Stat 5102 Second Midterm Exam **April 10, 2013**

Name ______________________________ Student ID __________________

The exam is closed book and closed notes. You may use one $8\frac{1}{2} \times 11$ sheet of paper with formulas, etc. You may also use the handouts on "brand name distributions" and Greek letters. Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from.

IID means independent and identically distributed; PDF means probability density function; MLE means maximum likelihood estimate.

The points for the questions total to 100. There are 6 pages and 5 problems.

1. [20 pts.] Suppose X_1, ..., X_n are IID from the distribution having PMF

$$f_\theta(x) = (1 - \theta)\theta^{x-1}, \qquad x = 1, 2, \ldots,$$

where θ is an unknown parameter satisfying $0 < \theta < 1$.

(a) Find the log likelihood for θ.

(b) Find the MLE for θ.

(c) Show that your MLE is the unique global maximizer of the log likelihood.

2. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID Exp(θ^2) and $Y_1, \ldots, Y_n$ are IID Exp(θ) and the X's and Y's are independent of each other, where $\theta > 0$ is the unknown parameter.

   (a) Find the log likelihood for θ.

   (b) Find the MLE for θ. **Hint:** the roots of the quadratic polynomial $ax^2 + bx + c$ are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

   (c) Find the asymptotic distribution of your MLE. The mean and variance of the asymptotic normal distribution should be expressed as functions of θ and n only.

3. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID NegBin$(7, p)$ where p is an unknown parameter. Suppose the prior distribution for p is Beta(α_1, α_2), where α_1 and α_2 are hyperparameters satisfying $\alpha_1 > 0$ and $\alpha_2 > 0$. Find the posterior distribution for p.

4. [20 pts.] Suppose $X_1, \ldots, X_n$ are IID with PDF

$$f(x \mid \theta) = \frac{\theta}{2(1 + |x|)^{\theta+1}}, \qquad -\infty < x < \infty,$$

where $\theta > 0$ is the unknown parameter. Suppose we use the improper prior distribution with density function

$$g(\theta) = \frac{1}{\theta}$$

(a) Find the posterior distribution for θ. **Hint:** $a^b = e^{b \log(a)}$.

(b) For what values of the data $x_1, \ldots, x_n$ does your answer to part (a) make sense? Be sure to explain your answer.

5. [20 pts.] Show that the $\text{NegBin}(r, p)$ distribution is an exponential family when r is considered known and p is the only parameter. Identify the natural parameter and the natural statistic.