Stat 5102 Second Midterm Exam **April 8, 2015**

Name ______________________________ Student ID ____________________

The exam is closed book and closed notes. You may use one $8\frac{1}{2} \times 11$ sheet of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. You may use a calculator.

Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from. Leave no undone integrals in your answers, but other than that requirement there is no unique "correct" simplification. Any correct answer gets full credit, except as explicitly stated in questions.

Abbreviations used: independent and identically distributed (IID), maximum likelihood estimate (MLE), probability density function (PDF), probability mass function (PMF).

The points for the questions total to 100. There are 6 pages and 5 problems.

1. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID from the distribution having PDF

$$f_\theta(x) = \frac{\theta x^{\theta - 1}}{(1 + x)^{\theta + 1}}, \qquad 0 < x < \infty,$$

where θ is an unknown parameter satisfying $0 < \theta < \infty$.

(a) Find the log likelihood for θ.

(b) Find the MLE for θ.

(c) Show that your MLE is the unique global maximizer of the log likelihood.

2. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID Poi(θ^3) where $\theta > 0$ is the unknown parameter.

   (a) Find the log likelihood for θ.

   (b) Find the MLE for θ.

   (c) Find the (expected) Fisher information for θ.

(d) Find the asymptotic distribution of your MLE. The mean and variance of the asymptotic normal distribution should be expressed as functions of θ and n only.

3. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID from the distribution having PDF

$$f_\theta(x) = \frac{\theta x^{\theta-1}}{(1+x)^{\theta+1}}, \qquad 0 < x < \infty,$$

where θ is an unknown parameter satisfying $0 < \theta < \infty$. Suppose the prior distribution for θ is $\text{Gam}(\alpha, \lambda)$, where α and λ are hyperparameters satisfying $\alpha > 0$ and $\lambda > 0$. Find the posterior distribution for θ.
Hint: $a^b = e^{b \log(a)}$.

4. [20 pts.] Suppose X is Poi(μ). We have only one observation. And suppose the prior distribution for μ is proportional to μ^{-10}, an improper prior.

(a) Find the posterior distribution for μ.

(b) For what values of the data x does your answer to part (a) make sense?

5. [20 pts.] Suppose X and Y are integer-valued random variables satisfying $0 \leq Y \leq X < \infty$, which have joint PMF

$$f_{\mu,\theta}(x, y) = \frac{\mu^x e^{-\mu}}{x!} \cdot \binom{x}{y} \theta^y (1-\theta)^{x-y}$$

where μ and θ are unknown parameters satisfying $0 < \mu < \infty$ and $0 < \theta < 1$.

Show that this distribution is an exponential family. Identify the natural parameter vector and the natural statistic vector.