Stat 5101 Final Exam **December 16, 2014**

Name ______________________ Student ID ______________

The exam is closed book and closed notes. You may use three $8\frac{1}{2} \times 11$ sheets of paper with formulas, etc. You may use the handout on "brand name distributions". You may use a calculator.

Put all of your work on this test form (use the back if necessary). Show your work or give an explanation of your answer. No credit for numbers with no indication of where they came from. Leave no undone integrals in your answers, but other than that requirement there is no unique "correct" simplification. Any correct answer gets full credit, except as explicitly stated in questions.

The points for the questions total to 200. There are 10 pages and 8 problems.

1. [25 pts.] Suppose Suppose $X_1, \ldots, X_N$ are IID $\text{Gam}(\alpha, \lambda)$ random variables, where N is a positive-integer-valued random variable that has mean μ and variance σ^2 and is independent of all of the X_i. Let

$$Y = \sum_{i=1}^{N} X_i.$$

   (a) Find $E(Y)$.

(b) Find $\mathrm{var}(Y)$.

2. [25 pts.] Define

$$h_\theta(x) = \frac{1 + (1 + x)^2}{(1 + x^2)^\theta}, \qquad -\infty < x < \infty.$$

(a) For what values of the real parameter θ does there exist a constant $c(\theta)$ that

$$f_\theta(x) = c(\theta) h_\theta(x), \qquad -\infty < x < \infty,$$

is a PDF?

(b) If X is a random variable having this PDF, for what values of θ does the expectation of X exist?

(c) If X is a random variable having this PDF, for what values of θ does the variance of X exist?

3. [25 pts.] Suppose X_1, X_2, ... are IID $\text{Exp}(\lambda)$ random variables and

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad (*)$$

What is the approximate normal distribution of $\log(\overline{X}_n)$ when n is large?

4. [25 pts.] Suppose X_1, X_2, $\ldots$ are IID $\text{Gam}(\theta, \theta^2)$ random variables, where $\theta > 0$ is a real parameter. What is the variance stabilizing transformation: for what function g does $g(\overline{X}_n)$ have approximate nondegenerate normal distribution for large n with variance that is a constant function of the parameter θ? As usual, $\overline{X}_n$ is defined by equation $(*)$ in problem 3.

5. [25 pts.] Suppose clicks from a Geiger counter (a type of radiation detector) can be considered a Poisson process with rate 0.3 clicks per second.

   (a) What is the distribution of the number of clicks in a one minute interval? Specify the distribution completely, giving the value of the parameter as well as the name.

   (b) What is the mean number of clicks in a one minute interval?

   (c) What is the variance of the number of clicks in a one minute interval?

(d) What is the distribution of the time interval between clicks? Specify the distribution completely, giving the value of the parameter as well as the name.

(e) What is the probability of there being no clicks in a specified five second interval?

6. [25 pts.] Suppose X is has the Beta$(\theta, 2)$ distribution. Find the PDF of the random variable $Y = 1/(1 - X)$. The definition of a function describes the domain as well as the rule. For full credit, your answer must not contain any gamma functions.

7. [25 pts.] This

$$F_\theta(x) = \begin{cases} 0, & -\infty < x \leq -1 \\ \frac{1+2\theta+2\theta x+2x+x^2}{2(1+2\theta)}, & -1 < x \leq 0 \\ \frac{1+2\theta+2\theta x+2x-x^2}{2(1+2\theta)}, & 0 < x \leq 1 \\ 1, & 1 < x < \infty \end{cases}$$

is a DF, where $\theta > 0$ is a parameter. Find the corresponding PDF.

8. [25 pts.] Suppose the random vector (X, Y) has the PDF

$$f(x, y) = \frac{2}{\pi} \cdot (1 - x^2 - y^2), \qquad x^2 + y^2 < 1.$$

Hint: the domain the set of (x, y) pairs such that $x^2 + y^2 < 1$ (be careful about limits of integration: if x is fixed, then y goes from what to what?).

(a) Find the conditional PDF of Y given X. The definition of a function describes the domain as well as the rule.

(b) Find the conditional expectation of Y given X. Hint: symmetry.

(c) Find the conditional variance of Y given X.