Stat 5101 Notes: Brand Name Distributions

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Contents

1 Discrete Uniform Distribution

Abbreviation DiscUnif(n).

Type Discrete.

Rationale Equally likely outcomes.

Sample Space The interval 1, 2, …, n of the integers.

Probability Mass Function

$$f(x) = \frac{1}{n}, \qquad x = 1, 2, \ldots, n$$

Moments

$$\begin{aligned} E(X) &= \frac{n+1}{2} \\ \operatorname{var}(X) &= \frac{n^2-1}{12} \end{aligned}$$

2 General Discrete Uniform Distribution

Type Discrete.

Sample Space Any finite set S.

Probability Mass Function

$$f(x) = \frac{1}{n}, \qquad x \in S,$$

where n is the number of elements of S.

3 Uniform Distribution

Abbreviation Unif(a, b).

Type Continuous.

Rationale Continuous analog of the discrete uniform distribution.

Parameters Real numbers a and b with $a < b$.

Sample Space The interval (a, b) of the real numbers.

Probability Density Function

$$f(x) = \frac{1}{b - a}, \qquad a < x < b$$

Moments

$$\begin{aligned} E(X) &= \frac{a + b}{2} \\ \operatorname{var}(X) &= \frac{(b - a)^2}{12} \end{aligned}$$

Relation to Other Distributions Beta$(1, 1) =$ Unif$(0, 1)$.

4 General Uniform Distribution

Type Continuous.

Sample Space Any open set S in $\mathbb{R}^n$.

Probability Density Function

$$f(x) = \frac{1}{c}, \qquad x \in S$$

where c is the measure (length in one dimension, area in two, volume in three, etc.) of the set S.

5 Bernoulli Distribution

Abbreviation Ber(p).

Type Discrete.

Rationale Any zero-or-one-valued random variable.

Parameter Real number $0 \le p \le 1$.

Sample Space The two-element set $\{0, 1\}$.

Probability Mass Function

$$f(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Moments

$$\begin{aligned} E(X) &= p \\ \operatorname{var}(X) &= p(1 - p) \end{aligned}$$

Addition Rule If X_1, …, X_k are IID Ber(p) random variables, then $X_1 + \cdots + X_k$ is a Bin(k, p) random variable.

Degeneracy If $p = 0$ the distribution is concentrated at 0. If $p = 1$ the distribution is concentrated at 1.

Relation to Other Distributions $\text{Ber}(p) = \text{Bin}(1, p)$.

6 Binomial Distribution

Abbreviation $\text{Bin}(n, p)$.

Type Discrete.

Rationale Sum of n IID Bernoulli random variables.

Parameters Real number $0 \le p \le 1$. Integer $n \ge 1$.

Sample Space The interval 0, 1, ..., n of the integers.

Probability Mass Function

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0, 1, \ldots, n$$

Moments

$$\begin{aligned} E(X) &= np \\ \text{var}(X) &= np(1-p) \end{aligned}$$

Addition Rule If X_1, ..., X_k are independent random variables, X_i being $\text{Bin}(n_i, p)$ distributed, then $X_1 + \cdots + X_k$ is a $\text{Bin}(n_1 + \cdots + n_k, p)$ random variable.

Normal Approximation If np and $n(1-p)$ are both large, then

$$\text{Bin}(n, p) \approx \mathcal{N}\bigl(np, np(1-p)\bigr)$$

Poisson Approximation If n is large but np is small, then

$$\text{Bin}(n, p) \approx \text{Poi}(np)$$

Theorem The fact that the probability mass function sums to one is equivalent to the **binomial theorem:** for any real numbers a and b

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

Degeneracy If $p = 0$ the distribution is concentrated at 0. If $p = 1$ the distribution is concentrated at n.

Relation to Other Distributions $\mathrm{Ber}(p) = \mathrm{Bin}(1, p)$.

7 Hypergeometric Distribution

Abbreviation $\mathrm{Hypergeometric}(A, B, n)$.

Type Discrete.

Rationale Sample of size n without replacement from finite population of B zeros and A ones.

Sample Space The interval $\max(0, n - B)$, ..., $\min(n, A)$ of the integers.

Probability Mass Function

$$f(x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}, \qquad x = \max(0, n - B), \ldots, \min(n, A)$$

Moments

$$\begin{aligned} E(X) &= np \\ \mathrm{var}(X) &= np(1 - p) \cdot \frac{N - n}{N - 1} \end{aligned}$$

where

$$\begin{aligned} p &= \frac{A}{A + B} \\ N &= A + B \end{aligned} \tag{7.1}$$

Binomial Approximation If n is small compared to either A or B, then

$$\mathrm{Hypergeometric}(n, A, B) \approx \mathrm{Bin}(n, p)$$

where p is given by (7.1).

Normal Approximation If n is large, but small compared to either A or B, then

$$\text{Hypergeometric}(n, A, B) \approx \mathcal{N}\big(np, np(1-p)\big)$$

where p is given by (7.1).

Theorem The fact that the probability mass function sums to one is equivalent to

$$\sum_{x=\max(0,n-B)}^{\min(A,n)} \binom{A}{x}\binom{B}{n-x} = \binom{A+B}{n}$$

8 Poisson Distribution

Abbreviation Poi(μ)

Type Discrete.

Rationale Counts in a Poisson process.

Parameter Real number $\mu > 0$.

Sample Space The non-negative integers 0, 1, ...

Probability Mass Function

$$f(x) = \frac{\mu^x}{x!} e^{-\mu}, \qquad x = 0, 1, \ldots$$

Moments

$$E(X) = \mu$$
$$\operatorname{var}(X) = \mu$$

Addition Rule If X_1, ..., X_k are independent random variables, X_i being Poi(μ_i) distributed, then $X_1 + \cdots + X_k$ is a Poi($\mu_1 + \cdots + \mu_k$) random variable.

Normal Approximation If μ is large, then

$$\text{Poi}(\mu) \approx \mathcal{N}(\mu, \mu)$$

Theorem The fact that the probability mass function sums to one is equivalent to the Maclaurin series for the exponential function: for any real number x

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

9 Geometric Distribution

Abbreviation Geo(p).

Type Discrete.

Rationales

- Discrete lifetime of object that does not age.
- Waiting time or interarrival time in sequence of IID Bernoulli trials.
- Inverse sampling.
- Discrete analog of the exponential distribution.

Parameter Real number $0 < p \le 1$.

Sample Space The non-negative integers 0, 1, ...

Probability Mass Function

$$f(x) = p(1-p)^x \qquad x = 0, 1, \dots$$

Moments

$$E(X) = \frac{1-p}{p}$$
$$\operatorname{var}(X) = \frac{1-p}{p^2}$$

Addition Rule If X_1, ..., X_k are IID Geo(p) random variables, then $X_1 + \cdots + X_k$ is a NegBin(k, p) random variable.

Theorem The fact that the probability mass function sums to one is equivalent to the geometric series: for any real number s such that $|s| < 1$

$$\sum_{k=0}^{\infty} s^k = \frac{1}{1-s}.$$

Degeneracy If $p = 1$ the distribution is concentrated at 0.

10 Negative Binomial Distribution

Abbreviation $\text{NegBin}(r, p)$.

Type Discrete.

Rationale

- Sum of IID geometric random variables.
- Inverse sampling.
- Gamma mixture of Poisson distributions.

Parameters Real number $0 < p \leq 1$. Integer $r \geq 1$.

Sample Space The non-negative integers 0, 1, ...

Probability Mass Function

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \qquad x = 0, 1, \ldots$$

Moments

$$E(X) = \frac{r(1-p)}{p}$$
$$\text{var}(X) = \frac{r(1-p)}{p^2}$$

Addition Rule If X_1, ..., X_k are independent random variables, X_i being $\text{NegBin}(r_i, p)$ distributed, then $X_1 + \cdots + X_k$ is a $\text{NegBin}(r_1 + \cdots + r_k, p)$ random variable.

Normal Approximation If $r(1-p)$ is large, then

$$\text{NegBin}(r,p) \approx \mathcal{N}\left(\frac{r(1-p)}{p}, \frac{r(1-p)}{p^2}\right)$$

Degeneracy If $p = 1$ the distribution is concentrated at 0.

Extended Definition The definition makes sense for noninteger r if binomial coefficients are defined by

$$\binom{r}{k} = \frac{r \cdot (r-1) \cdots (r-k+1)}{k!}$$

which for integer r agrees with the standard definition.

Also

$$\binom{r+x-1}{x} = (-1)^x \binom{-r}{x} \tag{10.1}$$

which explains the name "negative binomial."

Theorem The fact that the probability mass function sums to one is equivalent to the **generalized binomial theorem:** for any real number s such that $-1 < s < 1$ and any real number m

$$\sum_{k=0}^{\infty} \binom{m}{k} s^k = (1+s)^m. \tag{10.2}$$

If m is a nonnegative integer, then $\binom{m}{k}$ is zero for $k > m$, and we get the ordinary binomial theorem.

Changing variables from m to $-m$ and from s to $-s$ and using (10.1) turns (10.2) into

$$\sum_{k=0}^{\infty} \binom{m+k-1}{k} s^k = \sum_{k=0}^{\infty} \binom{-m}{k} (-s)^k = (1-s)^{-m}$$

which has a more obvious relationship to the negative binomial density summing to one.

11 Normal Distribution

Abbreviation $\mathcal{N}(\mu, \sigma^2)$.

Type Continuous.

Rationale

- Limiting distribution in the central limit theorem.
- Error distribution that turns the method of least squares into maximum likelihood estimation.

Parameters Real numbers μ and $\sigma^2 > 0$.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \qquad -\infty < x < \infty$$

Moments

$$\begin{aligned} E(X) &= \mu \\ \operatorname{var}(X) &= \sigma^2 \\ E\{(X-\mu)^3\} &= 0 \\ E\{(X-\mu)^4\} &= 3\sigma^4 \end{aligned}$$

Linear Transformations If X is $\mathcal{N}(\mu, \sigma^2)$ distributed, then $aX + b$ is $\mathcal{N}(a\mu + b, a^2\sigma^2)$ distributed.

Addition Rule If $X_1, \ldots, X_k$ are independent random variables, X_i being $\mathcal{N}(\mu_i, \sigma_i^2)$ distributed, then $X_1 + \cdots + X_k$ is a $\mathcal{N}(\mu_1 + \cdots + \mu_k, \sigma_1^2 + \cdots + \sigma_k^2)$ random variable.

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_{-\infty}^{\infty} e^{-z^2/2} \, dz = \sqrt{2\pi}$$

Relation to Other Distributions If Z is $\mathcal{N}(0,1)$ distributed, then Z^2 is $\operatorname{Gam}(\frac{1}{2}, \frac{1}{2}) = \operatorname{chi}^2(1)$ distributed. Also related to Student t, Snedecor F, and Cauchy distributions (for which see).

12 Exponential Distribution

Abbreviation Exp(λ).

Type Continuous.

Rationales

- Lifetime of object that does not age.
- Waiting time or interarrival time in Poisson process.
- Continuous analog of the geometric distribution.

Parameter Real number $\lambda > 0$.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty$$

Cumulative Distribution Function

$$F(x) = 1 - e^{-\lambda x}, \qquad 0 < x < \infty$$

Moments

$$\begin{aligned} E(X) &= \frac{1}{\lambda} \\ \operatorname{var}(X) &= \frac{1}{\lambda^2} \end{aligned}$$

Addition Rule If X_1, ..., X_k are IID Exp(λ) random variables, then $X_1 + \cdots + X_k$ is a Gam(k, λ) random variable.

Relation to Other Distributions $\text{Exp}(\lambda) = \text{Gam}(1, \lambda)$.

13 Gamma Distribution

Abbreviation Gam(α, λ).

Type Continuous.

Rationales

- Sum of IID exponential random variables.
- Conjugate prior for exponential, Poisson, or normal precision family.

Parameter Real numbers $\alpha > 0$ and $\lambda > 0$.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\lambda x}, \qquad 0 < x < \infty$$

where $\Gamma(\alpha)$ is defined by (13.1) below.

Moments

$$\begin{aligned} E(X) &= \frac{\alpha}{\lambda} \\ \operatorname{var}(X) &= \frac{\alpha}{\lambda^2} \end{aligned}$$

Addition Rule If X_1, $\ldots$, X_k are independent random variables, X_i being $\text{Gam}(\alpha_i, \lambda)$ distributed, then $X_1 + \cdots + X_k$ is a $\text{Gam}(\alpha_1 + \cdots + \alpha_k, \lambda)$ random variable.

Normal Approximation If α is large, then

$$\text{Gam}(\alpha, \lambda) \approx \mathcal{N}\left(\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda^2}\right)$$

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_0^\infty x^{\alpha - 1} e^{-\lambda x} \, dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

the case $\lambda = 1$ is the definition of the *gamma function*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha - 1} e^{-x} \, dx \tag{13.1}$$

Relation to Other Distributions

- $\text{Exp}(\lambda) = \text{Gam}(1, \lambda)$.
- $\text{chi}^2(\nu) = \text{Gam}(\frac{\nu}{2}, \frac{1}{2})$.
- If X and Y are independent, X is $\Gamma(\alpha_1, \lambda)$ distributed and Y is $\Gamma(\alpha_2, \lambda)$ distributed, then $X/(X + Y)$ is $\text{Beta}(\alpha_1, \alpha_2)$ distributed.
- If Z is $\mathcal{N}(0, 1)$ distributed, then Z^2 is $\text{Gam}(\frac{1}{2}, \frac{1}{2})$ distributed.

Facts About Gamma Functions Integration by parts in (13.1) establishes the **gamma function recursion formula**

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \qquad \alpha > 0 \tag{13.2}$$

The relationship between the $\text{Exp}(\lambda)$ and $\text{Gam}(1, \lambda)$ distributions gives

$$\Gamma(1) = 1$$

and the relationship between the $\mathcal{N}(0, 1)$ and $\text{Gam}(\frac{1}{2}, \frac{1}{2})$ distributions gives

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Together with the recursion (13.2) these give for any positive integer n

$$\Gamma(n + 1) = n!$$

and

$$\Gamma(n + \tfrac{1}{2}) = \left(n - \tfrac{1}{2}\right)\left(n - \tfrac{3}{2}\right) \cdots \tfrac{3}{2} \cdot \tfrac{1}{2}\sqrt{\pi}$$

14 Beta Distribution

Abbreviation $\text{Beta}(\alpha_1, \alpha_2)$.

Type Continuous.

Rationales

- Ratio of gamma random variables.
- Conjugate prior for binomial or negative binomial family.

Parameter Real numbers $\alpha_1 > 0$ and $\alpha_2 > 0$.

Sample Space The interval $(0, 1)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1} \qquad 0 < x < 1$$

where $\Gamma(\alpha)$ is defined by (13.1) above.

Moments

$$\begin{aligned} E(X) &= \frac{\alpha_1}{\alpha_1 + \alpha_2} \\ \operatorname{var}(X) &= \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2(\alpha_1 + \alpha_2 + 1)} \end{aligned}$$

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_0^1 x^{\alpha_1 - 1}(1 - x)^{\alpha_2 - 1}\, dx = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

Relation to Other Distributions

- If X and Y are independent, X is $\Gamma(\alpha_1, \lambda)$ distributed and Y is $\Gamma(\alpha_2, \lambda)$ distributed, then $X/(X + Y)$ is $\operatorname{Beta}(\alpha_1, \alpha_2)$ distributed.
- $\operatorname{Beta}(1, 1) = \operatorname{Unif}(0, 1)$.

15 Multinomial Distribution

Abbreviation $\operatorname{Multi}(n, \mathbf{p})$.

Type Discrete.

Rationale Multivariate analog of the binomial distribution.

Parameters Real vector $\mathbf{p}$ in the parameter space

$$\left\{ \mathbf{p} \in \mathbb{R}^k : 0 \le p_i, \ i = 1, \ldots, k, \text{ and } \sum_{i=1}^{k} p_i = 1 \right\} \tag{15.1}$$

(real vectors whose components are nonnegative and sum to one).

Sample Space The set of vectors

$$S = \left\{ \mathbf{x} \in \mathbb{Z}^k : 0 \le x_i, \ i = 1, \ldots, k, \text{ and } \sum_{i=1}^{k} x_i = n \right\} \tag{15.2}$$

(integer vectors whose components are nonnegative and sum to n).

Probability Mass Function

$$f(\mathbf{x}) = \binom{n}{\mathbf{x}} \prod_{i=1}^{k} p_i^{x_i}, \qquad \mathbf{x} \in S$$

where

$$\binom{n}{\mathbf{x}} = \frac{n!}{\prod_{i=1}^{k} x_i!}$$

is called a *multinomial coefficient*.

Moments

$$\begin{aligned} E(X_i) &= np_i \\ \operatorname{var}(X_i) &= np_i(1 - p_i) \\ \operatorname{cov}(X_i, X_j) &= -np_i p_j, \qquad i \ne j \end{aligned}$$

Moments (Vector Form)

$$\begin{aligned} E(\mathbf{X}) &= n\mathbf{p} \\ \operatorname{var}(\mathbf{X}) &= n\mathbf{M} \end{aligned}$$

where

$$\mathbf{M} = \mathbf{P} - \mathbf{p}\mathbf{p}^T$$

where $\mathbf{P}$ is the diagonal matrix whose vector of diagonal elements is $\mathbf{p}$.

Addition Rule If $\mathbf{X}_1$, ..., $\mathbf{X}_k$ are independent random vectors, $\mathbf{X}_i$ being $\text{Multi}(n_i, \mathbf{p})$ distributed, then $\mathbf{X}_1 + \cdots + \mathbf{X}_k$ is a $\text{Multi}(n_1 + \cdots + n_k, \mathbf{p})$ random variable.

Normal Approximation If n is large and $\mathbf{p}$ is not near the boundary of the parameter space (15.1), then

$$\text{Multi}(n, \mathbf{p}) \approx \mathcal{N}(n\mathbf{p}, n\mathbf{M})$$

Theorem The fact that the probability mass function sums to one is equivalent to the **multinomial theorem:** for any vector $\mathbf{a}$ of real numbers

$$\sum_{\mathbf{x} \in S} \binom{n}{\mathbf{x}} \prod_{i=1}^{k} a_i^{x_i} = (a_1 + \cdots + a_k)^n$$

Degeneracy If a vector $\mathbf{a}$ exists such that $\mathbf{M}\mathbf{a} = 0$, then $\text{var}(\mathbf{a}^T \mathbf{X}) = 0$.

In particular, the vector $\mathbf{u} = (1, 1, \ldots, 1)$ always satisfies $\mathbf{M}\mathbf{u} = 0$, so $\text{var}(\mathbf{u}^T \mathbf{X}) = 0$. This is obvious, since $\mathbf{u}^T \mathbf{X} = \sum_{i=1}^{k} X_i = n$ by definition of the multinomial distribution, and the variance of a constant is zero. This means a multinomial random vector of dimension k is “really” of dimension no more than $k - 1$ because it is concentrated on a hyperplane containing the sample space (15.2).

Marginal Distributions Every univariate marginal is binomial

$$X_i \sim \text{Bin}(n, p_i)$$

Not, strictly speaking marginals, but random vectors formed by collapsing categories are multinomial. If A_1, ..., A_m is a partition of the set $\{1, \ldots, k\}$ and

$$Y_j = \sum_{i \in A_j} X_i, \qquad j = 1, \ldots, m$$

$$q_j = \sum_{i \in A_j} p_i, \qquad j = 1, \ldots, m$$

then the random vector $\mathbf{Y}$ has a $\text{Multi}(n, \mathbf{q})$ distribution.

Conditional Distributions If $\{i_1, \ldots, i_m\}$ and $\{i_{m+1}, \ldots, i_k\}$ partition the set $\{1, \ldots, k\}$, then the conditional distribution of X_{i_1}, ..., X_{i_m} given $X_{i_{m+1}}$, ..., X_{i_k} is $\text{Multi}(n - X_{i_{m+1}} - \cdots - X_{i_k}, \mathbf{q})$, where the parameter vector $\mathbf{q}$ has components

$$q_j = \frac{p_{i_j}}{p_{i_1} + \cdots + p_{i_m}}, \qquad j = 1, \ldots, m$$

Relation to Other Distributions

- Each marginal of a multinomial is binomial.
- If X is $\text{Bin}(n, p)$, then the vector $(X, n - X)$ is $\text{Multi}\big(n, (p, 1 - p)\big)$.

16 Bivariate Normal Distribution

Abbreviation See multivariate normal below.

Type Continuous.

Rationales See multivariate normal below.

Parameters Real vector $\boldsymbol{\mu}$ of dimension 2, real symmetric positive semi-definite matrix $\mathbf{M}$ of dimension 2×2 having the form

$$\mathbf{M} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < +1$.

Sample Space The Euclidean space $\mathbb{R}^2$.

Probability Density Function

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2\pi} \det(\mathbf{M})^{-1/2} \exp\left(-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{M}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right) \\ &= \frac{1}{2\pi\sqrt{1 - \rho^2}\sigma_1\sigma_2} \exp\Bigg(-\frac{1}{2(1 - \rho^2)} \Bigg[\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 \\ &\qquad -2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2\Bigg]\Bigg), \qquad \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

Moments

$$
\begin{aligned}
E(X_i) &= \mu_i, \qquad i = 1, 2 \\
\operatorname{var}(X_i) &= \sigma_i^2, \qquad i = 1, 2 \\
\operatorname{cov}(X_1, X_2) &= \rho \sigma_1 \sigma_2 \\
\operatorname{cor}(X_1, X_2) &= \rho
\end{aligned}
$$

Moments (Vector Form)

$$
\begin{aligned}
E(\mathbf{X}) &= \boldsymbol{\mu} \\
\operatorname{var}(\mathbf{X}) &= \mathbf{M}
\end{aligned}
$$

Linear Transformations See multivariate normal below.

Addition Rule See multivariate normal below.

Marginal Distributions X_i is $\mathcal{N}(\mu_i, \sigma_i^2)$ distributed, $i = 1$, 2.

Conditional Distributions The conditional distribution of X_2 given X_1 is

$$
\mathcal{N}\Bigl(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), (1 - \rho^2)\sigma_2^2\Bigr)
$$

17 Multivariate Normal Distribution

Abbreviation $\mathcal{N}(\boldsymbol{\mu}, \mathbf{M})$

Type Continuous.

Rationales

- Multivariate analog of the univariate normal distribution.
- Limiting distribution in the multivariate central limit theorem.

Parameters Real vector $\boldsymbol{\mu}$ of dimension k, real symmetric positive semi-definite matrix $\mathbf{M}$ of dimension $k \times k$.

Sample Space The Euclidean space $\mathbb{R}^k$.

Probability Density Function If $\mathbf{M}$ is (strictly) positive definite,

$$f(\mathbf{x}) = (2\pi)^{-k/2} \det(\mathbf{M})^{-1/2} \exp\left(-\tfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{M}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right), \qquad \mathbf{x} \in \mathbb{R}^k$$

Otherwise there is no density ($\mathbf{X}$ is concentrated on a hyperplane).

Moments (Vector Form)

$$\begin{aligned} E(\mathbf{X}) &= \boldsymbol{\mu} \\ \operatorname{var}(\mathbf{X}) &= \mathbf{M} \end{aligned}$$

Linear Transformations If $\mathbf{X}$ is $\mathcal{N}(\boldsymbol{\mu}, \mathbf{M})$ distributed, then $\mathbf{a} + \mathbf{B}\mathbf{X}$, where $\mathbf{a}$ is a constant vector and $\mathbf{B}$ is a constant matrix of dimensions such that the vector addition and matrix multiplication make sense, has the $\mathcal{N}(\mathbf{a} + \mathbf{B}\boldsymbol{\mu}, \mathbf{B}\mathbf{M}\mathbf{B}^T)$ distribution.

Addition Rule If $\mathbf{X}_1, \ldots, \mathbf{X}_k$ are independent random vectors, $\mathbf{X}_i$ being $\mathcal{N}(\boldsymbol{\mu}_i, \mathbf{M}_i)$ distributed, then $\mathbf{X}_1 + \cdots + \mathbf{X}_k$ is a $\mathcal{N}(\boldsymbol{\mu}_1 + \cdots + \boldsymbol{\mu}_k, \mathbf{M}_1 + \cdots + \mathbf{M}_k)$ random variable.

Degeneracy If a vector $\mathbf{a}$ exists such that $\mathbf{M}\mathbf{a} = 0$, then $\operatorname{var}(\mathbf{a}^T\mathbf{X}) = 0$.

Partitioned Vectors and Matrices The random vector and parameters are written in *partitioned form*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \tag{17.1a}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \tag{17.1b}$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_2 \end{pmatrix} \tag{17.1c}$$

when $\mathbf{X}_1$ consists of the first r elements of $\mathbf{X}$ and $\mathbf{X}_2$ of the other $k - r$ elements and similarly for $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$.

Marginal Distributions Every marginal of a multivariate normal is normal (univariate or multivariate as the case may be). In partitioned form, the (marginal) distribution of $\mathbf{X}_1$ is $\mathcal{N}(\boldsymbol{\mu}_1, \mathbf{M}_{11})$.

Conditional Distributions Every conditional of a multivariate normal is normal (univariate or multivariate as the case may be). In partitioned form, the conditional distribution of $\mathbf{X}_1$ given $\mathbf{X}_2$ is

$$\mathcal{N}(\boldsymbol{\mu}_1 + \mathbf{M}_{12}\mathbf{M}_{22}^{-}[\mathbf{X}_2 - \boldsymbol{\mu}_2], \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^{-}\mathbf{M}_{21})$$

where the notation $\mathbf{M}_{22}^{-}$ denotes the inverse of the matrix $\mathbf{M}_{22}^{-}$ if the matrix is invertible and otherwise any generalized inverse.

18 Chi-Square Distribution

Abbreviation $\text{chi}^2(\nu)$ or $\chi^2(\nu)$.

Type Continuous.

Rationales

- Sum of squares of IID standard normal random variables.
- Sampling distribution of sample variance when data are IID normal.
- Asymptotic distribution in Pearson chi-square test.
- Asymptotic distribution of log likelihood ratio.

Parameter Real number $\nu > 0$ called "degrees of freedom."

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{(\frac{1}{2})^{\nu/2}}{\Gamma(\frac{\nu}{2})} x^{\nu/2-1} e^{-x/2}, \qquad 0 < x < \infty.$$

Moments

$$E(X) = \nu$$
$$\text{var}(X) = 2\nu$$

Addition Rule If X_1, ..., X_k are independent random variables, X_i being $\text{chi}^2(\nu_i)$ distributed, then $X_1 + \cdots + X_k$ is a $\text{chi}^2(\nu_1 + \cdots + \nu_k)$ random variable.

Normal Approximation If ν is large, then

$$\text{chi}^2(\nu) \approx \mathcal{N}(\nu, 2\nu)$$

Relation to Other Distributions

- $\text{chi}^2(\nu) = \text{Gam}(\frac{\nu}{2}, \frac{1}{2})$.
- If X is $\mathcal{N}(0,1)$ distributed, then X^2 is $\text{chi}^2(1)$ distributed.
- If Z and Y are independent, X is $\mathcal{N}(0,1)$ distributed and Y is $\text{chi}^2(\nu)$ distributed, then $X/\sqrt{Y/\nu}$ is $t(\nu)$ distributed.
- If X and Y are independent and are $\text{chi}^2(\mu)$ and $\text{chi}^2(\nu)$ distributed, respectively, then $(X/\mu)/(Y/\nu)$ is $F(\mu,\nu)$ distributed.

19 Student's t Distribution

Abbreviation $t(\nu)$.

Type Continuous.

Rationales

- Sampling distribution of pivotal quantity $\sqrt{n}(\overline{X}_n - \mu)/S_n$ when data are IID normal.
- Marginal for μ in conjugate prior family for two-parameter normal data.

Parameter Real number $\nu > 0$ called "degrees of freedom."

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\sqrt{\nu\pi}} \cdot \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \cdot \frac{1}{\left(1 + \frac{x^2}{\nu}\right)^{(\nu+1)/2}}, \qquad -\infty < x < +\infty$$

Moments If $\nu > 1$, then

$$E(X) = 0.$$

Otherwise the mean does not exist. If $\nu > 2$, then

$$\operatorname{var}(X) = \frac{\nu}{\nu - 2}.$$

Otherwise the variance does not exist.

Normal Approximation If ν is large, then

$$t(\nu) \approx \mathcal{N}(0, 1)$$

Relation to Other Distributions

- If X and Y are independent, X is $\mathcal{N}(0, 1)$ distributed and Y is $\text{chi}^2(\nu)$ distributed, then $X/\sqrt{Y/\nu}$ is $t(\nu)$ distributed.
- If X is $t(\nu)$ distributed, then X^2 is $F(1, \nu)$ distributed.
- $t(1) = \text{Cauchy}(0, 1)$.

20 Snedecor's F Distribution

Abbreviation $F(\mu, \nu)$.

Type Continuous.

Rationale

- Ratio of sums of squares for normal data (test statistics in regression and analysis of variance).

Parameters Real numbers $\mu > 0$ and $\nu > 0$ called "numerator degrees of freedom" and "denominator degrees of freedom," respectively.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\Gamma(\frac{\mu+\nu}{2}) \mu^{\mu/2} \nu^{\nu/2}}{\Gamma(\frac{\mu}{2}) \Gamma(\frac{\nu}{2})} \cdot \frac{x^{\mu/2-1}}{(\mu x + \nu)^{(\mu+\nu)/2}}, \qquad 0 < x < +\infty$$

Moments If $\nu > 2$, then

$$E(X) = \frac{\nu}{\nu - 2}.$$

Otherwise the mean does not exist.

Relation to Other Distributions

- If X and Y are independent and are $\text{chi}^2(\mu)$ and $\text{chi}^2(\nu)$ distributed, respectively, then $(X/\mu)/(Y/\nu)$ is $F(\mu, \nu)$ distributed.
- If X is $t(\nu)$ distributed, then X^2 is $F(1, \nu)$ distributed.

21 Cauchy Distribution

Abbreviation Cauchy(μ, σ).

Type Continuous.

Rationales

- Very heavy tailed distribution.
- Counterexample to law of large numbers.

Parameters Real numbers μ and $\sigma > 0$.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\pi\sigma} \cdot \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < +\infty$$

Moments No moments exist.

Addition Rule If X_1, …, X_k are IID Cauchy(μ, σ) random variables, then $\overline{X}_n = (X_1 + \cdots + X_k)/n$ is also Cauchy(μ, σ).

Relation to Other Distributions

- $t(1) = \text{Cauchy}(0, 1)$.

22 Laplace Distribution

Abbreviation Laplace(μ, σ).

Type Continuous.

Rationales The sample median is the maximum likelihood estimate of the location parameter.

Parameters Real numbers μ and $\sigma > 0$, called the mean and standard deviation, respectively.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left|\frac{x-\mu}{\sigma}\right|\right), \qquad -\infty < x < \infty$$

Moments

$$\begin{aligned} E(X) &= \mu \\ \text{var}(X) &= \sigma^2 \end{aligned}$$