Statistics 5101, Fall 2000, Geyer

Homework Solutions #5

Problem L4-29

(a) This density is symmetric about zero, hence the mean is zero. Hence there is no difference between central moments and ordinary moments and $\operatorname{var}(Y) = E(Y^2)$. Now

$$\begin{aligned}
E(Y^2) &= \int_{-\infty}^{\infty} y^2 \frac{1}{2} e^{-|y|} \, dy \\
&= 2 \int_0^{\infty} y^2 \frac{1}{2} e^{-|y|} \, dy \\
&= \int_0^{\infty} y^2 e^{-y} \, dy \\
&= \Gamma(3) \\
&= 2! \\
&= 2
\end{aligned}$$

(b) This density is symmetric about zero, hence the mean is zero and $\operatorname{var}(Y) = E(Y^2)$. Now

$$\begin{aligned}
E(Y^2) &= \int_{-1}^{1} y^2 (1 - |y|) \, dy \\
&= 2 \int_0^{\infty} y^2 (1 - y) \, dx \\
&= 2 \left[\frac{y^3}{3} - \frac{y^4}{4} \right]_0^1 \\
&= 2 \left(\frac{1}{3} - \frac{1}{4} \right) \\
&= \frac{1}{6}
\end{aligned}$$

(c) This density is symmetric about 1/2, hence the mean is 1/2. Also

$$E(Y^2) = \int_0^1 y^2 6y(1 - y) \, dy = 6 \int_0^1 y^3 (1 - y) \, dy = 6 \left[\frac{y^4}{4} - \frac{y^5}{5} \right]_0^1 = \frac{6}{20}$$

Then

$$\operatorname{var}(Y) = E(Y^2) - E(Y)^2 = \frac{6}{20} - \left(\frac{1}{2} \right)^2 = \frac{1}{20}$$

Problem L4-40ab

(a)

$$E(X) = 1 \times \frac{1}{2} + 3\frac{1}{2} = 2$$

$$E(Y) = 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 2\frac{1}{3} = 1$$

$$E(X^2) = 1^2 \times \frac{1}{2} + 3^2\frac{1}{2} = 5$$

$$E(Y^2) = 0^2 \times \frac{1}{3} + 1^2 \times \frac{1}{3} + 2^2\frac{1}{3} = \frac{5}{3}$$

$$\operatorname{var}(X) = E(X^2) - E(X)^2 = 5 - 2^2 = 1$$

$$\operatorname{var}(Y) = E(Y^2) - E(Y)^2 = \frac{5}{3} - 1^2 = \frac{2}{3}$$

$$E(XY) = (1 \times 2)\frac{1}{4} + (3 \times 1)\frac{1}{3} + (3 \times 2)\frac{1}{12} = 2$$

$$\operatorname{cov}(X, Y) = E(XY) - E(X)E(Y) = 2 - 2 \times 1 = 0.$$

(the last result is obvious from symmetry).

(b)

$$\rho_{X,Y} = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y} = 0$$

Problem N2-21

Since $X_1 + \cdots + X_n = 0$, we also have $\operatorname{var}(X_1 + \cdots + X_n) = 0$, but

$$\operatorname{var}(X_1 + \cdots + X_n) = n \operatorname{var}(X_1) + n(n-1) \operatorname{cov}(X_1, X_2)$$

by Theorem 2.22 in the notes. Hence

$$\operatorname{cov}(X_1, X_2) = -\frac{1}{n-1} \operatorname{var}(X_1)$$

and

$$\operatorname{cor}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\operatorname{sd}(X_1)^2} = \frac{\operatorname{cov}(X_1, X_2)}{\operatorname{var}(X_1)} = -\frac{1}{n-1}$$

Problem N2-22

Almost exactly the same calculation as the preceeding problem, except one starts with the inequality

$$\operatorname{var}(X_1 + \cdots + X_n) \ge 0$$

and consequently derives an inequality.

Problem N2-24

This was CANCELLED, because it turned out to be messier than I thought.

$$\begin{aligned}
\operatorname{var}(\overline{X}_n) &= \operatorname{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\operatorname{var}\left(\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\operatorname{cov}(X_i, X_j) \\
&= \frac{\sigma^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\rho^{|i-j|}
\end{aligned}$$

We can get rid of one sum. There are n terms with $i = j$ hence ρ^0, and there are $2(n-1)$ terms with $i = j \pm 1$ hence ρ^1, and there are $2(n-2)$ terms with $i = j \pm 2$ hence ρ^2, and so forth to 2 terms with $i = 1$ and $j = n$ or vice versa hence ρ^{n-1}, thus

$$\operatorname{var}(\overline{X}_n) = \frac{\sigma^2}{n}\left(1 + 2\sum_{k=1}^{n-1}\frac{n-k}{n}\rho^k\right)$$

but this does not simplify any further, at least not using the geometric series.

If anyone is wondering how I ever thought this was simple, I was recalling that the limit as n goes to infinity is simple
Using the linear combination form for variance, we have

$$\lim_{n\to\infty} n\operatorname{var}(\overline{X}_n) = \sigma^2\left(1 + 2\sum_{k=1}^{\infty}\rho^k\right)$$

because

$$\frac{n-k}{n} \to 1, \qquad \text{as } n \to \infty$$

and

$$
\begin{aligned}
\sigma^2 \left(1 + 2 \sum_{k=1}^{\infty} \rho^k \right) &= \sigma^2 \left(-1 + 2 \sum_{k=0}^{\infty} \rho^k \right) \\
&= \sigma^2 \left(-1 + 2 \frac{1}{1 - \rho} \right) \\
&= \sigma^2 \frac{1 + \rho}{1 - \rho}
\end{aligned}
$$

But we need to cover more material before we can get this far with this problem.

Problem N2-25

First we need to do the analogous equation for covariance, which isn't given in the notes or in Lindgren.

$$
\begin{aligned}
cov(a + bX, c + dY) &= E\{(a + bX - \mu_{a+bX})(c + dY - \mu_{c+dY})\} \\
&= E\{bd(X - \mu_X)(Y - \mu_Y)\} \\
&= bdE\{(X - \mu_X)(Y - \mu_Y)\} \\
&= bd \operatorname{cov}(X, Y)
\end{aligned}
$$

Then

$$
\begin{aligned}
\operatorname{cor}(a + bX, c + dY) &= \frac{\operatorname{cov}(a + bX, c + dY)}{\operatorname{sd}(a + bX) \operatorname{sd}(c + dY)} \\
&= \frac{bd}{|bd|} \cdot \frac{\operatorname{cov}(X, Y)}{\operatorname{sd}(X) \operatorname{sd}(Y)} \\
&= \operatorname{sign}(bd) \operatorname{cor}(X, Y)
\end{aligned}
$$

Problem N2-28

(a)

$$
\begin{aligned}
E\{X(X - 1)\} &= \sum_{k=0}^{\infty} k(k-1) \frac{\mu^k}{k!} e^{-\mu} \\
&= \mu^2 \sum_{k=2}^{\infty} \frac{\mu^{k-2}}{(k-2)!} e^{-\mu} \\
&= \mu^2
\end{aligned}
$$

(b) We want to use

$$
\operatorname{var}(X) = E(X^2) - E(X)^2
$$

and we can get $E(X^2)$ from part (a)

$$E\{X(X-1)\} = E(X^2) - E(X) = \mu^2$$

so

$$E(X^2) = \mu^2 + \mu$$

and

$$\operatorname{var}(X) = (\mu^2 + \mu) - \mu^2 = \mu$$

Problem N2-30

Note the density is

$$f(x) = \frac{1}{b-a}, \qquad a < x < b$$

because the length of the interval is $b - a$.

This is symmetric about the midpoint of the interval $(a+b)/2$, so that is the mean.

Then

$$E(X^2) = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{(b^3 - a^3)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

and

$$\begin{aligned}
\operatorname{var}(X) &= E(X^2) - E(X)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{a^2 - 2ab + b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$

Problem N2-32

(a)

$$\begin{aligned}
E(X^p) &= \int_0^\infty x f(x)\,dx \\
&= \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha+p-1} e^{-\lambda x}\,dx \\
&= \frac{\Gamma(\alpha+p)}{\lambda^p \Gamma(\alpha)} \int_0^\infty \frac{\lambda^{\alpha+p}}{\Gamma(\alpha+p)} x^{\alpha+p-1} e^{-\lambda x}\,dx \\
&= \frac{\Gamma(\alpha+p)}{\lambda^p \Gamma(\alpha)}
\end{aligned}$$

and this cannot be simplified if p is not an integer.

(b) Using part (a) and the recursion formula for the gamma function, (B.2) in the appendix on "brand name distributions" of the notes, twice

$$E(X^2) = \frac{\Gamma(\alpha + 2)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha + 1)\Gamma(\alpha + 1)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha + 1)\alpha\Gamma(\alpha)}{\lambda^2 \Gamma(\alpha)} = \frac{(\alpha + 1)\alpha}{\lambda^2}$$

and

$$\operatorname{var}(X) = E(X^2) - E(X)^2 = \frac{(\alpha + 1)\alpha}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

Problem N2-33

(a) The integral

$$\int_1^\infty x^k \frac{3}{x^4} \, dx = 3 \int_1^\infty x^{k-4} \, dx$$

exists when $k - 4 < -1$, that is, when $k < 3$. If $k \geq 3$, the integral does not exist (or is $+\infty$).

The question asked about positive integers, so the answer is $k = 1$ or 2.

(b) For $k < 3$

$$E(X^k) = 3 \int_1^\infty x^{k-4} \, dx = \left. \frac{3x^{-4+k+1}}{-4 + k + 1} \right|_1^\infty = \frac{3}{3 - k}$$

Note (not a part of the problem, but an interesting point) that the formula

$$E(X^k) = \frac{3}{3 - k}$$

is *completely bogus* for $k > 3$. The formula gives a finite negative number for the expectation, which is ridiculous, the expectation of a positive random variable being positive. Of course, the expectation doesn't exist when $k > 3$, but (the point!) *you can't tell that from looking at the formula for $E(X^k)$ derived in this section.* You have to do the thinking in part (a) not just plow ahead to part (b).

Problem N2-34

(a) The integral

$$\int_0^1 x^k \frac{1}{2\sqrt{x}} \, dx = \frac{1}{2} \int_0^1 x^{k-1/2} \, dx$$

exists when $k - 1/2 > -1$, which is true for all positive k.

Thus $E(X^k)$ exists for $k = 1, 2, \ldots$ (all positive integers).

(b)

$$E(X^k) = \frac{1}{2} \int_0^1 x^{k-1/2} \, dx = \frac{1}{2} \left[\frac{x^{k+1/2}}{k + 1/2} \right]_0^1 = \frac{1}{2(k + 1/2)} = \frac{1}{2k + 1}$$