Statistics 5101, Fall 2000, Geyer

Homework Solutions #4

Problem 4-8

(a) This density is symmetric about 0, which is thus the mean.

(b) This density is symmetric about 0, which is thus the mean.

(c) This density is symmetric about 1/2, which is thus the mean.

Problem 4-44

(a)

$$\operatorname{var}(2X + 3Y - Z) = 2^2 \operatorname{var}(X) + 3^2 \operatorname{var}(Y) + 1^2 \operatorname{var}(Z) = 14$$

(b)

$$\operatorname{cov}(X - 2Y, 3X + Y + 2Z) = \operatorname{cov}(X, 3X) + \operatorname{cov}(-2Y, Y) = 3 \operatorname{var}(X) - 2 \operatorname{var}(Y) = 1$$

The other covariances vanish, because X, Y and Z are independent.

Problem 4-49

$$\begin{aligned}
\operatorname{cov}(X + Y, X - Y) &= \operatorname{cov}(X, X - Y) + \operatorname{cov}(Y, X - Y) \\
&= \operatorname{cov}(X, X) - \operatorname{cov}(X, Y) + \operatorname{cov}(Y, X) - \operatorname{cov}(Y, Y) \\
&= \operatorname{cov}(X, X) - \operatorname{cov}(Y, Y) \\
&= \operatorname{var}(X) - \operatorname{var}(Y)
\end{aligned}$$

and this equals zero if and only if $\sigma_X = \sigma_Y$.

Problem N2-3

Take $a = 1$ and $b = -1$ in Theorem 2.1 (linearity of expectation).

Problem N2-5

$$
\begin{aligned}
E(\overline{X}_n) &= E\left(\frac{X_1 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n}\big[E(X_1) + \cdots + E(X_n)\big] \\
&= \frac{1}{n} \cdot n\mu \\
&= \mu
\end{aligned}
$$

Problem N2-10

There are two things to be proved. First, since $X - a$ and $a - X$ are equal in distribution, they have the same moments, in particular,

$$
\begin{aligned}
E(X - a) &= E(a - X) \\
E(X) - a &= a - E(X) \\
2E(X) &= 2a \\
E(X) &= a
\end{aligned}
$$

That proves the first part.

The second part starts the same way except with k-th moments for k odd.

$$
\begin{aligned}
E\{(X - a)^3\} &= E\{(a - X)^3\} \\
E\{(X - a)^3\} &= E\{-(X - a)^3\} \\
E\{(X - a)^3\} &= -E\{(X - a)^3\}
\end{aligned}
$$

because $(-1)^k = -1$ if k is odd. Since the only number that is its own negative is zero,

$$
E\{(X - a)^3\} = 0,
$$

and this is what was to be proved because $\mu = a$ by the first part, so this is the k-th central moment.

Problem N2-11

(a) The inverse transformation $X = a + Y$ has derivative 1, so

$$
f_Y(y) = f_X(a + y)
$$

(b) The inverse transformation $X = a - Z$ has derivative -1, so

$$
f_Z(z) = f_X(a - z)
$$

(c) The two functions defined in parts (a) and (b) are the same if and only if they have the same values for the same argument, say t

$$\begin{aligned} f_Y(t) &= f_Z(t) \\ f_X(a+t) &= f_X(a-t) \end{aligned}$$

which is what was to be proved.

Problem N2-12

(all parts) Since these are symmetric distributions, the medians are the same as the means calculated in Problem 4-8.

Problem N2-14

(a) Since X is either zero or one and $0^k = 0$ and $1^k = 1$ for all k, it follows that $X^k = X$ for all k, and

$$E(X^k) = E(X) = \mu$$

(b) Since $0 \le X \le 1$, it follows that

$$0 \le E(X) \le 1$$

by monotonicity of probability (Theorem 2.8 in the notes).

(c)

$$\operatorname{var}(X) = E(X^2) - E(X)^2 = \mu - \mu^2 = \mu(1-\mu)$$

Problem N2-16

$$\operatorname{var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \operatorname{var}(X_i)$$

(the covariance terms are all zero if the variables are uncorrelated).

Problem N2-17

Note: There is no need to do this problem if you do N2-17 first. Both parts are special cases of the general formula derived in N2-17. Conversely, if you do this first, N2-17 can be done easily.

The first part:

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{\mu-\mu}{\sigma} = 0$$

and

$$\text{var}(Z) = \text{var}\left(\frac{X - \mu}{\sigma}\right) = \frac{\sigma^2}{\sigma^2} = 1$$

The second part:

$$E(X) = E(\mu + \sigma Z) = \mu + \sigma E(Z) = \mu + \sigma \cdot 0 = \mu$$

and

$$\text{var}(X) = \text{var}(\mu + \sigma Z) = \sigma^2 \, \text{var}(Z) = \sigma^2 \cdot 1 = 1$$

Problem N2-18

We need to solve the equations

$$\begin{aligned} \mu_Y &= a + b\mu_X \\ \sigma_Y^2 &= b^2 \sigma_X^2 \end{aligned}$$

for a and b. Solve the second and then plug into the first

$$\begin{aligned} b &= \frac{\sigma_Y}{\sigma_X} \\ a &= \mu_Y - b\mu_X = \mu_Y - \frac{\sigma_Y}{\sigma_X}\mu_X \end{aligned}$$

On the other hand, we could have used the solution to N2-17. First standardize, then "unstandardize"

$$Y = \mu_Y + \sigma_Y Z = \mu_Y + \sigma_Y \frac{X - \mu_X}{\sigma_X} = \mu_Y + \frac{\sigma_Y}{\sigma_X}(X - \mu_X)$$

which is the same solution as obtained by solving simultaneous equations.