Stat 8111 **Final Exam** December '15

Ten people took this exam. The scores ranged from 91 to 52 with a mean of 77.9.

1. Let f_0 and f_1 be two probability densities on the line. Consider the parametric family of densities defined by $\theta \in [0, 1]$

$$f_\theta(x) = (1-\theta)f_0(x) + \theta f_1(x).$$

Let g be an everywhere positive density on $[0, 1]$. Under squared error loss find the Bayes estimator of θ against the prior g. Briefly explain why this estimator is admissible.

2. Let $X_1, \ldots, X_n$ be iid random variables with common density function

$$f_\theta(x) = \begin{cases} \theta x \exp(-\theta x^2/2) & \text{for} \quad x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\theta \in (0, \infty) = \Theta$ is a unknown parameter.

i) Assuming a convex loss function show that there exists a best unbiased estimator of $\gamma(\theta) = \sqrt{\theta}$.

ii) For testing H:$\theta \le \theta_o$ against K:$\theta > \theta_o$ show that there exists UMP level α test.

3. Let $X_1, \ldots, X_n$ be iid Bernoulli(θ) random variables. Consider the prior distribution for θ which puts mass λ at the point $\theta = 0.5$ were $0 < \lambda < 1$ and distributes the remaining probability uniformly over the unit interval. Find the posterior probability that $\theta = 0.5$ when $\sum_{i=1}^n x_i = k$. Show that this posterior probability is greater than λ if and only if

$$\binom{n}{k}\left(\frac{1}{2}\right)^n > \frac{1}{n+1}$$

4. Let X be a random vector with a family of possible distributions $\{P_{\theta,\lambda} : \theta \in \Theta, \lambda \in \Lambda\}$ indexed by the parameter θ and λ. Let T, some function of X, be a statistic whose distribution depends on (θ, λ) only through θ. In addition assume for each fixed λ T is sufficient for θ, i.e. T is partially sufficient for θ. Show that for testing H:$\theta = \theta_1, \lambda \in \Lambda$ against K:$\theta = \theta_2, \lambda \in \Lambda$ there exists a UMP level α test.

5. Let X be Normal$(\theta, 1)$ with $\theta \in (-\infty, \infty) = \Theta$. Suppose we wish to estimate $\gamma(\theta) = \theta$ with squared error loss. Let g be a nonnegative function defined on Θ for which the integral

$$\int \theta \exp[-(x-\theta)^2/2] g(\theta)\, d\theta$$

exists. Then g defines a generalized Bayes estimator, say δ, given by

$$\delta(x) = \frac{\int \theta \exp[-(x-\theta)^2/2] g(\theta)\, d\theta}{\int \exp[-(x-\theta)^2/2] g(\theta)\, d\theta}.$$

Note δ is a proper Bayes estimator if $\int g(\theta)\, d\theta < \infty$.

Prove the following three theorems. In your arguments you can assume that interchanging the order of integration is always justified.

Theorem 1. δ is generalized Bayes against g if and only if

$$\exp[\int_0^x (\delta(y) - y)\, dy] = \int_{-\infty}^{\infty} \exp[-(x-\theta)^2/2] g(\theta)\, d\theta.$$

Hint: Begin by taking the log of each side of the above equation.

Theorem 2. If δ is generalized Bayes against g then it is proper Bayes if and only if

$$\int_{-\infty}^{\infty} \exp[\int_0^x (\delta(y) - y)\, dy]\, dx < \infty.$$

Theorem 3. Let δ be generalized Bayes against g.

i) If there is an $M > 0$ such that for $|x| > M$ we have that

$$|\delta(x)| \leq |x| - (1+\epsilon)/|x|$$

for some $\epsilon > 0$ then δ is proper Bayes.

ii) If there is an $M > 0$ such that for $|x| > M$ we have that

$$|\delta(x)| \geq |x| - (1-\epsilon)/|x|$$

for some $\epsilon \geq 0$ then δ is not proper Bayes.

Answers

1. Let $m_i = E(\theta^i)$ and $h(x) = (1-m_1)f_0(x) + m_1 f_1(x)$. Then

$$p(\theta|x) = \frac{\{(1-\theta)f_0(x) + \theta f_1(x)\}g(\theta)}{h(x)}$$

and

$$E(\theta|x) = \frac{(m_1 - m_2)f_0(x) + m_2 f_1(x)}{h(x)}.$$

Since the risk function is a continuous function of θ and the prior is everywhere positive the estimator is admissible.

2. Since

$$f_\theta(x_1, \ldots, x_n) = \theta^n \prod x_i \exp(-\theta \sum x_i^2/2)$$

belongs to the one parameter exponential family $\sum x_i^2$ is complete and sufficient for θ.

Since $E_\theta[(2/\pi)^{1/2}(1/X_1)] = \sqrt{\theta}$ the conditional expectation of this unbiased estimator with respect to $\sum X_i^2$ is best unbiased.

Since for $\theta_1 < \theta_2$ we have that $f_{\theta_2}(x_1, \ldots, x_n)/f_{\theta_1}(x_1, \ldots, x_n)$ is a decreasing function in $\sum x_i^2$ the UMP level α test rejects when $\sum X_i^2$ is less than some constant.

3. The posterior probability is

$$\begin{aligned}
P(\theta = 1/2 \mid k) &= \frac{\lambda(1/2)^n}{\lambda(1/2)^n + (1-\lambda)\int_0^1 \theta^k (1-\theta)^{n-k}\, d\theta} \\
&= \frac{\lambda(1/2)^n}{\lambda(1/2)^n + (1-\lambda)((\Gamma(k+1)\Gamma(n-k+1))\Gamma(n+1))} \qquad (1) \\
&= \frac{\lambda(1/2)^n}{\lambda(1/2)^n + (1-\lambda)\, k!\, (n-k)!)/(n+1)!}
\end{aligned}$$

So this posterior probability is greater than λ if and only if

$$(1/2)^n > \lambda(1/2)^n + (1-\lambda)\, k!\, (n-k)!)/(n+1)!$$

and the result follows from simple algebra.

4. Let $\psi_0(T)$ be MP at level α among all tests which are just functions of T. Let $\phi(X)$ be any other level α test and let λ_1 be fixed. Let $\psi(T) = E_{\lambda_1}(\phi(X)|T)$. Since

$$E_{\theta_i}\psi(T) = E_{\theta_i,\lambda_1}\phi(X)$$

ψ is a level α test of H and its power does not exceed that of ψ_0.

5. Theorem 1 follows by taking the natural log of both sides and differentiating.

Theorem 2 follows from Theorem 1 and interchanging the order if integration.

Consider part i) of Theorem 3. Then for $y > M$

$$\delta(y) - y \le y - \frac{(1+\epsilon)}{y} - y = \frac{-(1+\epsilon)}{y}$$

and for $x > M$

$$\exp \int_M^x (\delta(y) - y)\, dy \le \exp \int_M^x \frac{-(1+\epsilon)}{y}\, dy = \frac{1}{x^{1+\epsilon}} - \frac{1}{M^{1+\epsilon}}.$$

For $x < M$ a similar bound exits and since δ is continuous and hence bounded on $[-M, M]$ the result follows from Theorem 2.

A similar argument proves part ii).