Stat 8111 **Final Exam** December '16

Eleven students took the exam, the scores were 92, 78, 4 in the 50's, 1 in the 40's, 1 in the 30's and 3 in the 20's.

1. i) Let $X_1, X_2, \ldots, X_n$ be iid each Bernoulli(θ) where $\theta \in \Theta = [0,1]$. What is the a complete sufficient statistic for θ. What real valued functions $\gamma(\theta)$ have unbiased estimators? For such a $\gamma(\theta)$ what is its best unbiased estimator?

ii) Suppose now $n = 3$ and $\Theta = \{0, 1/2, 1\}$. What is the a complete sufficient statistic for this problem? Justify your answer. What real valued functions $\gamma(\theta)$ have unbiased estimators? For such a $\gamma(\theta)$ what is its best unbiased estimator?

2. Let X be a discrete random variable taking on values in the set of non-negative integers. Suppose the probability functions f_θ for $0 < \theta < 1$ have the form

$$f_\theta(x) = \theta^x h(\theta)\lambda(x) \quad \text{for} \quad x = 0, 1, \ldots$$

Let $0 < \theta_0 < 1$ be given and consider testing

$$H : \theta \leq \theta_0 \quad \text{against} \quad K : \theta > \theta_0$$

Let d_0 be the decision we accept H and d_1 the decision we accept K. Let the loss function satisfy

$$\begin{aligned} L(d_0, \theta) &= 0 \quad \text{for} \quad \theta \leq \theta_0 \\ &= \theta - \theta_0 \quad \text{for} \quad \theta > \theta_0 \\ L(d_1, \theta) &= 0 \quad \text{for} \quad \theta \geq \theta_0 \\ &= \theta_0 - \theta \quad \text{for} \quad \theta < \theta_0 \end{aligned}$$

If g is a prior density for θ and f_g the corresponding marginal probability function for X. Show that a Bayes test is given by

$$\begin{aligned} \delta_g(x) &= d_1 \quad \text{when} \frac{\lambda(x) f_g(x+1)}{\lambda(x+1) f_g(x)} > \theta_0 \\ &= d_0 \quad \text{otherwise} \end{aligned}$$

3. Let $0 < a < b < \infty$ be fixed real numbers. Consider a decision problem where $D = [a, b]$ is the space of possible decisions and $\Theta = [a, b]$ is the space of possible parameter values. For some fixed $\lambda > 0$ consider the loss function

$$L(\theta, d) = \frac{2}{\lambda(\lambda+1)} \{ d[(\frac{d}{\theta})^\lambda - 1] + \lambda(\theta - d) \}$$

i) Show that $L(\theta, \theta) = 0$ and that for each fixed θ $L(\theta, \cdot)$ is strictly convex in d.

ii) If g is a prior density for θ find the no data Bayes estimator for g.

4. Suppose X is a discrete random variable taking on values 2, 3, 4, ... For $\theta \in \Theta$, a subset of real numbers, let $f_\theta(\cdot)$ be a probability function for X. Assume that $f(x|\theta) > 0$ for all x and all θ. Assume that this family has the monotone likelihood ratio (MLR) property in $T(X) = X$.

Let $Y = W(X)$ where

$$\begin{aligned} W(j) &= j \quad \text{if } j \text{ is odd} \\ &= j + 1 \quad \text{if } j \text{ is even} \end{aligned}$$

Show that the family of distributions for Y has the MLR property in $T(Y) = Y$

5. Let X be a random variable with a family of possible distributions indexed by the real parameter θ. Consider testing $H : \theta \leq 0$ against $K : \theta > 0$. Let ϕ be a test and $E_\theta \phi(X)$ its power function. Let g be a prior density for θ. Let

$$P_g(\text{type } i \text{ error of } \phi) = \int E_\theta \phi(X) g(\theta)\, d\theta$$

$$P_g(\text{type } ii \text{ error of } \phi) = \int (1 - E_\theta \phi(X)) g(\theta)\, d\theta$$

Let $0 < \alpha < 1$ be fixed. We say that a test is prior most powerful (PMP) at level α if

$$\text{subject to } P_g(\text{type } i \text{ error of } \phi) \leq \alpha \qquad \text{it minimizes } P_g(\text{type } ii \text{ error of } \phi)$$

i) Find the form of the PMP level α test.

ii) Suppose the prior g belongs to a class of possible prior densities, say G. We say that a test is uniformly prior most powerful (UPMP) for the family G if

$$\text{subject to } P_g(\text{type } i \text{ error of } \phi) \leq \alpha \text{ for all } g \in G$$

$$\text{it minimizes } P_g(\text{type } ii \text{ error of } \phi) \text{ uniformly for } g \in G$$

For the rest of the problem assume that X is a normal random variable with mean θ and variance one. For each $g \in G$ let ϕ_g be the PMP level α test found in part i). Find the form of this test.

Show that if there exists a $g^* \in G$ such that

$$\phi_{g^*}(x) = \inf_{g \in G} \phi_g(x) \ \text{ for all } x.$$

then ϕ_{g^*} is a UPMP level α test.

iii) Now suppose G is the family of normal distributions with a known variance, σ^2 but unknown mean $\mu \in [a, b]$ where $-\infty < a < b < \infty$ are specified real numbers. Show that the ϕ_{g^*} of part ii) must exist.

iv) Show that the ϕ_{g^*} of part ii) must exist when μ belongs to the set of real numbers.

Answers

1. i) $T(X_,, \ldots, X_n) = \sum_{i-1}^n X_i$ is complete sufficient. Since $E_\theta \delta(T)$ is a polynomial in θ of degree $\leq n$ and

$$E_\theta(\frac{T(T-1)\cdots(T-k+1)}{n(n-1)\cdots(n-k+1)}) = \theta^k$$

every such polynomial will have a best unbiased estimator.

ii) Let

$$\begin{aligned} T(x_1, x_2, x_3) &= 0 \quad \text{if} \quad \sum x_i = 0 \\ &= 1.5 \quad \text{if} \quad \sum x_i = 1 \text{ or } 2 \\ &= 3 \quad \text{if} \quad \sum x_i = 3 \end{aligned}$$

Note for $\theta \in \{0, 1/2, 1\}$

$$\begin{aligned} P_\theta(T = 0) &= (1-\theta)^3 \\ P_\theta(T = 1.5) &= 3\theta(1-\theta) \\ P_\theta(T = 3) &= \theta^3 \end{aligned}$$

so by factorization theorem T is sufficient. Easy to check that T is complete. $E_{\theta=0}\delta(T) = 0$ implies that $\delta(0) = 0$. In the same way $E_{\theta=1}\delta(T) = 0$ implies that $\delta(3) = 0$. So

$$E_{\theta=1/2}\delta(T) = (1/8)\delta(0) + (3/4)\delta(1.5) + (1/8)\delta(3) = 0$$

implies that $\delta(1.5) = 0$ and so T is complete.

If g is an arbitrary real valued function of θ then its best unbiased estimator is $\delta(0) = g(0)$, $\delta(3) = g(3)$ and

$$\delta(1.5) = \frac{4}{3}[g(1/2) - g(0)/8 - g(1)/8]$$

2. For the no data problem we have

$$\begin{aligned} E_g L(d_0, \theta) &= \int_{\theta_0}^1 (\theta - \theta_o) g(\theta)\, d\theta \\ &= \int_{\theta_0}^1 \theta g(\theta)\, d\theta - \theta_0 P_g(\theta > \theta_0) \end{aligned}$$

In the same way

$$E_g L(d_1, \theta) = \theta_0 P_g(\theta < \theta_0) - \int_0^{\theta_0} \theta g(\theta)\, d\theta$$

Then easy to check that $E_g L(d_1, \theta) < E_g L(d_0, \theta)$ when $\int_0^1 \theta g(\theta)\, d\theta > \theta_0$

Since

$$f_g(x) = \lambda(x) \int_0^1 \theta^x h(\theta) g(\theta)\, d\theta$$

we have

$$\begin{aligned} \int_0^1 \theta p_g(\theta|x) &= \frac{\lambda(x) \int_0^1 \theta^{x+1} h(\theta) g(\theta)\, d\theta}{f_g(x)} \\ &= \frac{\lambda(x) f_g(x+1)}{\lambda(x+1) f_g(x)} \end{aligned}$$

3. The first part of part i) is trivial for the second part we just need to differentiate the loss function twice to see that it is convex and that its minimum value occurs when $\theta = d$. When taking the derivative it is enough to consider the function

$$f(d) = \frac{d^{\lambda+1}}{\theta^{\lambda}} - (\lambda+1)d + \lambda\theta$$

and we see that

$$f'(d) = (\lambda+1)(\frac{d}{\theta})^{\lambda} - (\lambda+1)$$

and the results follow.

ii) Now for a fixed d we have

$$\int_{\Theta} L(\theta, d)g(\theta)\, d\theta = \frac{2}{\lambda(\lambda+1)}\{d^{\lambda+1}E(1/\theta^{\lambda}) - (\lambda+1)d + \lambda E(\theta)\}$$

Now differentiating and setting it equal to zero we find the minimizing d is given by

$$d = (E(1/\theta^{\lambda}))^{-1/\lambda}$$

4. Let a_1, b_1, a_2, b_2 be positive real numbers with $(b_1/a_1) \leq (b_2/a_2)$ then

$$\frac{b_1}{a_1} \leq \frac{b_1 + b_2}{a_1 + a_2} \leq \frac{b_2}{a_2}$$

To prove the first inequality note that

$$\frac{b_1}{a_1} \leq \frac{b_1 + b_2}{a_1 + a_2} \Longleftrightarrow a_1b_1 + a_2b_1 \leq a_1b_1 + a_1b_2 \Longleftrightarrow a_2b_1 \leq a_1b_2 \Longleftrightarrow \frac{b_1}{a_1} \leq \frac{b_2}{a_2}$$

and the second is proved in the same way.

Let $g(y|\theta)$ be the probability function for Y. Let $y_1 < y_2$ be positive even integers and let $\theta_1 < \theta_2$ then we have

$$\begin{aligned}
\frac{g(y_1|\theta_2)}{g(y_1|\theta_1)} &= \frac{f(y_1|\theta_2) + f(y_1+1|\theta_2)}{f(y_1|\theta_1) + f(y_1+1|\theta_1)} \\
&\leq \frac{f(y_1+1|\theta_2)}{f(y_1+1|\theta_1)} \\
&\leq \frac{f(y_2|\theta_2)}{f(y_1|\theta_1)} \\
&\leq \frac{f(y_2|\theta_2) + f(y_2+1|\theta_2)}{f(y_2|\theta_1) + f(y_2+1|\theta_1)} \\
&= \frac{g(y_2|\theta_2)}{g(y_2|\theta_1)}
\end{aligned}$$

Note the first and third inequality follow from the above remark and the second inequality from the MLR property.

5. i) Note

$$P_g(\text{type } i \text{ error of } \phi) = P_g(\theta \le 0) \int \phi(x) f_{0,g}(x)\, dx$$
$$P_g(\text{type } ii \text{ error of } \phi) = P_g(\theta > 0) \int (1 - \phi(X)) f_{1,g}\, dx$$

where $f_{0,g}$ and $f_{1,g}$ denote the conditional densities of X given $\theta \le 0$ and $\theta > 0$. So the NP Lemma tells us to reject H when $f_{1,g} > k f_{0,g}$ for some constant k.

ii) Consider tests of the form

$$\phi^c(x) = 1 \text{ when } x > c \quad \text{and} \quad \phi^c(x) = 0 \text{ when } x < c$$

The power function of such a test $E_\theta \phi^c(X)$ is a strictly increasing function of θ. Clearly every ϕ_g must be a ϕ^c for some choice of c, say $c(g)$. Note by assumption $c(g^*) \le c(g)$ for every other g in G. So for $\theta > 0$

$$P_\theta(\text{type II error of } \phi_{g^*}) \le P_\theta(\text{type II error of } \phi_g)$$

and hence

$$P_g(\text{type } ii \text{ error of } \phi_{g^*}) \le P_g(\text{type } ii \text{ error of } \phi_g)$$

iii) For the prior g corresponding to μ let ϕ_μ be the PMP level α test. Then clearly $c(\mu)$ is a continuous function of μ so it must achieve its minimum on a compact set.

iv) This follows since

$$\lim_{\mu \to \infty} c(\mu) = -\infty = \lim_{\mu \to -\infty} c(\mu)$$

The first equality is true since $P_\mu(\theta \le 0)$ goes to 0 as μ approaches ∞. To prove the second it is enough to show that for any real number a we have that $P_\mu(X \ge a \text{ and } \theta \le 0)$ goes to 0 as μ approaches $-\infty$. But this follows since the marginal distribution of X is normal with mean μ and variance $1 + \sigma^2$.