Stat 5101 Final Exam **December 21, 2020**

Name _____ Student ID _____

The exam is open book and open notes. You may also use the handouts on "brand name distributions" and Greek letters. You may use a calculator. You may use R on your own computer or Rweb (`https://rweb.webapps.cla.umn.edu/Rweb/Rweb.general.html`).

You **may not** obtain help from any person, computer application, or service other than material on the course web pages or your own notes and homework or R or Rweb. In particular, you are not allowed to use Mathematica or other computer algebra system, including the Wolfram Alpha web site.

Show your work or give an explanation of your answer. No credit for numbers or formulas with no indication of where they came from. Leave no undone derivatives or integrals or means or variances or unevaluated gamma functions in your answers, but other than that requirement there is no unique "correct" simplification. Any correct (and explained) answer gets full credit unless the question explicitly states otherwise.

This exam is on-line. Submit your solutions to the course Canvas site (under Assignments) by 4:30 PM.

Abbreviations used:

IID for independent and identically distributed

PDF for probability density function

PMF for probability mass function

Bin(n, p) for the binomial distribution with parameters n and p

Beta(α_1, α_2) for the beta distribution with parameters α_1 and α_2

The points for the questions total to 200. There are 4 pages and 8 problems.

1. [25 pts.] Suppose the conditional distribution of Y given X is $\text{Bin}(n, X)$, and the marginal distribution of X is $\text{Beta}(\alpha_1, \alpha_2)$.

   (a) Find $E(Y)$.

   (b) Find $\text{var}(Y)$.

2. [25 pts.] Define

$$h_\theta(x) = \frac{x^2 + \cos(x)}{1 + x^2 + x^{\theta/2}}, \qquad 0 < x < \infty.$$

   (a) For what values of the positive real parameter θ does there exist a constant $c(\theta)$ that

$$f_\theta(x) = c(\theta) h_\theta(x), \qquad 0 < x < \infty,$$

   is a PDF? Hint: consider the cases $0 < \theta < 4$ and $4 \leq \theta < \infty$ separately.

   (b) If X is a random variable having this PDF, for what values of $\theta > 0$ and $\beta > 0$ does the expectation of X^β exist?

3. [25 pts.] Suppose X is a $\text{Gam}(\alpha, \lambda)$ random variable. What is the approximate normal distribution of $1/\sqrt{X}$ when α is large?

4. [25 pts.] Suppose X_1, X_2, ... are IID $\mathcal{N}(\theta, \theta^2)$ random variables, where θ is a real parameter. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large n with variance that is a constant function of the parameter? As usual,

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

5. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f_\theta(x, y) = \frac{3(2 - \theta x^2 - y^2)}{5 - \theta} \qquad 0 < x < 1, \ 0 < y < 1,$$

where $0 < \theta < 1$ is a parameter.

(a) Find the conditional PDF of X given Y.

(b) Find the conditional expectation of X given Y.

6. [25 pts.] Suppose you have called a phone help line and the automated system says there are three people waiting. Assume this does not include the person currently being helped. Suppose the service times for this help line form a Poisson process with rate $\lambda = 12$ per hour.

   (a) What is the expected time you have to wait until you start talking to the help line person (until that person has dealt with four others)?

   (b) Same as part (a) except median rather than mean.

7. [25 pts.] Suppose X has the distribution with PDF given by

$$f_\theta(x) = (1 + \theta)x^{-2-\theta} \qquad 1 < x < \infty,$$

   where $\theta > 0$ is a parameter. What is the PDF of $Y = 1/X$? The definition of a function describes the domain as well as the rule.

8. [25 pts.] Suppose X is a random variable having PDF given by

$$f_\theta(x) = (1 + \theta)x^{-2-\theta}, \qquad 1 < x < \infty,$$

   where $\theta > 0$ is a parameter.

   (a) Find its distribution function (DF). Be sure to define the DF on the whole real line.

   (b) Find the median of the distribution of X.