Statistics 5101, Fall 2000, Geyer

Homework Solutions #9

Problem L12-32

A general normal random variable $\mathbf{X}$ has the form $\mathbf{X} = \mathbf{a} + \mathbf{B}\mathbf{Z}$, where $\mathbf{Z} \sim \mathcal{N}(0, \mathbf{I})$. In order for $\mathbf{X}$ to have mean zero, we need $\mathbf{a} = 0$. Thus $\mathbf{X} = \mathbf{B}\mathbf{Z}$, and in order for $\mathbf{X}$ to be nondegenerate, $\mathbf{B}$ must be invertible, in which case

$$\mathbf{M} = \operatorname{var}(\mathbf{X}) = \mathbf{B}\mathbf{B}'$$

and

$$\mathbf{M}^{-1} = (\mathbf{B}')^{-1}\mathbf{B}^{-1}$$

and

$$Q = \mathbf{X}'\mathbf{M}^{-1}\mathbf{X} = \mathbf{X}'(\mathbf{B}')^{-1}\mathbf{B}^{-1}\mathbf{X} = \mathbf{Z}'\mathbf{Z}$$

because $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{X}$.

Now

$$\mathbf{Z}'\mathbf{Z} = \sum_{i=1}^{p} Z_i^2$$

is the sum of squares of p independent standard normal random variables, hence $\text{chi}^2(p)$ distributed.

Problem N5-1

Using R for the computer work

```
> M <- matrix(c(3, 2, -1, 2, 3, 2, -1, 2, 3), nrow=3)
> M
     [,1] [,2] [,3]
[1,]    3    2   -1
[2,]    2    3    2
[3,]   -1    2    3
> eigen(M)
$values
[1]  5.3722813  4.0000000 -0.3722813

$vectors
          [,1]          [,2]       [,3]
[1,] 0.4544013 -7.071068e-01 -0.5417743
[2,] 0.7661846 -1.155753e-16  0.6426206
[3,] 0.4544013  7.071068e-01 -0.5417743
```

The negative eigenvalue -0.3722813 means this is not a positive semi-definite matrix, hence not a variance matrix.

Problem N5-2

Using R for the computer work

```
> M <- matrix(c(3, 2, -1/3, 2, 3, 2, -1/3, 2, 3), nrow=3)
> M
           [,1] [,2]       [,3]
[1,]  3.0000000    2 -0.3333333
[2,]  2.0000000    3  2.0000000
[3,] -0.3333333    2  3.0000000
> eigen(M)
$values
[1] 5.666667 3.333333 0.000000

$vectors
          [,1]          [,2]       [,3]
[1,] 0.4850713 -7.071068e-01 -0.5144958
[2,] 0.7276069 -2.473336e-19  0.6859943
[3,] 0.4850713  7.071068e-01 -0.5144958
```

Since all three eigenvalues are nonnegative, this is a possible covariance matrix, but since one of the eigenvalues is zero, **M** is not positive definite and the random variable is degenerate.

Problem N5-3

The only hard thing about this problem is realizing it is simple (and perhaps looking up how to integrate trig functions).

$$E(X) = \frac{1}{2\pi} \int_0^{2\pi} \sin(u) \, du = 0$$
$$E(Y) = \frac{1}{2\pi} \int_0^{2\pi} \cos(u) \, du = 0$$

Hence the mean vector is zero. Then

$$\operatorname{var}(X) = E(X^2) = \frac{1}{2\pi} \int_0^{2\pi} \sin^2(u) \, du = \frac{1}{2}$$
$$\operatorname{var}(Y) = E(Y^2) = \frac{1}{2\pi} \int_0^{2\pi} \cos^2(u) \, du = \frac{1}{2}$$
$$\operatorname{cov}(X, Y) = E(XY) = \frac{1}{2\pi} \int_0^{2\pi} \sin(u) \cos(u) \, du = 0$$

So

$$\operatorname{var}\left\{ \begin{pmatrix} X \\ Y \end{pmatrix} \right\} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Interesting Comment Not part of the problem, but we have here another example that uncorrelated does not imply independent. $\text{cov}(X, Y) = 0$, but having a degenerate joint distribution, X and Y are highly dependent. Indeed, since the vector (X, Y) is concentrated on the unit circle, $X^2 + Y^2 = 1$ with probability one (essentially because of the trig identity $\sin^2 u + \cos^2 u = 1$ for all u). Thus the conditional distribution of Y given X is concentrated at the two points $\pm\sqrt{1 - X^2}$. So X almost determines Y (it determines the absolute value, only the sign is still random).

Problem N5-4

In general

$$\mathbf{w}'\mathbf{M}\mathbf{w} = \sum_{k=1}^{n} \sum_{l=1}^{n} m_{kl} w_k w_l$$

If we choose $\mathbf{w}$ so that only w_i and w_j are nonzero, this becomes

$$m_{ii} w_i^2 + 2 m_{ij} w_i w_j + m_{jj} w_j^2$$

and this must be nonnegative for any choice of w_i and w_j

$$m_{ii} w_i^2 + 2 m_{ij} w_i w_j + m_{jj} w_j^2 \geq 0 \tag{1}$$

Try

$$\begin{aligned} w_i &= 1/\sqrt{m_{ii}} \\ w_j &= 1/\sqrt{m_{jj}} \end{aligned}$$

Then (1) becomes

$$1 + 2 \frac{m_{ij}}{\sqrt{m_{ii} m_{jj}}} + 1 \geq 0$$

which implies one of the inequalities to be proved. If instead make w_j negative, we get the same thing with a minus sign on the middle term, and that proves the other inequality.

Problem N5-5

If the dimension is $n = 1$, then the variance matrix is just a scalar $\mathbf{M} = \sigma^2$ and $\mathbf{M}^{-1} = 1/\sigma^2$ and $\det \mathbf{M} = \sigma^2$. Making all these changes turns (5.24) on p. 144 of the notes into (9) on p. 180 in Lindgren.

Problem N5-6

A vector $\mathbf{X}$ concentrated at the point $\mathbf{a}$ has

$$\begin{aligned} E(\mathbf{X}) &= \mathbf{a} \\ \text{var}(\mathbf{X}) &= 0 \end{aligned}$$

(the variance matrix being the zero matrix). Our definition of multivariate normal says there is a multivariate normal random vector with this mean and variance, namely the vector $\mathbf{Y} = \mathbf{a} + \mathbf{B}\mathbf{Z}$, where $\mathbf{Z}$ is a standard multivariate normal random vector and we choose $\mathbf{B}$ to also be the zero matrix. Clearly $\mathbf{Y}$ is also concentrated at $\mathbf{a}$ and thus is equal to $\mathbf{X}$ with probability one.

Problem N5-8

Any linear transformation of multivariate normal is again multivariate normal (Theorem 12 of Chapter 12 in Lindgren, inexplicably omitted from the notes). Thus $\mathbf{Y}$ is multivariate normal. Hence to figure out which normal, we only need to find its parameters (mean vector and variance matrix).

$$\begin{aligned} E(\mathbf{Y}) &= \mathbf{M}^{-1} E(\mathbf{X}) = 0 \\ \operatorname{var}(\mathbf{Y}) &= \mathbf{M}^{-1} \operatorname{var}(X) \mathbf{M}^{-1} = \mathbf{M}^{-1} \mathbf{M} \mathbf{M}^{-1} = \mathbf{M}^{-1} \end{aligned}$$

Thus $\mathbf{Y} \sim \mathcal{N}(0, \mathbf{M}^{-1})$.