

Stat 5101 (Geyer) Fall 2014
Homework Assignment 6
Due Wednesday, October 22, 2014

Solve each problem. Explain your reasoning. No credit for answers with no explanation. If the problem is a proof, then you need words as well as formulas. Explain why your formulas follow one from another.

6-1. The function

$$f(x) = 2x, \quad 0 < x < 1$$

is a PDF. Suppose X is a random variable having this PDF.

- (a) Calculate $E(X)$.
- (b) Calculate $E(X^2)$.
- (c) Calculate $\text{var}(X)$.

6-2. The function

$$f(x, y) = x + y, \quad 0 < x < 1, 0 < y < 1$$

is a PDF. Suppose (X, Y) is a random vector having this PDF.

- (a) Calculate $E(X)$.
- (b) Calculate $\text{var}(X)$.
- (c) Calculate $\text{cov}(X, Y)$.

(By symmetry, $E(Y) = E(X)$ and $\text{var}(Y) = \text{var}(X)$ so we do not need to calculate them.)

6-3. Suppose (X, Y) is a continuous random vector having PDF f . Say for each of the following definitions of f whether X and Y are independent or not.

- (a) $f(x, y) = 4xy, 0 < x < 1, 0 < y < 1$.
- (b) $f(x, y) = 8xy, 0 < x < y < 1$.
- (c) $f(x, y) = 144(x - 1/2)^2(y - 1/2)^2, 0 < x < 1, 0 < y < 1$.
- (d) $f(x, y) = 288(x - 1/2)^2(y - 1/2)^2, 0 < x < y < 1$.

6-4. Suppose X is a continuous random variable having PDF

$$f(x) = \begin{cases} 1+x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

- (a) Find $E(X)$.
- (b) Find $E(X^2)$.
- (c) Find $\text{var}(X^2)$.

Hint: Since the PDF has a case-splitting formula, you must split integrals into pieces

$$E\{g(X)\} = \int_{-1}^0 g(x)f(x) dx + \int_0^1 g(x)f(x) dx$$

such that the PDF is defined by one formula for each piece.

6-5. Suppose X is a continuous random variable having the $\text{Exp}(\lambda)$ distribution. Write $\mu = E(X)$.

- (a) Find $E\{(X - \mu)^3\}$.
- (b) Find $E\{(X - \mu)^4\}$.

6-6. Suppose (X, Y) is a continuous random vector having PDF

$$f(x, y) = 2, \quad 0 < x < y < 1$$

- (a) Find $E(X)$
- (b) Find $E(Y)$
- (c) Find $E(X^2)$
- (d) Find $E(Y^2)$
- (e) Find $E(XY)$
- (f) Find $\text{var}(X)$
- (g) Find $\text{var}(Y)$
- (h) Find $\text{cov}(X, Y)$

Hint: the limits of integration are a bit tricky.

$$E\{g(X, Y)\} = 2 \int_0^1 \int_0^y g(x, y) dx dy = 2 \int_0^1 \int_x^1 g(x, y) dy dx$$

6-7. Suppose X is a continuous random variable having the same PDF as in problem 6-4. Find its distribution function. Be sure to define the DF on the whole real line.

6-8. Suppose X is a continuous random variable having DF

$$F(x) = \begin{cases} 0, & x \leq 1 \\ 1 - 1/x, & x > 1 \end{cases}$$

Find its PDF. Define the PDF on the whole real line.

6-9. Suppose U has the Unif(0, 1) distribution. What is the PDF of

$$Y = -\frac{1}{\lambda} \log(U)$$

6-10. Suppose X has the Unif(-1, 1) distribution. What is the PDF of

$$Y = X^2$$

6-11. Suppose (X, Y) has the uniform distribution on the disk

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$$

What is the PDF of the random vector (R, T) which is (X, Y) expressed in polar coordinates?

Hint: The map from (r, t) to (x, y) is given by

$$\begin{aligned} x &= r \cos(t) \\ y &= r \sin(t) \end{aligned}$$