# Split Plots

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# What is a Split Plot?

Split plots are designs for factorial treatment structure.

They are useful when we want to vary one or more of the factors less often than the other factors (e.g., expensive to change, time consuming to change, logistically challenging to change, can only be applied to "large" units, etc).

There are several ways to think about split plots, each useful in different circumstances.

For example, you are blowing glass art figures and we are interested in factors that affect fragility. You can set the annealing oven to two different temperatures, and you can make three different sizes of figures.

The oven takes hours to come to temperature and hours to cool down. We do not want to change that frequently. Figure size, however, can be changed at will.

What we do is randomly assign temperatures to days. Then, within each day, we randomly choose an order for the three sizes of figures.

| A:2 | A:2 | A:1 | A:2 | A:1 | A:1 |
|---|---|---|---|---|---|
| B:3 | B:1 | B:3 | B:2 | B:1 | B:2 |
| B:1 | B:3 | B:1 | B:3 | B:2 | B:3 |
| B:2 | B:2 | B:2 | B:1 | B:3 | B:1 |

In this schematic, A is temperature, B is size, and the little columns represent days.

Temperature is assigned to days, and size is assigned to the tasks within a day.

This is nicely balanced, but all tasks within a day must have the same oven temperature.

# Unit Structure

Terminology of split plots comes from agriculture.

Units in a split plot have structure. We have big units, called whole plots. The whole plots comprise smaller units, called split plots.

In a sense, split plots are nested in whole plots.

In our example, days are the whole plots, and tasks within a day are the split plots.

You randomly assign the levels of one factor to the whole plots. This is the whole plot treatment factor.

Whole plot treatment factors are the hard-to-vary factors. In our example, temperature is the WP treatment factor.

Within each whole plot, you randomly assign the levels of the other factor to split plots. This is the split plot treatment factor.

Split plot treatment factors are the easy-to-vary factors. In our example, size is the SP treatment factor.

From a randomization perspective, whole plots act like units for the whole plot treatment factor.

From a randomization perspective, whole plots act like blocks for the split plot treatment factor.

Two sizes of units (one nested in the other) and two randomizations. That gives us a split plot design.

# Restricted Randomization

A second view of a split plot is through an equivalent view of the randomization.

Randomly assign the treatments (combinations of whole plot and split plot treatment factors) to the split plots subject to two restrictions:

- All split plots in the same whole plot get the same level of the whole plot treatment factor.
- All levels of the split plot treatment factor occur in each whole plot.

The restricted randomization is equivalent to the two randomizations of the unit structure approach.

This view is correct, but often not as insightful as the unit structure approach.

This view is most helpful when the whole plot is not physically apparent and it's really only the restricted randomization that leads us to recognize a split plot.

## Incomplete Blocks

A split plot design can also be viewed as an incomplete block design.

Whole plots are the incomplete blocks, and differences between the levels of the whole plot treatment factor are confounded with block (whole plot) differences.

However, the randomization at the whole plot level induces a random effect at the whole plot level (i.e., random blocks).

We get information about the whole plot treatment factor via interblock recovery.

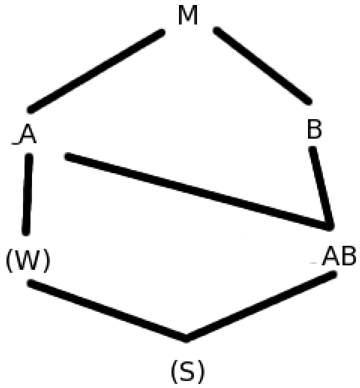
# Model

The model and analysis for a split plot are not that hard.

But that assumes that you know that you have a split plot experiment. Deciding that you (or someone else) have a split plot is probably the hardest bit.

From a model perspective, we get a random effect for each size of unit. In effect, the randomization to a unit is represented by a random effect at that unit level.

Thus we have a random whole plot term and a random split plot term (which cannot be distinguished from ordinary error).




We cannot distinguish (S) from (Error).

Note that this Hasse diagram looks just like the one we saw for cheese raters.

Different designs can lead to the same model structure.

We can just use lmer() or lme() with a random effect for the whole plots and proceed as usual.

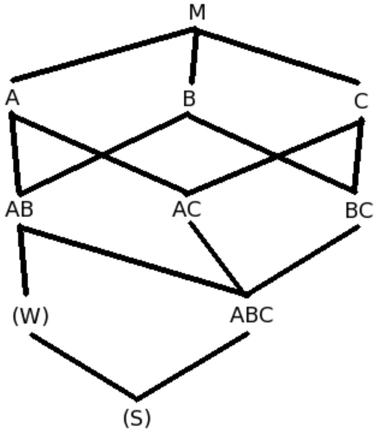
Comparisons at whole plot level are less precise than those at split plot level. Similarly, less power at whole plot levee.

# Generalizations

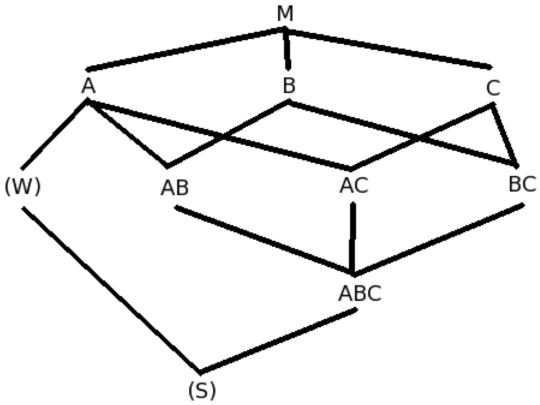
More than two factors. We can have multiple factors at whole plot level and/or split plot level.

The design at the whole plot level could be any one of our blocking designs. RCB is very common at WP level.

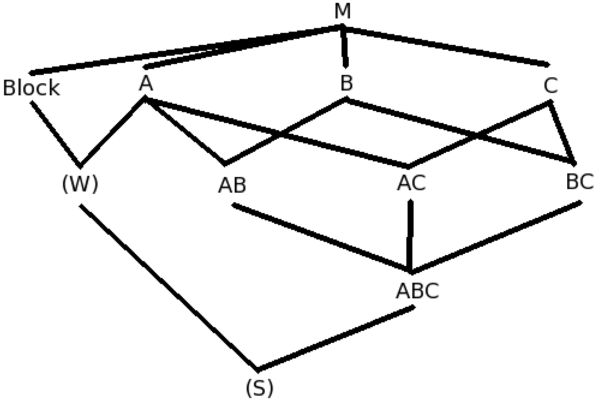
Can do additional balancing at split plot level. E.g., take a cross over design (replicated LS), then add a second factor at the whole plot (subject) level.



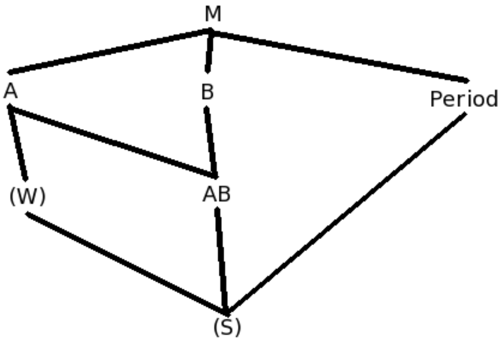
Two whole plot factors, one split plot factor.



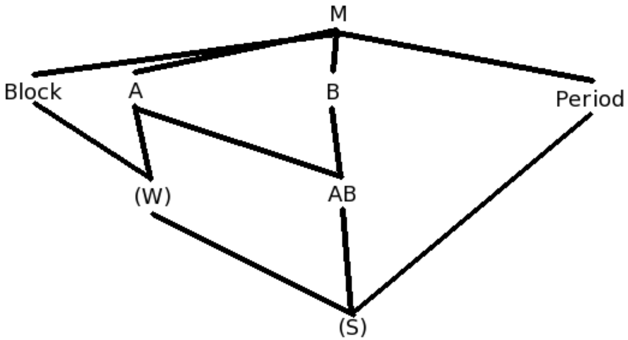

One whole plot factor, two split plot factors.




One whole plot factor, two split plot factors; RCB at the WP level.




Crossover design in B (Latin sq), but A randomly applied to subjects.




Crossover design in B (Latin sq), but A randomly applied to subjects in RCB fashion.

Some books only talk about split plots with whole plot blocking.

Some of these books use a model of random blocks that interact with the whole plot factor and the split plot factor. This is <u>not the same</u> as what I have described.

These books tend to have an engineering orientation, so I call this the industrial split plot model.

I don't use this model.

# Split split plot designs

Once you have the idea of splitting units into smaller units, you can split more than once.

A split split plot has three sizes of units: whole plots that are made up of split plots which are made up of split split plots.

Two levels of nesting in the unit structure: split split plots nest into split plots, and split plots nest into whole plots.

You need at least three factors: a whole plot treatment factor, a split plot treatment factor, and a split split plot treatment factor.

$A:5$

$\Downarrow$

| | | |
|---|---|---|
| $B:2 \Rightarrow$ | 5:2:2 | $\leftarrow C:2$ |
| | 5:2:3 | $\leftarrow C:3$ |
| | 5:2:1 | $\leftarrow C:1$ |
| $B:1 \Rightarrow$ | 5:1:1 | $\leftarrow C:1$ |
| | 5:1:2 | $\leftarrow C:2$ |
| | 5:1:3 | $\leftarrow C:3$ |
| $B:3 \Rightarrow$ | 5:3:3 | $\leftarrow C:3$ |
| | 5:3:2 | $\leftarrow C:2$ |
| | 5:3:1 | $\leftarrow C:1$ |

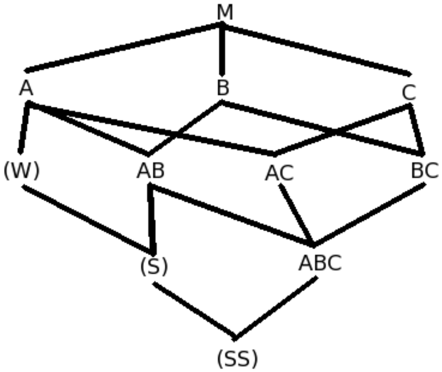
7 by 3 by 3 split split plot. This whole plot received level 5 of factor A; the three split plots and nine split split plots are assigned as shown.

With three levels of randomization and three sizes of units, we get three random terms: one for whole plots, one for split plots, and one for split split plots (indistinguishable from error).

We can have various kinds of blocking at the whole plot level.

We can have more than one factor at each randomization level.

Follow the randomization! Counting factors is not a way to distinguish between split plot designs and split split plot designs (or even CRD).




Split split plot with CRD at WP level.

# Split blocks/strip plots etc.

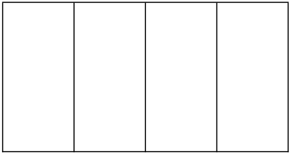
Once you get the idea of splitting (nesting) units, you could go all the way to a split split split split split plot if you wanted. I don't think I've seen beyond split split plot in the wild.

However, we now have unit structure. We have seen nesting units.

Can units cross? Yes, they can.

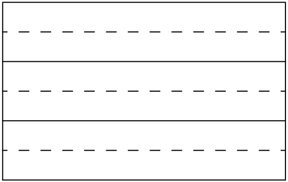
We can build designs with unit structures that have nesting, crossing, or both. Then we layer the treatment structure on top of that!

Randomly apply three different primers to three horizontal strips.

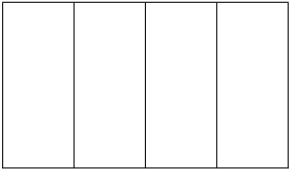


Randomly apply four different paints to four vertical strips.

The horizontal units cross the vertical units on the same wall.



Randomly apply three different varieties to horizontal strips. Randomly apply two different fertilizers to the two horizontal substrips.

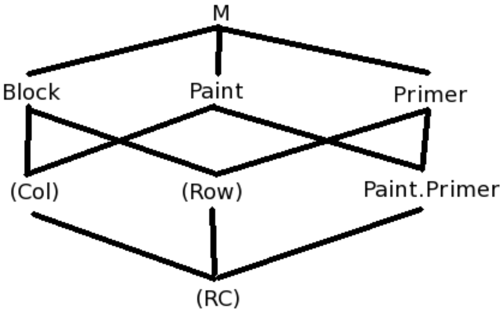


Randomly apply four irrigation levels to four vertical strips.

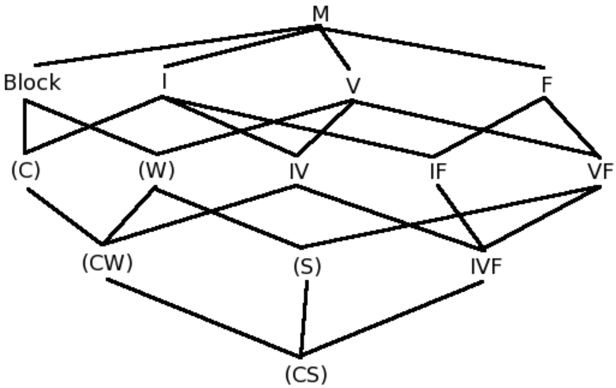
We have a blocked split plot in the horizontal units and an RCB in the vertical units, and the vertical units cross the horizontal units.

We typically need replication in blocks for this to work well. Above, blocks were the walls or the large chunks of land.

The basic model is to have a random effect for each kind of unit (randomization) and wherever units cross.




Split block, also called strip plot. Ignore paint, it's an RCB on primer; ignore primer, it's an RCB on paint.




Split plot crossing an RCB. Ignore irrigation, it's a split plot in V and F. Ignore F, it's a strip plot in I and V.

# Repeated Measures

Repeated measures look like split plots, but there is no randomization at the "split plot" level.

Typically the "split plot" treatment factor is time, and with repeated measures we just keep measuring the same unit repeatedly over time.

Time does not like to be randomized,[1] so it's not a split plot.

Another version arises when we can measure the same thing multiple ways. We literally just get multiple measurements.

[1] Insert generic Dr. Who reference.

In the repeated measures terminology:

- "Whole plots" are called subjects.
- "Whole plot" treatment factors are called grouping factors.
- "Split plot" treatment factors are called trial factors.

In our example, we prepare emulsions using three different emulsifiers. We then measure each separate emulsion over time.

Each emulsion is the “subject.” The emulsifiers form the grouping factor. Time is the trial factor.

Kind of looks like a split plot, but no randomization.

What is happening is that we have experimented at the subject level, but we observe a vector of responses across the trial level.

This vector of responses is probably correlated, not independent.

Some kind of correlation is potentially present among units we use in experimentation, but randomization of treatments to units scrambles the correlation to the point it can usually be ignored.

But, no randomization, no scrambling; the correlation comes through unaltered and potentially affecting results.

# Analysis

Potential approaches:

1. Full multivariate analysis.
2. Univariate summaries.
3. Univariate analysis.
4. Modified univariate analysis.
5. Model the correlation.

1. Full multivariate analysis. This requires a lot of data to work well and many techniques we have not discussed. Take Stat 5401 if you are interested in this approach.

2. Univariate summaries. Here you create some kind of statistic from the trial data for each subject, for example, the rate of change over time. You then treat this as the response for a subject and do standard analysis. By looking at different summaries you can examine different aspects of trial factor effects.

Univariate summaries are a legitimate approach, but you need to choose the right summary (or summaries), and you have to figure out the relationship if you have more than one summary.

3. Univariate analysis approach. This approach says assume there is a random subject effect and that this effect interacts with every trial factor. With just a single trial factor this is equivalent to the standard split plot analysis.

If nature has been very kind to you and the data at the trial level have a covariance that satisfies a special condition, then the univariate approach is legitimate.

If the trial factor has only two levels, then the univariate approach is always legitimate.

If you were unlucky and didn't get the special form of covariance, then tests at the trial factor level tend to be liberal.

The special condition (the Huynh-Feldt condition) is that all differences of repeated measures have the same variance.

One case that satisfies the HF condition is sphericity: all variances are the same and all correlations between trial levels within a subject are the same (the correlations don't have to be zero).

For multiple trial factors there is a generalization of sphericity called compound symmetry.

There is a Mauchly Test for the HF condition, but it is <u>very</u> dependent on normality.

4. Modified univariate analysis. The modifications are for the "treat it like a split plot" approach with old school mixed effects analysis. The modifications adjust the tests in an attempt to make them less liberal (but not conservative).

There is a Greenhouse-Geisser adjustment and a Huynh-Feldt adjustment. Both of these reduce the error DF for trial level tests by some factor estimated from the data.

5. Model the correlation. The approach is possible with REML computations; it models and estimates the correlation, and then takes the correlation into account.

We generally anticipate positive autocorrelation over time between the observations for a single subject (separate subjects still being independent). There are many potential models for this, but autoregressive of order one (AR1) is the simplest and most common.