

Stat 5101 Notes: Brand Name Distributions

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1 Discrete Uniform Distribution

Abbreviation DiscUnif(n).

Type Discrete.

Rationale Equally likely outcomes.

Sample Space The interval $1, 2, \dots, n$ of the integers.

Probability Mass Function

$$f(x) = \frac{1}{n}, \quad x = 1, 2, \dots, n$$

Moments

$$E(X) = \frac{n+1}{2}$$
$$\text{var}(X) = \frac{n^2-1}{12}$$

2 General Discrete Uniform Distribution

Type Discrete.

Sample Space Any finite set S .

Probability Mass Function

$$f(x) = \frac{1}{n}, \quad x \in S,$$

where n is the number of elements of S .

3 Uniform Distribution

Abbreviation Unif(a, b).

Type Continuous.

Rationale Continuous analog of the discrete uniform distribution.

Parameters Real numbers a and b with $a < b$.

Sample Space The interval (a, b) of the real numbers.

Probability Density Function

$$f(x) = \frac{1}{b-a}, \quad a < x < b$$

Moments

$$E(X) = \frac{a+b}{2}$$
$$\text{var}(X) = \frac{(b-a)^2}{12}$$

Relation to Other Distributions Beta(1, 1) = Unif(0, 1).

4 General Uniform Distribution

Type Continuous.

Sample Space Any open set S in \mathbb{R}^n .

Probability Density Function

$$f(x) = \frac{1}{c}, \quad x \in S$$

where c is the measure (length in one dimension, area in two, volume in three, etc.) of the set S .

5 Bernoulli Distribution

Abbreviation $\text{Ber}(p)$.

Type Discrete.

Rationale Any zero-or-one-valued random variable.

Parameter Real number $0 \leq p \leq 1$.

Sample Space The two-element set $\{0, 1\}$.

Probability Mass Function

$$f(x) = \begin{cases} p, & x = 1 \\ 1 - p, & x = 0 \end{cases}$$

Moments

$$\begin{aligned} E(X) &= p \\ \text{var}(X) &= p(1 - p) \end{aligned}$$

Addition Rule If X_1, \dots, X_k are IID $\text{Ber}(p)$ random variables, then $X_1 + \dots + X_k$ is a $\text{Bin}(k, p)$ random variable.

Relation to Other Distributions $\text{Ber}(p) = \text{Bin}(1, p)$.

6 Binomial Distribution

Abbreviation $\text{Bin}(n, p)$.

Type Discrete.

Rationale Sum of IID Bernoulli random variables.

Parameters Real number $0 \leq p \leq 1$. Integer $n \geq 1$.

Sample Space The interval $0, 1, \dots, n$ of the integers.

Probability Mass Function

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$$

Moments

$$\begin{aligned} E(X) &= np \\ \text{var}(X) &= np(1-p) \end{aligned}$$

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\text{Bin}(n_i, p)$ distributed, then $X_1 + \dots + X_k$ is a $\text{Bin}(n_1 + \dots + n_k, p)$ random variable.

Normal Approximation If np and $n(1-p)$ are both large, then

$$\text{Bin}(n, p) \approx \mathcal{N}(np, np(1-p))$$

Poisson Approximation If n is large but np is small, then

$$\text{Bin}(n, p) \approx \text{Poi}(np)$$

Theorem The fact that the probability mass function sums to one is equivalent to the **binomial theorem**: for any real numbers a and b

$$\sum_{k=0}^n \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

Degeneracy If $p = 0$ the distribution is concentrated at 0. If $p = 1$ the distribution is concentrated at n .

Relation to Other Distributions $\text{Ber}(p) = \text{Bin}(1, p)$.

7 Hypergeometric Distribution

Abbreviation Hypergeometric(A, B, n).

Type Discrete.

Rationale Sample of size n without replacement from finite population of B zeros and A ones.

Sample Space The interval $\max(0, n - B), \dots, \min(n, A)$ of the integers.

Probability Mass Function

$$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}}, \quad x = \max(0, n - B), \dots, \min(n, A)$$

Moments

$$\begin{aligned} E(X) &= np \\ \text{var}(X) &= np(1-p) \cdot \frac{N-n}{N-1} \end{aligned}$$

where

$$\begin{aligned} p &= \frac{A}{A+B} \\ N &= A+B \end{aligned} \tag{7.1}$$

Binomial Approximation If n is small compared to either A or B , then

$$\text{Hypergeometric}(n, A, B) \approx \text{Bin}(n, p)$$

where p is given by (7.1).

Normal Approximation If n is large, but small compared to either A or B , then

$$\text{Hypergeometric}(n, A, B) \approx \mathcal{N}(np, np(1-p))$$

where p is given by (7.1).

Theorem The fact that the probability mass function sums to one is equivalent to

$$\sum_{x=\max(0,n-B)}^{\min(A,n)} \binom{A}{x} \binom{B}{n-x} = \binom{A+B}{n}$$

8 Poisson Distribution

Abbreviation $\text{Poi}(\mu)$

Type Discrete.

Rationale Counts in a Poisson process.

Parameter Real number $\mu > 0$.

Sample Space The non-negative integers $0, 1, \dots$

Probability Mass Function

$$f(x) = \frac{\mu^x}{x!} e^{-\mu}, \quad x = 0, 1, \dots$$

Moments

$$\begin{aligned} E(X) &= \mu \\ \text{var}(X) &= \mu \end{aligned}$$

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\text{Poi}(\mu_i)$ distributed, then $X_1 + \dots + X_k$ is a $\text{Poi}(\mu_1 + \dots + \mu_k)$ random variable.

Normal Approximation If μ is large, then

$$\text{Poi}(\mu) \approx \mathcal{N}(\mu, \mu)$$

Theorem The fact that the probability mass function sums to one is equivalent to the Maclaurin series for the exponential function: for any real number x

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

9 Geometric Distribution

Abbreviation $\text{Geo}(p)$.

Type Discrete.

Rationales

- Discrete lifetime of object that does not age.
- Waiting time or interarrival time in sequence of IID Bernoulli trials.
- Inverse sampling.
- Discrete analog of the exponential distribution.

Parameter Real number $0 < p < 1$.

Sample Space The non-negative integers $0, 1, \dots$

Probability Mass Function

$$f(x) = p(1 - p)^x \quad x = 0, 1, \dots$$

Moments

$$E(X) = \frac{1 - p}{p}$$
$$\text{var}(X) = \frac{1 - p}{p^2}$$

Addition Rule If X_1, \dots, X_k are IID $\text{Geo}(p)$ random variables, then $X_1 + \dots + X_k$ is a $\text{NegBin}(k, p)$ random variable.

Theorem The fact that the probability mass function sums to one is equivalent to the geometric series: for any real number s such that $-1 < s < 1$

$$\sum_{k=0}^{\infty} s^k = \frac{1}{1 - s}.$$

10 Negative Binomial Distribution

Abbreviation $\text{NegBin}(r, p)$.

Type Discrete.

Rationale

- Sum of IID geometric random variables.
- Inverse sampling.
- Gamma mixture of Poisson distributions.

Parameters Real number $0 \leq p \leq 1$. Integer $r \geq 1$.

Sample Space The non-negative integers $0, 1, \dots$

Probability Mass Function

$$f(x) = \binom{r+x-1}{x} p^r (1-p)^x, \quad x = 0, 1, \dots$$

Moments

$$E(X) = \frac{r(1-p)}{p}$$
$$\text{var}(X) = \frac{r(1-p)}{p^2}$$

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\text{NegBin}(r_i, p)$ distributed, then $X_1 + \dots + X_k$ is a $\text{NegBin}(r_1 + \dots + r_k, p)$ random variable.

Normal Approximation If $r(1-p)$ is large, then

$$\text{NegBin}(r, p) \approx \mathcal{N}\left(\frac{r(1-p)}{p}, \frac{r(1-p)}{p^2}\right)$$

Extended Definition The definition makes sense for noninteger r if binomial coefficients are defined by

$$\binom{r}{k} = \frac{r \cdot (r-1) \cdots (r-k+1)}{k!}$$

which for integer r agrees with the standard definition.

Also

$$\binom{r+x-1}{x} = (-1)^x \binom{-r}{x} \quad (10.1)$$

which explains the name “negative binomial.”

Theorem The fact that the probability mass function sums to one is equivalent to the **generalized binomial theorem**: for any real number s such that $-1 < s < 1$ and any real number m

$$\sum_{k=0}^{\infty} \binom{m}{k} s^k = (1+s)^m. \quad (10.2)$$

If m is a nonnegative integer, then $\binom{m}{k}$ is zero for $k > m$, and we get the ordinary binomial theorem.

Changing variables from m to $-m$ and from s to $-s$ and using (10.1) turns (10.2) into

$$\sum_{k=0}^{\infty} \binom{m+k-1}{k} s^k = \sum_{k=0}^{\infty} \binom{-m}{k} (-s)^k = (1-s)^{-m}$$

which has a more obvious relationship to the negative binomial density summing to one.

11 Normal Distribution

Abbreviation $\mathcal{N}(\mu, \sigma^2)$.

Type Continuous.

Rationale

- Limiting distribution in the central limit theorem.
- Error distribution that turns the method of least squares into maximum likelihood estimation.

Parameters Real numbers μ and $\sigma^2 > 0$.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

Moments

$$\begin{aligned} E(X) &= \mu \\ \text{var}(X) &= \sigma^2 \\ E\{(X - \mu)^3\} &= 0 \\ E\{(X - \mu)^4\} &= 3\sigma^4 \end{aligned}$$

Linear Transformations If X is $\mathcal{N}(\mu, \sigma^2)$ distributed, then $aX + b$ is $\mathcal{N}(a\mu + b, a^2\sigma^2)$ distributed.

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\mathcal{N}(\mu_i, \sigma_i^2)$ distributed, then $X_1 + \dots + X_k$ is a $\mathcal{N}(\mu_1 + \dots + \mu_k, \sigma_1^2 + \dots + \sigma_k^2)$ random variable.

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_{-\infty}^{\infty} e^{-z^2/2} dz = \sqrt{2\pi}$$

Relation to Other Distributions If Z is $\mathcal{N}(0, 1)$ distributed, then Z^2 is $\text{Gam}(\frac{1}{2}, \frac{1}{2})$ distributed.

12 Exponential Distribution

Abbreviation $\text{Exp}(\lambda)$.

Type Continuous.

Rationales

- Lifetime of object that does not age.
- Waiting time or interarrival time in Poisson process.
- Continuous analog of the geometric distribution.

Parameter Real number $\lambda > 0$.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \lambda e^{-\lambda x}, \quad 0 < x < \infty$$

Cumulative Distribution Function

$$F(x) = 1 - e^{-\lambda x}, \quad 0 < x < \infty$$

Moments

$$E(X) = \frac{1}{\lambda}$$
$$\text{var}(X) = \frac{1}{\lambda^2}$$

Addition Rule If X_1, \dots, X_k are IID $\text{Exp}(\lambda)$ random variables, then $X_1 + \dots + X_k$ is a $\text{Gam}(k, \lambda)$ random variable.

Relation to Other Distributions $\text{Exp}(\lambda) = \text{Gam}(1, \lambda)$.

13 Gamma Distribution

Abbreviation $\text{Gam}(\alpha, \lambda)$.

Type Continuous.

Rationales

- Sum of IID exponential random variables.
- Conjugate prior for exponential, Poisson, or normal precision family.

Parameter Real numbers $\alpha > 0$ and $\lambda > 0$.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad 0 < x < \infty$$

where $\Gamma(\alpha)$ is defined by (13.1) below.

Moments

$$\begin{aligned} E(X) &= \frac{\alpha}{\lambda} \\ \text{var}(X) &= \frac{\alpha}{\lambda^2} \end{aligned}$$

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\text{Gam}(\alpha_i, \lambda)$ distributed, then $X_1 + \dots + X_k$ is a $\text{Gam}(\alpha_1 + \dots + \alpha_k, \lambda)$ random variable.

Normal Approximation If α is large, then

$$\text{Gam}(\alpha, \lambda) \approx \mathcal{N}\left(\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda^2}\right)$$

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_0^\infty x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^\alpha}$$

the case $\lambda = 1$ is the definition of the *gamma function*

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \tag{13.1}$$

Relation to Other Distributions

- $\text{Exp}(\lambda) = \text{Gam}(1, \lambda)$.
- $\text{chi}^2(\nu) = \text{Gam}(\frac{\nu}{2}, \frac{1}{2})$.
- If X and Y are independent, X is $\Gamma(\alpha_1, \lambda)$ distributed and Y is $\Gamma(\alpha_2, \lambda)$ distributed, then $X/(X + Y)$ is $\text{Beta}(\alpha_1, \alpha_2)$ distributed.
- If Z is $\mathcal{N}(0, 1)$ distributed, then Z^2 is $\text{Gam}(\frac{1}{2}, \frac{1}{2})$ distributed.

Facts About Gamma Functions Integration by parts in (13.1) establishes the **gamma function recursion formula**

$$\Gamma(\alpha + 1) = \alpha\Gamma(\alpha), \quad \alpha > 0 \quad (13.2)$$

The relationship between the $\text{Exp}(\lambda)$ and $\text{Gam}(1, \lambda)$ distributions gives

$$\Gamma(1) = 1$$

and the relationship between the $\mathcal{N}(0, 1)$ and $\text{Gam}(\frac{1}{2}, \frac{1}{2})$ distributions gives

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Together with the recursion (13.2) these give for any positive integer n

$$\Gamma(n + 1) = n!$$

and

$$\Gamma(n + \frac{1}{2}) = (n - \frac{1}{2})(n - \frac{3}{2}) \cdots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi}$$

14 Beta Distribution

Abbreviation $\text{Beta}(\alpha_1, \alpha_2)$.

Type Continuous.

Rationales

- Ratio of gamma random variables.
- Conjugate prior for binomial or negative binomial family.

Parameter Real numbers $\alpha_1 > 0$ and $\alpha_2 > 0$.

Sample Space The interval $(0, 1)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} (1-x)^{\alpha_2-1} \quad 0 < x < 1$$

where $\Gamma(\alpha)$ is defined by (13.1) above.

Moments

$$E(X) = \frac{\alpha_1}{\alpha_1 + \alpha_2}$$
$$\text{var}(X) = \frac{\alpha_1 \alpha_2}{(\alpha_1 + \alpha_2)^2 (\alpha_1 + \alpha_2 + 1)}$$

Theorem The fact that the probability density function integrates to one is equivalent to the integral

$$\int_0^1 x^{\alpha_1-1} (1-x)^{\alpha_2-1} dx = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}$$

Relation to Other Distributions

- If X and Y are independent, X is $\Gamma(\alpha_1, \lambda)$ distributed and Y is $\Gamma(\alpha_2, \lambda)$ distributed, then $X/(X + Y)$ is $\text{Beta}(\alpha_1, \alpha_2)$ distributed.
- $\text{Beta}(1, 1) = \text{Unif}(0, 1)$.

15 Multinomial Distribution

Abbreviation $\text{Multi}(n, \mathbf{p})$.

Type Discrete.

Rationale Multivariate analog of the binomial distribution.

Parameters Real vector \mathbf{p} in the parameter space

$$\left\{ \mathbf{p} \in \mathbb{R}^k : 0 \leq p_i, i = 1, \dots, k, \text{ and } \sum_{i=1}^k p_i = 1 \right\} \quad (15.1)$$

Sample Space The set of vectors with integer coordinates

$$S = \left\{ \mathbf{x} \in \mathbb{Z}^k : 0 \leq x_i, i = 1, \dots, k, \text{ and } \sum_{i=1}^k x_i = n \right\} \quad (15.2)$$

Probability Mass Function

$$f(\mathbf{x}) = \binom{n}{\mathbf{x}} \prod_{i=1}^k p_i^{x_i}, \quad \mathbf{x} \in S$$

where

$$\binom{n}{\mathbf{x}} = \frac{n!}{\prod_{i=1}^k x_i!}$$

is called a *multinomial coefficient*.

Moments

$$\begin{aligned} E(X_i) &= np_i \\ \text{var}(X_i) &= np_i(1 - p_i) \\ \text{cov}(X_i, X_j) &= -np_i p_j, \quad i \neq j \end{aligned}$$

Moments (Vector Form)

$$\begin{aligned} E(\mathbf{X}) &= n\mathbf{p} \\ \text{var}(\mathbf{X}) &= n\mathbf{M} \end{aligned}$$

where

$$\mathbf{M} = \text{diag}(\mathbf{p}) - \mathbf{p}\mathbf{p}'$$

is the matrix with elements $m_{ij} = \text{cov}(X_i, X_j)/n$.

Addition Rule If $\mathbf{X}_1, \dots, \mathbf{X}_k$ are independent random vectors, \mathbf{X}_i being $\text{Multi}(n_i, \mathbf{p})$ distributed, then $\mathbf{X}_1 + \dots + \mathbf{X}_k$ is a $\text{Multi}(n_1 + \dots + n_k, \mathbf{p})$ random variable.

Normal Approximation If n is large and \mathbf{p} is not near the boundary of the parameter space (15.1), then

$$\text{Multi}(n, \mathbf{p}) \approx \mathcal{N}(n\mathbf{p}, n\mathbf{M})$$

Theorem The fact that the probability mass function sums to one is equivalent to the **multinomial theorem**: for any vector \mathbf{a} of real numbers

$$\sum_{\mathbf{x} \in S} \left[\binom{n}{\mathbf{x}} \prod_{i=1}^k a_i^{x_i} \right] = (a_1 + \cdots + a_k)^n$$

Degeneracy If there exists a vector \mathbf{a} such that $\mathbf{M}\mathbf{a} = 0$, then $\text{var}(\mathbf{a}'\mathbf{X}) = 0$.

In particular, the vector $\mathbf{u} = (1, 1, \dots, 1)$ always satisfies $\mathbf{M}\mathbf{u} = 0$, so $\text{var}(\mathbf{u}'\mathbf{X}) = 0$. This is obvious, since $\mathbf{u}'\mathbf{X} = \sum_{i=1}^k X_i = n$ by definition of the multinomial distribution, and the variance of a constant is zero. This means a multinomial random vector of dimension k is “really” of dimension no more than $k - 1$ because it is concentrated on a hyperplane containing the sample space (15.2).

Marginal Distributions Every univariate marginal is binomial

$$X_i \sim \text{Bin}(n, p_i)$$

Not, strictly speaking marginals, but random vectors formed by collapsing categories are multinomial. If A_1, \dots, A_m is a partition of the set $\{1, \dots, k\}$ and

$$Y_j = \sum_{i \in A_j} X_i, \quad j = 1, \dots, m$$

$$q_j = \sum_{i \in A_j} p_i, \quad j = 1, \dots, m$$

then the random vector \mathbf{Y} has a $\text{Multi}(n, \mathbf{q})$ distribution.

Conditional Distributions If $\{i_1, \dots, i_m\}$ and $\{i_{m+1}, \dots, i_k\}$ partition the set $\{1, \dots, k\}$, then the conditional distribution of X_{i_1}, \dots, X_{i_m} given $X_{i_{m+1}}, \dots, X_{i_k}$ is $\text{Multi}(n - X_{i_{m+1}} - \cdots - X_{i_k}, \mathbf{q})$, where the parameter vector \mathbf{q} has components

$$q_j = \frac{p_{i_j}}{p_{i_1} + \cdots + p_{i_m}}, \quad j = 1, \dots, m$$

Relation to Other Distributions

- Each marginal of a multinomial is binomial.
- If X is $\text{Bin}(n, p)$, then the vector $(X, n - X)$ is $\text{Multi}(n, (p, 1 - p))$.

16 Bivariate Normal Distribution

Abbreviation See multivariate normal below.

Type Continuous.

Rationales See multivariate normal below.

Parameters Real vector $\boldsymbol{\mu}$ of dimension 2, real symmetric positive semi-definite matrix \mathbf{M} of dimension 2×2 having the form

$$\mathbf{M} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$ and $-1 < \rho < +1$.

Sample Space The Euclidean space \mathbb{R}^2 .

Probability Density Function

$$\begin{aligned} f(\mathbf{x}) &= \frac{1}{2\pi} \det(\mathbf{M})^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{M}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right) \\ &= \frac{1}{2\pi\sqrt{1 - \rho^2}\sigma_1\sigma_2} \exp\left(-\frac{1}{2(1 - \rho^2)} \left[\left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 \right. \right. \\ &\quad \left. \left. - 2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 \right] \right), \quad \mathbf{x} \in \mathbb{R}^2 \end{aligned}$$

Moments

$$\begin{aligned} E(X_i) &= \mu_i, & i &= 1, 2 \\ \text{var}(X_i) &= \sigma_i^2, & i &= 1, 2 \\ \text{cov}(X_1, X_2) &= \rho\sigma_1\sigma_2 \\ \text{cor}(X_1, X_2) &= \rho \end{aligned}$$

Moments (Vector Form)

$$E(\mathbf{X}) = \boldsymbol{\mu}$$
$$\text{var}(\mathbf{X}) = \mathbf{M}$$

Linear Transformations See multivariate normal below.

Addition Rule See multivariate normal below.

Marginal Distributions X_i is $\mathcal{N}(\mu_i, \sigma_i^2)$ distributed, $i = 1, 2$.

Conditional Distributions The conditional distribution of X_2 given X_1 is

$$\mathcal{N}\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x_1 - \mu_1), (1 - \rho^2)\sigma_2^2\right)$$

17 Multivariate Normal Distribution

Abbreviation $\mathcal{N}(\boldsymbol{\mu}, \mathbf{M})$

Type Continuous.

Rationales

- Multivariate analog of the univariate normal distribution.
- Limiting distribution in the multivariate central limit theorem.

Parameters Real vector $\boldsymbol{\mu}$ of dimension k , real symmetric positive semi-definite matrix \mathbf{M} of dimension $k \times k$.

Sample Space The Euclidean space \mathbb{R}^k .

Probability Density Function If \mathbf{M} is (strictly) positive definite,

$$f(\mathbf{x}) = (2\pi)^{-k/2} \det(\mathbf{M})^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \mathbf{M}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right), \quad \mathbf{x} \in \mathbb{R}^k$$

Otherwise there is no density (\mathbf{X} is concentrated on a hyperplane).

Moments (Vector Form)

$$\begin{aligned} E(\mathbf{X}) &= \boldsymbol{\mu} \\ \text{var}(\mathbf{X}) &= \mathbf{M} \end{aligned}$$

Linear Transformations If \mathbf{X} is $\mathcal{N}(\boldsymbol{\mu}, \mathbf{M})$ distributed, then $\mathbf{A}\mathbf{X} + \mathbf{b}$, where \mathbf{A} is a constant matrix and \mathbf{b} is a constant vector of dimensions such that the matrix multiplication and vector addition make sense, has the $\mathcal{N}(\mathbf{A}\boldsymbol{\mu} + \mathbf{b}, \mathbf{A}\mathbf{M}\mathbf{A}')$ distribution.

Addition Rule If $\mathbf{X}_1, \dots, \mathbf{X}_k$ are independent random vectors, \mathbf{X}_i being $\mathcal{N}(\boldsymbol{\mu}_i, \mathbf{M}_i)$ distributed, then $\mathbf{X}_1 + \dots + \mathbf{X}_k$ is a $\mathcal{N}(\boldsymbol{\mu}_1 + \dots + \boldsymbol{\mu}_k, \mathbf{M}_1 + \dots + \mathbf{M}_k)$ random variable.

Degeneracy If there exists a vector \mathbf{a} such that $\mathbf{M}\mathbf{a} = 0$, then $\text{var}(\mathbf{a}'\mathbf{X}) = 0$.

Partitioned Vectors and Matrices The random vector and parameters are written in *partitioned form*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \quad (17.1a)$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \quad (17.1b)$$

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_2 \end{pmatrix} \quad (17.1c)$$

when \mathbf{X}_1 consists of the first r elements of \mathbf{X} and \mathbf{X}_2 of the other $k - r$ elements and similarly for $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$.

Marginal Distributions Every marginal of a multivariate normal is normal (univariate or multivariate as the case may be). In partitioned form, the (marginal) distribution of \mathbf{X}_1 is $\mathcal{N}(\boldsymbol{\mu}_1, \mathbf{M}_{11})$.

Conditional Distributions Every conditional of a multivariate normal is normal (univariate or multivariate as the case may be). In partitioned form, the conditional distribution of \mathbf{X}_1 given \mathbf{X}_2 is

$$\mathcal{N}(\boldsymbol{\mu}_1 + \mathbf{M}_{12}\mathbf{M}_{22}^-[\mathbf{X}_2 - \boldsymbol{\mu}_2], \mathbf{M}_{11} - \mathbf{M}_{12}\mathbf{M}_{22}^- \mathbf{M}_{21})$$

where the notation \mathbf{M}_{22}^- denotes the inverse of the matrix \mathbf{M}_{22} if the matrix is invertible and otherwise any generalized inverse.

18 Chi-Square Distribution

Abbreviation $\text{chi}^2(\nu)$ or $\chi^2(\nu)$.

Type Continuous.

Rationales

- Sum of squares of IID standard normal random variables.
- Sampling distribution of sample variance when data are IID normal.

Parameter Real number $\nu > 0$ called “degrees of freedom.”

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\left(\frac{1}{2}\right)^{\nu/2}}{\Gamma\left(\frac{\nu}{2}\right)} x^{\nu/2-1} e^{-x/2}, \quad 0 < x < \infty.$$

Moments

$$\begin{aligned} E(X) &= \nu \\ \text{var}(X) &= 2\nu \end{aligned}$$

Addition Rule If X_1, \dots, X_k are independent random variables, X_i being $\text{chi}^2(\nu_i)$ distributed, then $X_1 + \dots + X_k$ is a $\text{chi}^2(\nu_1 + \dots + \nu_k)$ random variable.

Normal Approximation If ν is large, then

$$\text{chi}^2(\nu) \approx \mathcal{N}(\nu, 2\nu)$$

Relation to Other Distributions

- $\text{chi}^2(\nu) = \text{Gam}\left(\frac{\nu}{2}, \frac{1}{2}\right)$.
- If X is $\mathcal{N}(0, 1)$ distributed, then X^2 is $\text{chi}^2(1)$ distributed.
- If Z and Y are independent, X is $\mathcal{N}(0, 1)$ distributed and Y is $\text{chi}^2(\nu)$ distributed, then $X/\sqrt{Y/\nu}$ is $t(\nu)$ distributed.
- If X and Y are independent and are $\text{chi}^2(\mu)$ and $\text{chi}^2(\nu)$ distributed, respectively, then $(X/\mu)/(Y/\nu)$ is $F(\mu, \nu)$ distributed.

19 Student's t Distribution

Abbreviation $t(\nu)$.

Type Continuous.

Rationales

- Sampling distribution of pivotal quantity $\sqrt{n}(\bar{X}_n - \mu)/S_n$ when data are IID normal.
- Marginal for μ in conjugate prior family for two-parameter normal data.

Parameter Real number $\nu > 0$ called “degrees of freedom.”

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\sqrt{\nu\pi}} \cdot \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \cdot \frac{1}{\left(1 + \frac{x^2}{\nu}\right)^{(\nu+1)/2}}, \quad -\infty < x < +\infty$$

Moments If $\nu > 1$, then

$$E(X) = 0.$$

Otherwise the mean does not exist. If $\nu > 2$, then

$$\text{var}(X) = \frac{\nu}{\nu - 2}.$$

Otherwise the variance does not exist.

Normal Approximation If ν is large, then

$$t(\nu) \approx \mathcal{N}(0, 1)$$

Relation to Other Distributions

- If X and Y are independent, X is $\mathcal{N}(0, 1)$ distributed and Y is $\text{chi}^2(\nu)$ distributed, then $X/\sqrt{Y/\nu}$ is $t(\nu)$ distributed.
- If X is $t(\nu)$ distributed, then X^2 is $F(1, \nu)$ distributed.
- $t(1) = \text{Cauchy}(0, 1)$.

20 Snedecor's F Distribution

Abbreviation $F(\mu, \nu)$.

Type Continuous.

Rationale

- Ratio of sums of squares for normal data (test statistics in regression and analysis of variance).

Parameters Real numbers $\mu > 0$ and $\nu > 0$ called “numerator degrees of freedom” and “denominator degrees of freedom,” respectively.

Sample Space The interval $(0, \infty)$ of the real numbers.

Probability Density Function

$$f(x) = \frac{\Gamma(\frac{\mu+\nu}{2})\mu^{\mu/2}\nu^{\nu/2}}{\Gamma(\frac{\mu}{2})\Gamma(\frac{\nu}{2})} \cdot \frac{x^{\mu/2+1}}{(\mu x + \nu)^{(\mu+\nu)/2}}, \quad 0 < x < +\infty$$

Moments If $\nu > 2$, then

$$E(X) = \frac{\nu}{\nu - 2}.$$

Otherwise the mean does not exist.

Relation to Other Distributions

- If X and Y are independent and are $\text{chi}^2(\mu)$ and $\text{chi}^2(\nu)$ distributed, respectively, then $(X/\mu)/(Y/\nu)$ is $F(\mu, \nu)$ distributed.
- If X is $t(\nu)$ distributed, then X^2 is $F(1, \nu)$ distributed.

21 Cauchy Distribution

Abbreviation Cauchy(μ, σ).

Type Continuous.

Rationales

- Very heavy tailed distribution.
- Counterexample to law of large numbers.

Parameters Real numbers μ and $\sigma > 0$, called the “location” and “scale” parameter, respectively.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{1}{\pi\sigma} \cdot \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < +\infty$$

Moments No moments exist.

Addition Rule If X_1, \dots, X_k are IID Cauchy(μ, σ) random variables, then $\bar{X}_n = (X_1 + \dots + X_k)/n$ is also Cauchy(μ, σ).

Relation to Other Distributions

- $t(1) = \text{Cauchy}(0, 1)$.

22 Laplace Distribution

Abbreviation Laplace(μ, σ).

Type Continuous.

Rationales Median is maximum likelihood estimate of location parameter.

Parameters Real numbers μ and $\sigma > 0$, called the mean and standard deviation, respectively.

Sample Space The real numbers.

Probability Density Function

$$f(x) = \frac{\sqrt{2}}{2\sigma} \exp\left(-\sqrt{2}\left|\frac{x-\mu}{\sigma}\right|\right), \quad -\infty < x < \infty$$

Moments

$$E(X) = \mu$$
$$\text{var}(X) = \sigma^2$$