Stat 5101 Final Exam **May 13, 2019**

Name ______________________________ Student ID ____________________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may use the handouts on "brand name distributions" and Greek letters. You may use a calculator. No other electronic devices are allowed.

Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers or formulas with no indication of where they came from. Leave no undone integrals or means or variances or unevaluated gamma functions in your answers, but other than that requirement there is no unique "correct" simplification. Any correct (and explained) answer gets full credit unless the question explicitly states otherwise.

Abbreviations used: independent and identically distributed (IID), probability density function (PDF), probability mass function (PMF).

The points for the questions total to 200. There are 9 pages and 8 problems.

1. [25 pts.] Suppose X_1, ..., X_N are IID $\mathcal{N}(\mu, \sigma^2)$ random variables, where N is a Poi(λt) random variable that is independent of all of the X_i. Let

$$Y = \sum_{i=1}^{N} X_i,$$

with the convention that $N = 0$ means $Y = 0$.

(a) Find $E(Y)$.

(b) Find $\mathrm{var}(Y)$.

2. [25 pts.] Define

$$h_\theta(x) = \frac{1}{\theta + x^{1/2} + x^{9/2}}, \qquad 0 < x < \infty.$$

(a) For what values of the positive real parameter θ does there exist a constant $c(\theta)$ that

$$f_\theta(x) = c(\theta) h_\theta(x), \qquad 0 < x < \infty,$$

is a PDF?

(b) If X is a random variable having this PDF, for what values of θ and for what $\beta > 0$ does the expectation of X^β exist?

3. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID Poi(μ) random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad (*)$$

What is the approximate normal distribution of $1/(1 + \overline{X}_n)$ when n is large?

4. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID $\text{Gam}(\alpha, \alpha^{1/3})$ random variables, where $\alpha > 0$ is a real parameter. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large n with variance that is a constant function of the parameter α? As usual, $\overline{X}_n$ is defined by equation $(*)$ in problem 3.

5. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f_\theta(x, y) = \frac{\theta^5 (x^2 + y^3) e^{-\theta(x+y)}}{2(3 + \theta)} \qquad 0 < x < \infty,\ 0 < y < \infty,$$

where $\theta > 0$ is a parameter. Hint: the theorem associated with the gamma distribution (brand name distributions handout).

(a) Find the conditional PDF of X given Y.

(b) Find the conditional expectation of X given Y.

6. [25 pts.] Suppose campus connector bus arrivals form a Poisson process with rate $\lambda = 15$ per hour (in the middle of the day). (They don't really, but just suppose for the purposes of this problem.) What is the probability that your wait for a bus is longer than 6 minutes?

7. [25 pts.] If X has the distribution with PDF given by

$$f_\theta(x) = \frac{c\theta^{2/5}x^2}{\theta + x^5}, \qquad 0 < x < \infty,$$

where

$$c = \frac{5}{\pi\sqrt{2 - 2/\sqrt{5}}},$$

and where $\theta > 0$ is a parameter, then what is the PDF of $Y = 1/(1+X)$? The definition of a function describes the domain as well as the rule.

8. [25 pts.] Suppose the conditional distribution of Y given X has PDF

$$f(y|x) = 2\pi^{-1/2} x^{3/2} y^{1/2} e^{-xy}, \qquad 0 < y < \infty,$$

and the marginal distribution of X is $\text{Gam}(\alpha, \lambda)$, where μ, α, and λ are real parameters such that $\alpha > 0$ and $\lambda > 0$. What is the conditional distribution of X given Y?

Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of y, α and λ.