**Stat 5101 Final Exam** **May 12, 2016**

Name ______________________ Student ID ______________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may use the handouts on "brand name distributions" and Greek letters. You may use a calculator.

Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers or formulas with no indication of where they came from. Leave no undone integrals or means or variances in your answers, but other than that requirement there is no unique "correct" simplification. Any correct answer gets full credit unless the question explicitly states otherwise.

Abbreviations used: independent and identically distributed (IID), probability density function (PDF), probability mass function (PMF).

The points for the questions total to 200. There are 9 pages and 8 problems.

1. [25 pts.] Suppose Suppose $X_1, \ldots, X_N$ are IID random variables having mean $\mu$ and standard deviation $\sigma$, where $N$ is a $\text{NegBin}(r, p)$ random variable that is independent of all of the $X_i$. Let

$$Y = \sum_{i=1}^{N} X_i,$$

with the convention that $N = 0$ means $Y = 0$.

(a) Find $E(Y)$.

(b) Find $\mathrm{var}(Y)$.

2. [25 pts.] Define

$$h_\theta(x) = \frac{1 + \cos(\theta x) + |x|}{(1 + x^2)^3}, \qquad -\infty < x < \infty.$$

(a) For what values of the positive real parameter $\theta$ does there exist a constant $c(\theta)$ that

$$f_\theta(x) = c(\theta) h_\theta(x), \qquad -\infty < x < \infty,$$

is a PDF?

(b) If $X$ is a random variable having this PDF, for what values of $\theta$ and for what $\beta > 0$ does the expectation of $|X|^\beta$ exist?

3. [25 pts.] Suppose $X_1$, $X_2$, $\ldots$ are IID Beta$(\theta, 1)$ random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \tag{$*$}$$

What is the approximate normal distribution of $\log(\overline{X}_n) - \log(1 - \overline{X}_n)$ when $n$ is large?

4. [25 pts.] Suppose $X_1$, $X_2$, $\ldots$ are IID $\mathcal{N}(\mu, \mu)$ random variables, where $\mu > 0$ is a real parameter. What is the variance stabilizing transformation: for what function $g$ does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large $n$ with variance that is a constant function of the parameter $\mu$? As usual, $\overline{X}_n$ is defined by equation $(*)$ in problem 3.

5. [25 pts.] Suppose the random vector $(X, Y)$ has the PDF

$$f_\theta(x, y) = \frac{6\theta}{(1 + \theta x + y)^4}, \qquad 0 < x < \infty,\ 0 < y < \infty,$$

where $\theta > 0$ is a parameter.

(a) Find the conditional PDF of $X$ given $Y$.

(b) Find the conditional expectation of $X$ given $Y$. (Hint: do the integral either by substitution or by integration by parts).

6. [25 pts.] Suppose $X$ is has the distribution with PDF

$$f_\lambda(x) = \frac{\lambda e^{-\lambda|x|}}{2}, \qquad -\infty < x < \infty,$$

where $\lambda > 0$ is a parameter.

(a) Find the DF of this distribution.

(b) Find the median of this distribution.

(c) Find the 0.25 quantile of this distribution.

7. [25 pts.] If $X$ has the $\mathcal{N}(\mu, \sigma^2)$ distribution and $Y = e^X$, then what is the PDF of $Y$? The definition of a function describes the domain as well as the rule.

8. [25 pts.] Suppose the conditional distribution of $Y$ given $X$ is $\mathcal{N}(\mu, 1/X)$ and the marginal distribution of $X$ is $\text{Gam}(\alpha, \lambda)$, where $\mu$, $\alpha$, and $\lambda$ are real parameters such that $\alpha > 0$ and $\lambda > 0$. What is the conditional distribution of $X$ given $Y$?

   Since this is a brand name distribution, no integrals need be done, it is enough to name the distribution and give its parameters as a function of $y$, $\mu$, $\alpha$ and $\lambda$.