/(\theta(1-\theta))$ and the prior distribution for θ is the uniform distribution.

4. Let $\Theta = (0, 1]$, $D = [0, 1]$ and

$$L(\theta, d) = \min\left\{\frac{(\theta - d)^2}{\theta^2}, 2\right\}$$

Let X be binomial(n, θ) where n is known. Show that the rule $\delta^*(x) = 0$ for $x = 0, 1, \ldots, n$ is the unique minimax decision rule.

5. Let X be a random variable with two possible density functions $f_{\theta_1}(\cdot)$ and $f_{\theta_2}(\cdot)$. Let $\pi = P(\theta = \theta_1)$ be the prior probability for θ_1 and $\pi(x) = P(\theta = \theta_1 \mid X = x)$. Let

$$h(x) = \pi f_{\theta_1}(x) + (1-\pi) f_{\theta_2}(x)$$

Show that

$$\int \pi(x) h(x)\, dx = \pi$$
$$\int \pi(x) f_{\theta_1}(x)\, dx \geq \pi$$

6. Let Z be a real valued random variable with $E(|Z|) < \infty$. Let D be the real line and

$$\begin{aligned} L(z,d) &= k_1(z-d) \quad \text{for} \quad d \leq z \\ &= k_2(d-z) \quad \text{for} \quad d \geq z \end{aligned}$$

where k_1 and k_2 are positive constants. Let d^* be a number which satisfies

$$P(Z \leq d^*) \geq \frac{k_1}{k_1+k_2} \quad \text{and} \quad P(Z \geq d^*) \geq \frac{k_2}{k_1+k_2}$$

Show that

$$E(L(Z,d^*)) = \inf_{\infty < d < \infty} E(L(Z,d))$$

7. Let $\Theta = \{\theta_1, \ldots, \theta_n\}$ be a set of real numbers. Let g be a prior distribution on Θ, that is $g(\theta_i) \geq 0$ and $\sum_{i=1}^n g(\theta_i) = 1$. Let D be the real numbers. Let $L(d,\theta) = W(|d-\theta|)$ where W is a non-negative function defined on $[0,\infty]$. Let $\phi(d) = \sum_{i=1}^n W(|d-\theta_i|)g(\theta_i)$

i) Show that if W' and W'' exist on $[0,\infty)$, with $W'(0)=0$, $W''(0)=0$ and $W''(a) > 0$ for $a > 0$ then ϕ is minimized at the unique point d^* satisfying

$$\sum_{\theta_i > d^*} W'(\theta_i - d^*) g(\theta_i) = \sum_{\theta_i < d^*} W'(d^* - \theta_i) g(\theta_i)$$

ii) Suppose W' and W'' exist on $(0,\infty$ with $W''(a) < 0$ for $a > 0$ and that W is right continuous at zero. Show that ϕ is minimized at a point d^* where $d^* = \theta_i$ for some choice of i.