HW 1

1. Let θ be uniformly distributed on [0,1] and let F be a distribution function. Define

$$G(y) = \sup\{x : F(x) \leq y\}$$

Show that the random variable $G(\theta)$ has F as its distribution function.

2. Let F and G be two 1-dimensional distribution functions. Let

$$H_0(x, y) = \max\{F(x) + G(y) - 1, 0\} \text{ and } H_1(x, y) = \min\{F(x), G(y)\}$$

Show that H_0 and H_1 are 2-dimensional distribution functions, each with F and G as their marginals. Furthermore show that if H is another 2-dimensional distribution function with marginals F and G then

$$H_0(x, y) \leq H(x, y) \leq H_1(x, y) \text{ for all } (x, y)$$

3. Let Y be Poisson(λ) and let X given $Y = y$ be binomial(y, p) where $0 < p < 1$ is fixed. Note the distribution binomial($0, p$) puts mass 1 at 0. Find the marginal distribution of X.

4. Let $X = (X_1, \ldots, X_n)$ where X_i's are independent and X_i is Poisson(λ_i) where $\lambda_i > 0$. Find the conditional distribution of X given $\sum_{i=1}^{n} X_i = k$

5. Let X_1 and X_2 be independent random Normal(0,1) random variables. find the distribution of $Y = X_1/X_2$.

6. Let $X_1, \ldots, X_n$ be independent random variables where X_i is gamma(α_i, β) and where the α_i's and β are all greater than zero. For $i = 1, \ldots, n-1$ let $Y_i = X_i/Y_n$ where $Y_n = \sum_{i=1}^{n} X_i$.

i) Find the joint distribution of $Y_1, \ldots, Y_n$ and note that $Y_1, \ldots, Y_{n-1}$ are independent of Y_n.

ii) Show that the joint density of $Y_1, \ldots, Y_{n-1}$ is given by

$$f(y_1, \ldots, y_{n-1}) = \frac{\Gamma(\sum_{i=1}^{n-1} \alpha_i)}{\prod_{i=1}^{n-1} \Gamma(\alpha_i)} \prod_{i=1}^{n-1} y_i^{\alpha_i - 1} \{(1 - \sum_{i=1}^{n-1} x_i)^{\alpha_i - 1}\} \text{ for } (y_1, \ldots, y_{n-1}) \in \Delta_{n-1}$$

where

$$\Delta_n = \{(y_1, \ldots, y_{n-1}) : y_i \geq 0 \text{ and } \sum_{i=1}^{n-1} y_1 \leq 1\}$$

iii) Find the marginal distribution of $Y_1, \ldots, Y_k$ where $1 \leq k \leq n-1$.

iv) Find $E(Y_i)$, $Var(Y_i)$ and $cov(Y_i, Y_j)$